MECHANICAL VIBRATIONS AND STRUCTURAL DYNAMICS (R15A0368)

COURSE FILE

IV B. Tech I Semester

(2018-2019)

Prepared By

Mr. G Dheeraj Asst. Prof

Department of Aeronautical Engineering

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

Affiliated to JNTU, Hyderabad, Approved by AICTE - Accredited by NBA & NAAC – ‘A’ Grade - ISO 9001:2015 Certified)

Maisammaguda, Dhulapally (Post Via. Kompally), Secunderabad – 500100, Telangana State, India.

MRCET VISION

- To become a model institution in the fields of Engineering, Technology and Management.
- To have a perfect synchronization of the ideologies of MRCET with challenging demands of International Pioneering Organizations.

MRCET MISSION

To establish a pedestal for the integral innovation, team spirit, originality and competence in the students, expose them to face the global challenges and become pioneers of Indian vision of modern society.

MRCET QUALITY POLICY.

- To pursue continual improvement of teaching learning process of Undergraduate and Post Graduate programs in Engineering & Management vigorously.
- To provide state of art infrastructure and expertise to impart the quality education.

PROGRAM OUTCOMES
(PO's)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design / development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multi disciplinary environments.
12. **Life- long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

DEPARTMENT OF AERONAUTICAL ENGINEERING

VISION

Department of Aeronautical Engineering aims to be indispensable source in Aeronautical Engineering which has a zeal to provide the value driven platform for the students to acquire knowledge and empower themselves to shoulder higher responsibility in building a strong nation.

MISSION

The primary mission of the department is to promote engineering education and research. To strive consistently to provide quality education, keeping in pace with time and technology. Department passions to integrate the intellectual, spiritual, ethical and social development of the students for shaping them into dynamic engineers.

QUALITY POLICY STATEMENT

Impart up-to-date knowledge to the students in Aeronautical area to make them quality engineers. Make the students experience the applications on quality equipment and tools. Provide systems, resources and training opportunities to achieve continuous improvement. Maintain global standards in education, training and services.

PROGRAM EDUCATIONAL OBJECTIVES – Aeronautical Engineering

1. **PEO1 (PROFESSIONALISM & CITIZENSHIP):** To create and sustain a community of learning in which students acquire knowledge and learn to apply it professionally with due consideration for ethical, ecological and economic issues.
2. **PEO2 (TECHNICAL ACCOMPLISHMENTS):** To provide knowledge based services to satisfy the needs of society and the industry by providing hands on experience in various technologies in core field.
3. **PEO3 (INVENTION, INNOVATION AND CREATIVITY):** To make the students to design, experiment, analyze, and interpret in the core field with the help of other multi disciplinary concepts wherever applicable.
4. **PEO4 (PROFESSIONAL DEVELOPMENT):** To educate the students to disseminate research findings with good soft skills and become a successful entrepreneur.
5. **PEO5 (HUMAN RESOURCE DEVELOPMENT):** To graduate the students in building national capabilities in technology, education and research

PROGRAM SPECIFIC OUTCOMES – Aeronautical Engineering

1. To mould students to become a professional with all necessary skills, personality and sound knowledge in basic and advance technological areas.
2. To promote understanding of concepts and develop ability in design manufacture and maintenance of aircraft, aerospace vehicles and associated equipment and develop application capability of the concepts sciences to engineering design and processes.
3. Understanding the current scenario in the field of aeronautics and acquire ability to apply knowledge of engineering, science and mathematics to design and conduct experiments in the field of Aeronautical Engineering.
4. To develop leadership skills in our students necessary to shape the social, intellectual, business and technical worlds.

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

IV Year B. Tech, ANE-I Sem

L	T/P/D	C
5	1/-/-	4

(R15A0368) MECHANICAL VIBRATIONS AND STRUCTURAL DYNAMICS

Objectives:

- To gain fundamental knowledge on vibration and related systems in the context of Aircraft Structures
- To give Exposure on damped and undamped vibratory systems.
- Basic knowledge on dynamic balancing of rotor system

UNIT-I

FUNDAMENTALS OF VIBRATION: Brief history of vibration, Importance of the study of vibration, basic concepts of vibration, classification of vibrations, vibration analysis procedure, spring elements, mass or inertia elements, damping elements, harmonic analysis. FREE VIBRATION OF SINGLE DEGREE OF FREEDOM SYSTEMS: Introduction, Free vibration of an undamped translational system, free vibration of an undamped torsional system, stability conditions, Raleigh's energy method, free vibration with viscous damping, free vibration with coulomb damping, free vibration with hysteretic damping.

UNIT-II

HARMONICALLY EXITED VIBRATIONS: Introduction, Equation of motion, response of an undamped system under harmonic force, Response of a damped system under harmonic force, Response of a damped system under harmonic motion of the base, Response of a damped system under rotating unbalance, forced vibration with coulomb damping, forced vibration with hysteresis damping.

UNIT-III

VIBRATION UNDER GENERAL FORCING CONDITIONS: Introduction, Response under a general periodic force, Response under a periodic force of irregular form, Response under a non periodic force, convolution integral. Two Degree of Freedom Systems: Introduction, Equation of motion for forced vibration, free vibration analysis of an undamped system, Torsional system, Coordinate coupling and principal coordinates, forced vibration analysis.

UNIT-IV

MULTIDEGREE OF FREEDOM SYSTEMS: Introduction, Modeling of Continuous systems as multi degree of freedom systems, Using Newtons second law to derive equations of motion, Influence coefficients, Free and Forced vibration of undamped systems, Forced vibration of viscously damped systems. Determination Of Natural Frequencies and Mode Shapes: Introduction, Dunkerleys formula, Rayleighs method, Holzers method, Matrix iteration method, Jacobi;s method.

UNIT-V

CONTINUOUS SYSTEMS: Transverse vibration of a spring or a cable, longitudinal vibration of bar or rod, Torsional vibration of a bar or rod, Lateral vibration of beams, critical speed of rotors.

Text Books:

1. Mechanical Vibrations by S.S.Rao.
2. Mechanical Vibrations by V.P.Singh

Reference Books:

1. Mechanical Vibrations by G.K. Grover
2. Mechanical Vibrations by W.T. Thomson
3. Mechanical vibrations: theory and application to structural dynamics, Michel Géradin, Daniel Rixen, John Wiley, 1997

Outcomes:

- Fundamental frequency of Multi- DOF systems can estimate by various methods.
- Effect of unbalance in rotating masses has been studied.
- How to determine eigenvalues and eigenvectors for a vibratory system has analysed

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF AERONAUTICAL ENGINEERING

MECHANICAL VIBRATION AND STRUCTURAL DYNAMICS

MODEL PAPER-I(R13)

MAXIMUM MARKS: 75

PART A **Max Marks: 25**

i. All questions in this section are compulsory
ii. Answer in TWO to FOUR sentences.

1. Find mass W, if the system has a natural frequency of 10 Hz shown in fig.1. Take K_1 = 2 N/mm, K_2 = 1.5 N/mm, K_3 = 3 N/mm and K_4 = K_5 = 1.5 N/mm. [3]

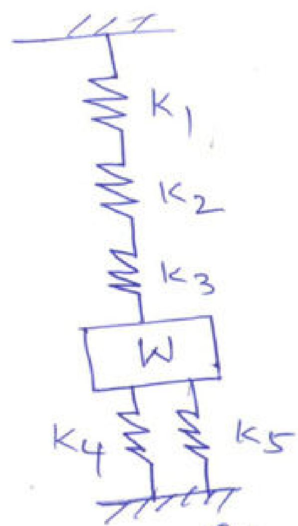


Fig. 1

2. What is vibration; write short notes on importance of vibration. [2]

3. What is meant by vibration isolation and transmissibility [3]

4. Derive the expression for natural frequency of undamped 2 DOF torsional vibration system. [3]

5. Write shortnotes on vibration isolation
[2]

6. What is meant by coordinate coupling explain briefly
[2]

7. Define (1) Fundamental frequency(2) Critical damping co-efficient (3) Time period
[3]

8. Explain briefly about Frahm's read Tachometer with near sketch.
[2]

9. What is meant by Eigenvalue and Eigenvector and explain with respect to vibration with an example.
[3]

10. Write short notes about self-excitation and stability analysis
[2]

PART B **Max Marks: 50**

i. Answer only one question among the two questions in choice.
ii. Each question answer (irrespective of the bits) carries 10M.

11. A weight attached to a spring of stiffness 625 N/m has a viscous damping device. When the weight is displaced and released, the period of vibration is found to be 2 seconds, and the ratio of consecutive amplitudes is 4 to 1. Determine the amplitude and phase when a Force F(t) = 20 Cos(5t) acts on the system [10]

OR

12. An unknown mass 'm' kg attached at the end of an unknown spring 'k' has a natural frequency of 100 cpm when 0.5 kg mass is added to 'm', the natural frequency is altered by 25% Determine the unknowns 'm' and 'k' ? ii) A spring mass system has a natural frequency of 10 rad/sec. The mass is pulled down from its static equilibrium position by 5 mm and given an upward velocity of 10 cm/sec, determine the ensuing motion. [10]

13. In a spring mass damper system the amplitude decays to half the original value in 4 oscillations and it takes 0.2 seconds to complete these oscillations. If the mass is set in

to free vibrations with an initial displacement of 5 mm and initial velocity of 0.5 m/sec, determine i) the subsequent motion ii) maximum amplitude of the mass iii) Time elapsed while the amplitude decays to less than or equal to 0.5 mm. [10]

OR

14. Why the vibration analysis for a vehicle free vibration due to engine balance for the single degree of freedom is required? Explain with an example. [10]

15 a) A uniform rod hangs freely from a hinge at the top. Using the three modes Φ1= x/l, Φ2= sin(x/l), and Φ3 = sin (2x/l), determine the characteristic equation by using the Rayleigh-Ritz method?

b). Determine the flexibility matrix for the spring-mass system shown in Fig.15.1 [5+5]

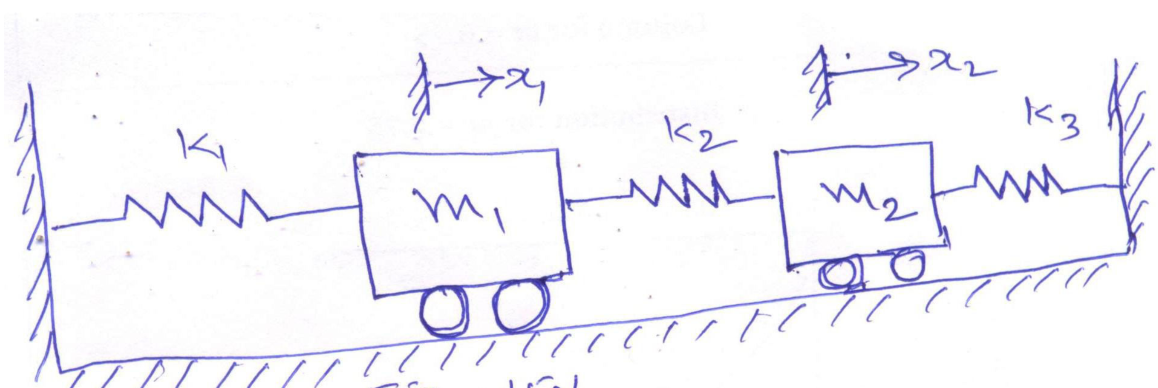


Fig 15.1

OR

16. Using Holzer's method, determine the natural frequencies and mode shapes of the torsional system of Fig. 16.1 when J = 1.0 kg-m^2 and K = 0.20 X 10^6 Nm/rad. [10]

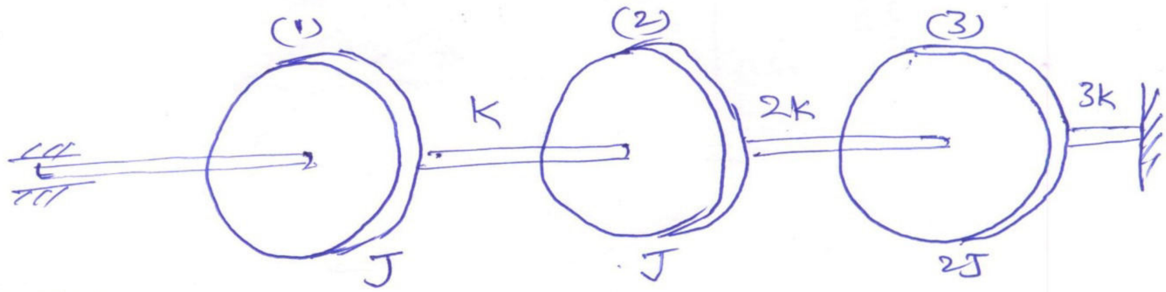


Fig 16.1

.

17. Using the Rayleigh-Ritz method, determine the first two natural frequencies and mode shapes for the longitudinal vibration of a uniform rod with a spring of stiffness k0 attached to the free end, as shown in Fig17.1. Use the first two normal modes of the fixed-free rod in longitudinal motion. [10]

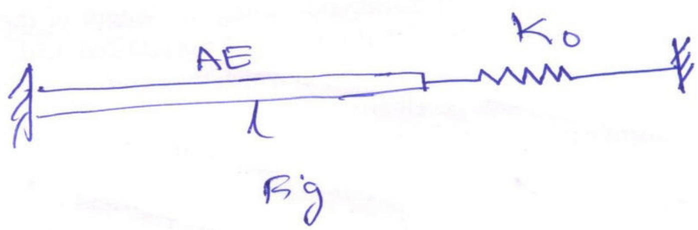


Fig. 17.1

OR

18. A machine of 20 kg mass is to be mounted on a vibrating base. The base vibration rages from 60 Hz to 75 Hz. And the amplitude varies from 2 mm to 3 mm. If the machine is to be isolated such that the amplitude is less than or equal to 0.5 mm determine the equivalent stiffness of the isolator to be used? [10]

19. Why the vibration analysis for a vehicle free vibration due to engine balance for the single degree of freedom is required? Explain with an example. [10]

OR

20. What is the need for vibration analysis for a vehicle free vibration due to road roughness for the single degree of freedom? Explain with an example. [10]

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF AERONAUTICAL ENGINEERING

MECHANICAL VIBRATION AND STRUCTURAL DYNAMICS

MODEL PAPER-II(R13)

MAXIMUM MARKS: 75

PART A **Max Marks: 25**

iii. All questions in this section are compulsory
iv. Answer in TWO to FOUR sentences.

1. Derive the expression for natural frequency of undamped free vibration system (3M)
2. Derive the equation of machine of undamped forced vibratory system (3M)
3. Explain briefly about hysteresis dumpily and coulomb dumpily. (2M)
4. Find the natural frequency of the system shown in fig-1. Take K_1=K_2=**1500 N/m,** K_3 **=2000 N/m and m= 5kg** (3M)

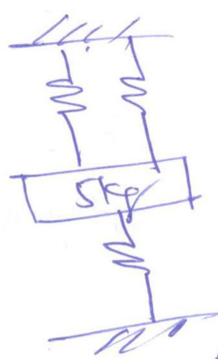


Fig. 1

5. Derive the expression for natural frequency of undamped 2 DOF spring –mass system. (3M)
6. Write short notes on inference coefficients (2M)
7. Write the procedure to derive equation of motion using Lagrange's equation (2M)

8. Define the term vibration and write different types of vibrations (2M)
9. Define (1) Logarithmic decrement
   (2) Periodic and a periodic motion
   (3) Potential energy (3M)

10. Write short notes an
    (1) Principal coordinates
    (2) Semi-definite system (2M)

PART B **Max Marks: 50**

iii. Answer only one question among the two questions in choice.
iv. Each question answer (irrespective of the bits) carries 10M.

11. What effect does a decrease in mass have on the frequency of a systems (10M)

(OR)

12. A cylinder of mass M radius' **r** 'rolls without slipping on a cylindrical surface of radius '**R**'. Find the natural frequency for small oscillation about the lowest point. (10M)
13. Find the steady state response of undamped single DOR systems subjected to the force
    F(t)= $\boldsymbol{F_0 e^{iwt}}$ by using the method of Laplace transformation (10M)

(OR)

14. Two rotors A & B are attached to the ends of a shaft 800mm long. The mass of the rotor '**A**' is 600 kg and is radius of gyration is 500mm. The corresponding values of rotor **B** 700kg and 600mm respectively. The shaft is 90mm diameter for the first 300mm, 150mm for next 180mm length and 120mm for the remaining length. Modulus of rigidity of the shaft material is 0.8x10^5MN/m^2. Find
    1) The position of the node.
    2) The frequency of torsional vibration (10M)

15. A uniform bar of length **l** is fixed at one end and the free end is stretched uniformly $\boldsymbol{l_0}$ and released at t=o. find the resulting longitudinal vibration. (10M)

(OR)

16. A uniform circled shaft of length l is fixed at the two ends. at its middle point a torque **To** is applied which twists it by θ_0 radians at the middle point. If the torque is released suddenly. Find the subsequent motion. (10M)

17. Compare the mode shape of a rotating shaft with a stationary shaft assuming that the shaft is rotating on a soft bearing (10M)

(OR)

18. (a) what is a principal coordinate

(b) the equation of motion of a two degrees of freedom system is given by

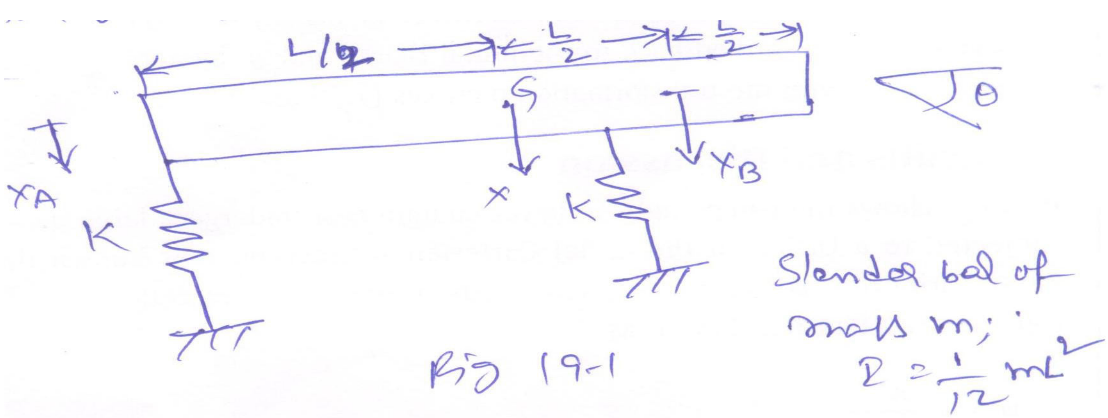
$$\begin{bmatrix} m & 0 \\ 0 & \frac{mL^2}{12} \end{bmatrix} + \begin{bmatrix} 2K & -\frac{KL}{4} \\ -\frac{KL}{4} & \frac{5KL^2}{16} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigenvectors for the above system given by $X_1 = \begin{bmatrix} 1 \\ \frac{1.43}{L} \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ \frac{-8.42}{L} \end{bmatrix}$

Calculate the principal coordinates of the system. (10M)

19. a) What are static and dynamic couplings?

b) Derive the differential equations governing free vibration of the system shown in figure 19.1, comprising a straight slender balance Supported by two springs and discuss the coupling using **x and θ** as generalized coordinates (4+6M)



OR

20. What is the need for vibration analysis for a vehicle free vibration due to road roughness for the single degree of freedom? Explain with an example. (10M)

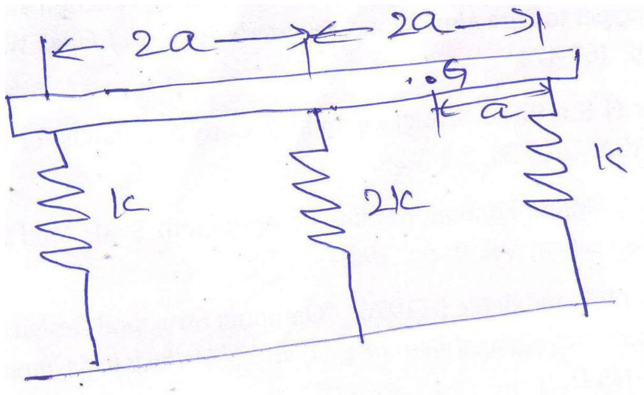


Fig 20.1

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF AERONAUTICAL ENGINEERING

MECHANICAL VIBRATION AND STRUCTURAL DYNAMICS

MODEL PAPER-III(R13)

MAXIMUM MARKS: 75

PART A **Max Marks: 25**

iAll questions in this section are compulsory

iiAnswer in TWO to FOUR sentences.

1. Derive the expression for natural frequency of undamped free vibration system (3M)
2. Derive the equation of machine of undamped forced vibratory system.(3M)
3. Explain briefly about hysteresis dumpily and coulomb dumpily.(2M)
4. Find the natural frequency of the system shown in fig-1. Take K_1=K_2=**1500 N/m,**K_3**=2000 N/m and m= 5kg (3M)**

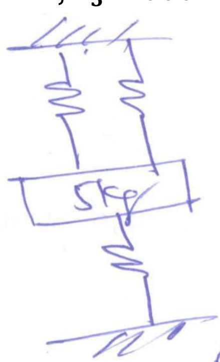


Fig.1

5. Derive the expression for natural frequency of undamped 2 DOF spring –mass system.(3M)
6. Write short notes on inference coefficients (2M)
7. Write the procedure to derive equation of motion using Lagrange's equation(2M)
8. Define the term vibration and write different types of vibrations(2M)
9. Define (1) Logarithmic decrement
   (2) Periodic and a periodic motion
   (3) Potential energy (3M)

10. Write short notes an
   (1) Principal coordinates

(2) Semi-definite system (2M)

PART B **Max Marks: 50**

I Answer only one question among the two questions in choice.

II Each question answer (irrespective of the bits) carries 10M.

11. What effect does a decrease in mass have on the frequency of a systems

(OR)

12. A cylinder of mass M radius' **r** 'rolls without slipping on a alindrical surface of radius '**R'**. Find the natural frequency for small oscillation about the lowest point.

13. Find the steady state response of undamped single DOR systems subjected to the force
F(t)= $\boldsymbol{F_0 e^{iwt}}$ by using the method of laplace transformation

(OR)

14. Two rotors A & B are attached to the ends of a shaft 800mm long. The mass of the rotor '**A**' is 600 kg and is radius of gyration is 500mm. The corresponding values of rotor **B** 700kg and 600mm respectively. The shaft is 90mm diameter for the first 300mm, 150mm for next 180mm length and 120mm for the remaining length. Modulus of rigidity of the shaft material is 0.8x10^5MN/m^2. Find

3) The position of the node.
4) The frequency of torsional vibration

15. A uniform bar of length **l** is fixed at one end and the free end is stretched uniformly $\boldsymbol{l_0}$ and released at t=o. find the resulting longitudinal vibration.

(OR)

16. A uniform circled shaft of length l is fixed at the two ends at its middle point a torque **To** is applied which twists it by θ_0 radians at the middle point. If the torque is released suddenly. Find the subsequent motion.

17. Compare the mode shape of a rotating shaft with a stationary shaft assuming that the shaft is rotating on a soft bearing.

(OR)

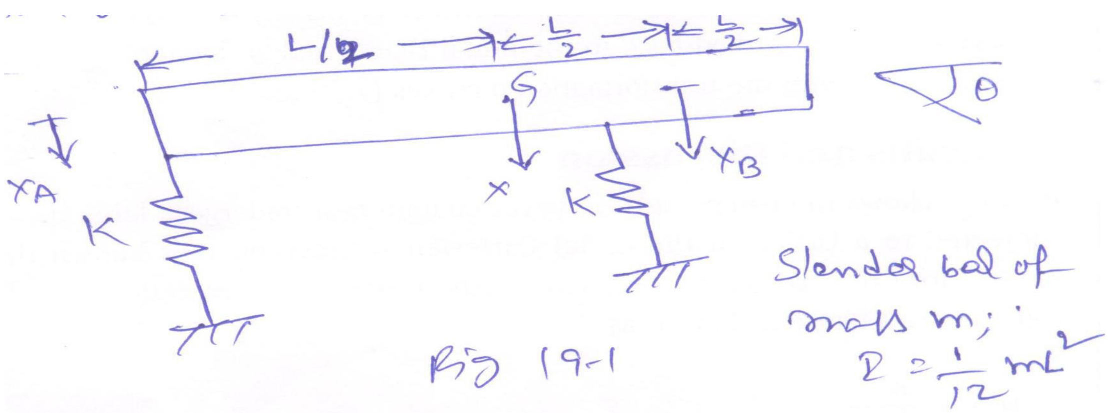
18. (a) what is a principal coordinate
(b) the equation of motion of a two degrees of freedom system is given by

$$\begin{bmatrix} m & 0 \\ 0 & \frac{mL^2}{12} \end{bmatrix} + \begin{bmatrix} 2K & -\frac{KL}{4} \\ -\frac{KL}{4} & \frac{5KL^2}{16} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigen vectors for the above system or given by $X1=\begin{bmatrix}1\\ \frac{1.43}{L}\end{bmatrix}$, $X2=\begin{bmatrix}1\\ \frac{-8.42}{L}\end{bmatrix}$

Calculate the principal coordinates of the system.

19. a) what are static and dynamic couplings?
b) derive the differential equations governing free vibration of the system shown in figure 19.1,comprising a straight slender balance Supported by two springs and discuss the coupling using **x and θ** as generalized coordinates



(OR)

20. The rigid beam shown in figure in its position of static equilibrium in the figure has a mass m and a mass moment of inertia $2ma^2$ about an axis perpendicular to the plane of the diagram and through its centre of gravity G. assuming no horizontal motion of G, derive the equation of motion considering the vertical displacement of CG and the rotation about the CG as the coordinates. Find the frequencies of small oscillations and the corresponding position of nodes. Identify the natural coordinates for decoupling the equations of motion.

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF AERONAUTICAL ENGINEERING

MECHANICAL VIBRATION AND STRUCTURAL DYNAMICS

MODEL PAPER-IV(R13)

MAXIMUM MARKS: 75

PART A **Max Marks: 25**

I All questions in this section are compulsory

II Answer in TWO to FOUR sentences.

1. Write a note on stiffness influence coefficients[2]
2. Derive the equation of motion of a simple spring mass system using energy method[3]
3. Define the terms SHM, resonance and time period[3]
4. Explain briefly about and coulomb dumping[3]
5. Write short notes on vibration isolation [2]
6. What is meant by static coupling in vibration system [3]
7. Write short notes on modeshapes with examples[2]
8. Write the procedure to find eigenvalue for the 3 DOF system[3]
9. Write short notes on transfer function in vibrations[2]
10. List out some vibration applications in airborne system[2]

PART B **Max Marks: 50**

I Answer only one question among the two questions in choice.

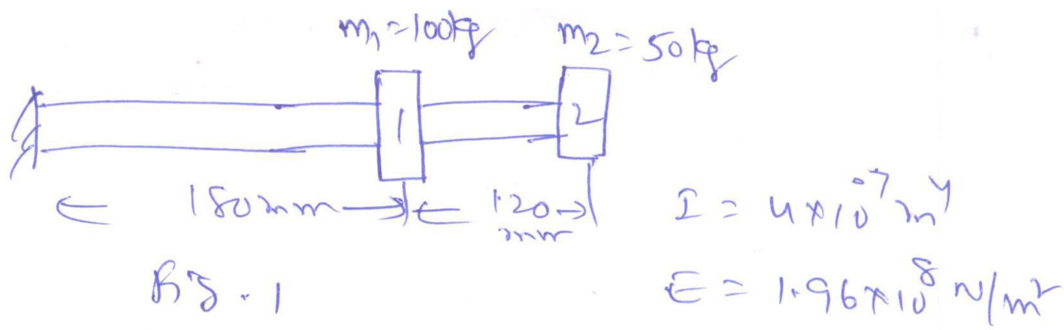
II Each question answer (irrespective of the bits) carries 10M.

1.

11. Define force transmissibility and obtain expression for
    i. Force transmissibility
    ii. Phase leg of transmitted force with impressed force.

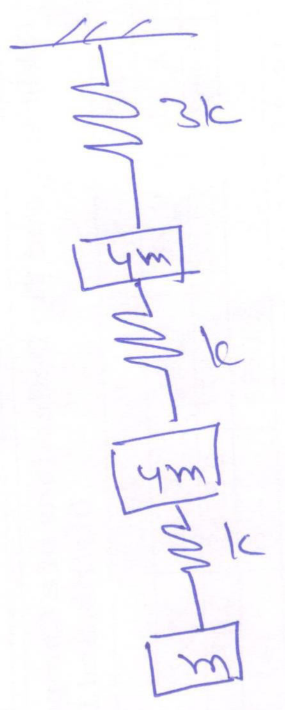
(OR)

12. A machine of mass 100kg cylinder at 600 rpm has a rotating **unbalance** of 100kg .mm. The machine is mounted on springs having stiffness 85 KN/m and negligible damping. The system is contained to more axially.
    a. Determine the steady state amplitude.
    b. If the damping is introduced to reduce the amplitude. By 50%, what should be the damping coefficient also find damping factor.
13. Find the fundamental natural frequency of Transverse vibration for the student shown in fig.1 by **dunkerles** method.



(OR)

14. Find the fundament material frequency for the system shown in fig.2 by the method of matrix iteration



15. a) State the types of damping and explain in brief viscous damping.
    b) A spring mass- dashpot system has mass 10kg and stiffness 40N/m. if the amplitude of free vibration decreases to 25% of original value after 5 c**ycles.** Determine the damping coefficient.

(OR)

16. Define logarithmic decrement show that logarithmic decrement can be expressed as **S=** $\frac{l}{n}$ **log e** $\frac{x_o}{x_n}$ (derive the expression), where x_o is amplitude at particular maximum and x_n is amplitude after on cycles.

17. A mass of 100 kg is suspended on a spring having a stiffness of 19600 N/m and is acted up on by a harmonic force of 39.2 N at the undamped natural frequency. The damping coefficient is 98 N-S/m, determine**.**
    a) Undamped natural frequency
    b) Amplitude of vibration of mass.
    c) Phase difference between force and displacement.

(OR)

18. A spring is tightly stretched between two supports as shown in fig.3. The tension T θ In the spring may be assumed to be constant for small displacement. Obtain the two natural frequencies for the system.

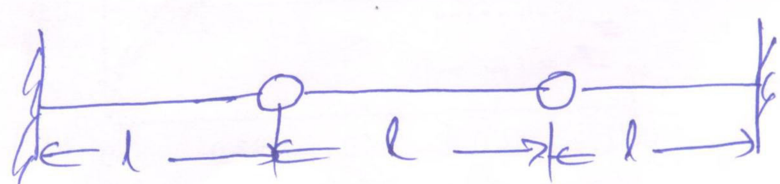


Fig. 3.

19. Derive the expression for longitudinal vibration of a bar.

(OR)

20. Derive the expression for vibration of string under tension.

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF AERONAUTICAL ENGINEERING

MECHANICAL VIBRATION AND STRUCTURAL DYNAMICS

MODEL PAPER-V(R13)

MAXIMUM MARKS: 75

PART A **Max Marks: 25**

I All questions in this section are compulsory

II Answer in TWO to FOUR sentences.

1. Write a note on influence coefficients[2]
2. Define the terms damping factor and logarithmic decrement[3]
3. What are continuous systems? explain[2]
4. Derive the equation of motion of a simple spring mass system using Newton's laws of motion[3]
5. Define the terms periodic motion, phase difference and DOF[2]
6. Explain briefly about modal analysis [3]
7. Differentiate discrete systems and distributed systems in vibrations[2]
8. Derive the equation of machine of undamped forced vibratory system[3]
9. Explain briefly about hysteresis dumping[2]
10. Explain briefly about dynamic coupling in vibration system[3]

PART B **Max Marks: 50**

I Answer only one question among the two questions in choice.

II Each question answer (irrespective of the bits) carries 10M.

11. Using **stodola** method find the fundamental natural frequency and mode shape of the system shown in fig .1

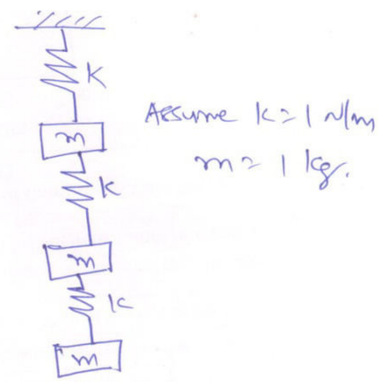


Fig. 1

(OR)

12. a). Derive the following terms
    i. Resonance
    ii. Simple harmonic motion
    iii. Time period

    b) Analyse the following motion

    X_1 =2 Cos (wf + 0.5)

    X_2 = 5 sin (wf +1.0)

13. A spring of an auto mobile frailer are compressed 0.1 under its own weight. Find the critical speed when the auto mobile is traveling over a road with a profile approximated by a sine wave of amplitude 0.08m and a wavelength of 14m. What will be the amplitude of ………. At 60 Km/hr.

(OR)

14. Determine the natural frequencies and mode shapes for a system shown in fig.2. J_1 and J_2 are mass moment inertias of the discs K_t isfor final stiffness of shaft.

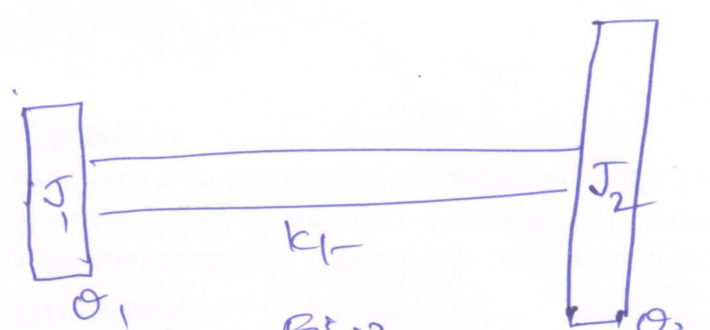


Fig. 2

15. A steel cantilever become carrying a weight of 100gms at the free end is used as frequency meter. The beam has a length of 10cm, weight of 0.5gm and thickness of 2mm. The internal friction is equivalent to a damping ratio of 0.05. When the fixed end of the beam is subjected to a harmonic displacement y(t)= 0.5 Cos wt cm, the maximum tip displacement is observed to be 2.5cm, find the forcing frequency w.

(OR)

16. Using **stodola** method find the fundamental natural frequency and mode shape of the system shown in fig .1

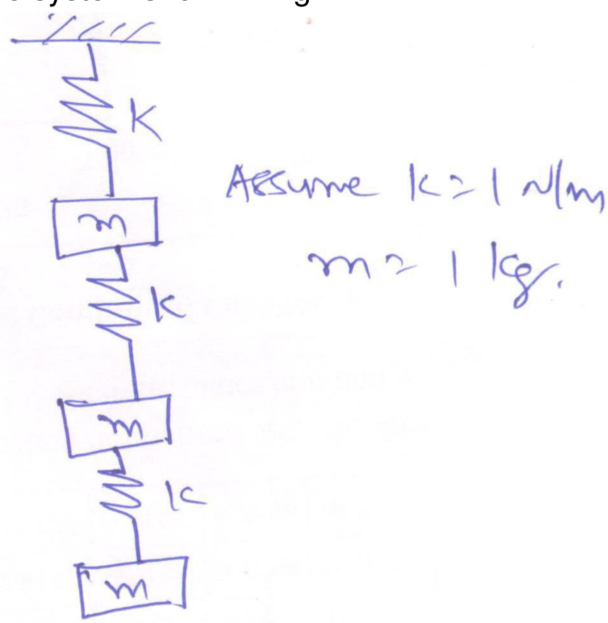


Fig. 1

17. a). Derive the following terms
    iv. Resonance
    v. Simple harmonic motion
    vi. Time period

    b) Analyse the following motion

    X_1 =2 Cos (wf + 0.5)

    X_2 = 5 sin (wf +1.0)

(OR)

18. A spring of an auto mobile trailer are compressed 0.1 under its own weight. Find the critical speed when the auto mobile is traveling over a road with a profile

approximated by a sine wave of amplitude 0.08m and a wavelength of 14m. What will be the amplitude of 5cm at 60 Km/hr.

19. Derive the expression for torsional vibration of a shaft.

(OR)

20. Derive the expression for transverse vibration of a beam.

List of Symbols

Symbol	Meaning	English Units	SI Units
$a, a_0, a_1, a_2, \ldots$	constants, lengths		
a_{ij}	flexibility coefficient	in./lb	m/N
$[a]$	flexibility matrix	in./lb	m/N
A	area	in^2	m^2
$A, A_0, A_1, \ldots$	constants		
$b, b_1, b_2, \ldots$	constants, lengths		
$B, B_1, B_2, \ldots$	constants		
$\vec{B}$	balancing weight	lb	N
$c, \underset{\sim}{c}$	viscous damping coefficient	lb-sec/in.	N · s/m
$c, c_0, c_1, c_2, \ldots$	constants		
c	wave velocity	in./sec	m/s
c_c	critical viscous damping constant	lb-sec/in.	N · s/m
c_i	damping constant of i th damper	lb-sec/in.	N · s/m
c_{ij}	damping coefficient	lb-sec/in.	N · s/m
$[c]$	damping matrix	lb-sec/in.	N · s/m
C, C_1, C_2, C_1', C_2'	constants		
d	diameter, dimension	in.	m
D	diameter	in.	m
$[D]$	dynamical matrix	sec^2	s^2
e	base of natural logarithms		
e	eccentricity	in.	m
$\vec{e}_x, \vec{e}_y$	unit vectors parallel to x and y directions		
E	Young's modulus	lb/in^2	Pa
$E[x]$	expected value of x		
f	linear frequency	Hz	Hz
f	force per unit length	lb/in.	N/m
$\underset{\sim}{f}, \boldsymbol{f}$	unit impulse	lb-sec	N · s
F, F_d	force	lb	N
F_0	amplitude of force $F(t)$	lb	N

Symbol	Meaning	English Units	SI Units
F_t, F_T	force transmitted	lb	N
F_t	force acting on ith mass	lb	N
$\vec{F}$	force vector	lb	N
$\underset{\sim}{F}$, $\boldsymbol{F}$	impulse	lb-sec	N · s
g	acceleration due to gravity	in./sec^2	m/s^2
$g(t)$	impulse response function		
G	shear modulus	lb/in^2	N/m^2
h	hysteresis damping constant	lb/in	N/m
$H(i\omega)$	frequency response function		
i	$\sqrt{-1}$		
I	area moment of inertia	in^4	m^4
$[I]$	identity matrix		
Im()	imaginary part of ()		
j	integer		
J	polar moment of inertia	in^4	m^4
$J, J_0, J_1, J_2, \ldots$	mass moment of inertia	lb-in./sec^2	kg · m^2
k, $\underset{\sim}{k}$	spring constant	lb/in.	N/m
k_i	spring constant of ith spring	lb/in.	N/m
k_t	torsional spring constant	lb-in/rad	N-m/rad
k_{ij}	stiffness coefficient	lb/in.	N/m
$[k]$	stiffness matrix	lb/in.	N/m
l, l_i	length	in.	m
m, $\underset{\sim}{m}$	mass	lb-sec^2/in.	kg
m_i	ith mass	lb-sec^2/in.	kg
m_{ij}	mass coefficient	lb-sec^2/in.	kg
$[m]$	mass matrix	lb-sec^2/in.	kg
M	mass	lb-sec^2/in.	kg
M	bending moment	lb-in.	N · m
$M_t, M_{t1}, M_{t2}, \ldots$	torque	lb-in.	N · m
M_{t0}	amplitude of $M_t(t)$	lb-in.	N · m
n	an integer		
n	number of degrees of freedom		
N	normal force	lb	N
N	total number of time steps		
p	pressure	lb/in^2	N/m^2
$p(x)$	probability density function of x		
$P(x)$	probability distribution function of x		
P	force, tension	lb	N
q_j	jth generalized coordinate		
$\vec{q}$	vector of generalized displacements		
$\dot{\vec{q}}$	vector of generalized velocities		
Q_j	jth generalized force		
r	frequency ratio $= \omega/\omega_n$		
$\vec{r}$	radius vector	in.	m

Symbol	Meaning	English Units	SI Units
Re( )	real part of ( )		
$R(\tau)$	autocorrelation function		
R	electrical resistance	ohm	ohm
R	Rayleigh's dissipation function	lb-in/sec	$\mathrm{N \cdot m/s}$
R	Rayleigh's quotient	$1/\mathrm{sec}^2$	$1/\mathrm{s}^2$
s	root of equation, Laplace variable		
S_a, S_d, S_v	acceleration, displacement, velocity spectrum		
$S_x(\omega)$	spectrum of x		
t	time	sec	s
t_i	ith time station	sec	s
T	torque	lb-in	N-m
T	kinetic energy	in.-lb	J
T_i	kinetic energy of ith mass	in.-lb	J
T_d, T_f	displacement, force transmissibility		
u_{ij}	an element of matrix $[U]$		
U, U_i	axial displacement	in.	m
U	potential energy	in.-lb	J
$\vec{U}$	unbalanced weight	lb	N
$[U]$	upper triangular matrix		
v, v_0	linear velocity	in./sec	m/s
V	shear force	lb	N
V	potential energy	in.-lb	J
V_i	potential energy of ith spring	in.-lb	J
w, w_1, w_2, ω_i	transverse deflections	in.	m
w_0	value of w at $t = 0$	in.	m
$\dot{w}_0$	value of $\dot{w}$ at $t = 0$	in./sec	m/s
w_n	nth mode of vibration		
W	weight of a mass	lb	N
W	total energy	in.-lb	J
W	transverse deflection	in.	m
W_i	value of W at $t = t_i$	in.	m
$W(x)$	a function of x		
x, y, z	cartesian coordinates, displacements	in.	m
x_0, $x(0)$	value of x at $t = 0$	in.	m
$\dot{x}_0$, $\dot{x}(0)$	value of $\dot{x}$ at $t = 0$	in./sec	m/s
x_j	displacement of jth mass	in.	m
x_j	value of x at $t = t_j$	in.	m
$\dot{x}_j$	value of $\dot{x}$ at $t = t_j$	in./sec	m/s
x_h	homogeneous part of $x(t)$	in.	m
x_p	particular part of $x(t)$	in.	m
$\vec{x}$	vector of displacements	in.	m
$\vec{x}_i$	value of $\vec{x}$ at $t = t_i$	in.	m
$\dot{\vec{x}}_i$	value of $\dot{\vec{x}}$ at $t = t_i$	in./sec	m/s
$\ddot{\vec{x}}_i$	value of $\ddot{\vec{x}}$ at $t = t_i$	$\mathrm{in./sec}^2$	$\mathrm{m/s}^2$

Symbol	Meaning	English Units	SI Units
$\vec{x}^{(i)}(t)$	ith mode		
X	amplitude of $x(t)$	in.	m
X_j	amplitude of $x_j(t)$	in.	m
$\vec{X}^{(i)}$	ith modal vector	in.	m
$X_i^{(j)}$	ith component of jth mode	in.	m
$[X]$	modal matrix	in.	m
$\vec{X}_r$	rth approximation to a mode shape		
y	base displacement	in.	m
Y	amplitude of $y(t)$	in.	m
z	relative displacement, $x - y$	in.	m
Z	amplitude of $z(t)$	in.	m
$Z(i\omega)$	mechanical impedance	lb/in.	N/m
α	angle, constant		
β	angle, constant		
β	hysteresis damping constant		
γ	specific weight	lb/ in^3	N/ m^3
δ	logarithmic decrement		
$\delta_1, \delta_2, \ldots$	deflections	in.	m
δ_{st}	static deflection	in.	m
δ_{ij}	Kronecker delta		
Δ	determinant		
ΔF	increment in F	lb	N
Δx	increment in x	in.	m
Δt	increment in time t	sec	s
ΔW	energy dissipated in a cycle	in.-lb	J
ε	a small quantity		
ε	strain		
ζ	damping ratio		
θ	constant, angular displacement		
θ_i	ith angular displacement	rad	rad
θ_0	value of θ at $t = 0$	rad	rad
$\dot{\theta}_0$	value of $\dot{\theta}$ at $t = 0$	rad/sec	rad/s
Θ	amplitude of $\theta(t)$	rad	rad
Θ_i	amplitude of $\theta_i(t)$	rad	rad
λ	eigenvalue $= 1/\omega^2$	sec^2	s^2
$[\lambda]$	transformation matrix		
μ	viscosity of a fluid	lb-sec/ in^2	kg/ m · s
μ	coefficient of friction		
μ_x	expected value of x		
ρ	mass density	lb-sec^2/ in^4	kg/ m^3
η	loss factor		
σ_x	standard deviation of x		
σ	stress	lb/ in^2	N/ m^2
τ	period of oscillation, time, time constant	sec	s

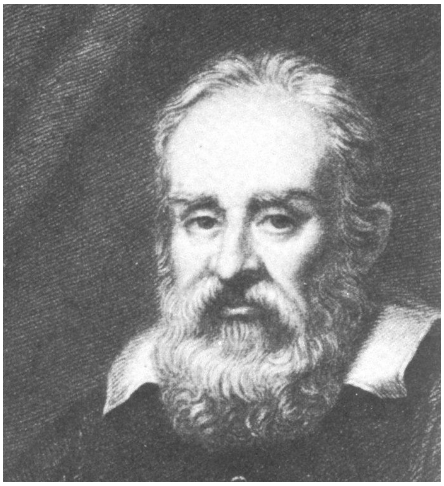
Symbol	Meaning	English Units	SI Units
τ	shear stress	lb/ in^2	N/ m^2
ϕ	angle, phase angle	rad	rad
ϕ_i	phase angle in ith mode	rad	rad
ω	frequency of oscillation	rad/sec	rad/s
ω_i	ith natural frequency	rad/sec	rad/s
ω_n	natural frequency	rad/sec	rad/s
ω_d	frequency of damped vibration	rad/sec	rad/s

Subscripts

Symbol	Meaning
cri	critical value
eq	equivalent value
i	ith value
L	left plane
max	maximum value
n	corresponding to natural frequency
R	right plane
0	specific or reference value
t	torsional

Operations

Symbol	Meaning
$(\dot{\ })$	$\dfrac{d(\ )}{dt}$
$(\ddot{\ })$	$\dfrac{d^2(\ )}{dt^2}$
$(\vec{\ })$	column vector ( )
[ ]	matrix
$[\ ]^{-1}$	inverse of [ ]
$[\ ]^T$	transpose of [ ]
$\Delta(\ )$	increment in ( )
$\mathscr{L}(\ )$	Laplace transform of ( )
$\mathscr{L}^{-1}(\ )$	inverse Laplace transform of ( )



Galileo Galilei (1564–1642), an Italian astronomer, philosopher, and professor of mathematics at the Universities of Pisa and Padua, in 1609 became the first man to point a telescope to the sky. He wrote the first treatise on modern dynamics in 1590. His works on the oscillations of a simple pendulum and the vibration of strings are of fundamental significance in the theory of vibrations.
(Courtesy of Dirk J. Struik, *A Concise History of Mathematics* (2nd rev. ed.), Dover Publications, Inc., New York, 1948.)

CHAPTER 1

Fundamentals of Vibration

Chapter Outline

This chapter introduces the subject of vibrations in a relatively simple manner. It begins with a brief history of the subject and continues with an examination of the importance of vibration. The basic concepts of degrees of freedom and of discrete and continuous systems are introduced, along with a description of the elementary parts of vibrating systems. The various classifications of vibration—namely, free and forced vibration, undamped and damped vibration, linear and nonlinear vibration, and deterministic and random vibration—are indicated. The various steps involved in vibration analysis of an engineering system are outlined, and essential definitions and concepts of vibration are introduced.

The concept of harmonic motion and its representation using vectors and complex numbers is described. The basic definitions and terminology related to harmonic motion, such as cycle, amplitude, period, frequency, phase angle, and natural frequency, are given. Finally, the harmonic analysis, dealing with the representation of any periodic function in terms of harmonic functions, using Fourier series, is outlined. The concepts of frequency spectrum, time- and frequency-domain representations of periodic functions, half-range expansions, and numerical computation of Fourier coefficients are discussed in detail.

Learning Objectives

After completing this chapter, the reader should be able to do the following:

- Describe briefly the history of vibration
- Indicate the importance of study of vibration
- Give various classifications of vibration
- State the steps involved in vibration analysis
- Compute the values of spring constants, masses, and damping constants
- Define harmonic motion and different possible representations of harmonic motion
- Add and subtract harmonic motions
- Conduct Fourier series expansion of given periodic functions
- Determine Fourier coefficients numerically using the MATLAB program

1.1 Preliminary Remarks

The subject of vibration is introduced here in a relatively simple manner. The chapter begins with a brief history of vibration and continues with an examination of its importance. The various steps involved in vibration analysis of an engineering system are outlined, and essential definitions and concepts of vibration are introduced. We learn here that all mechanical and structural systems can be modeled as mass-spring-damper systems. In some systems, such as an automobile, the mass, spring and damper can be identified as separate components (mass in the form of the body, spring in the form of suspension and damper in the form of shock absorbers). In some cases, the mass, spring and damper do not appear as separate components; they are inherent and integral to the system. For example, in an airplane wing, the mass of the wing is distributed throughout the wing. Also, due to its elasticity, the wing undergoes noticeable deformation during flight so that it can be modeled as a spring. In addition, the deflection of the wing introduces damping due to relative motion between components such as joints, connections and support as well as internal friction due to microstructural defects in the material. The chapter describes the modeling of spring, mass and damping elements, their characteristics and the combination of several springs, masses or damping elements appearing in a system. There follows a presentation of the concept of harmonic analysis, which can be used for the analysis of general periodic motions. No attempt at exhaustive treatment of the topics is made in Chapter 1; subsequent chapters will develop many of the ideas in more detail.

1.2 Brief History of the Study of Vibration

1.2.1 Origins of the Study of Vibration

People became interested in vibration when they created the first musical instruments, probably whistles or drums. Since then, both musicians and philosophers have sought out the rules and laws of sound production, used them in improving musical instruments, and passed them on from generation to generation. As long ago as 4000 B.C. [1.1], music had become highly developed and was much appreciated by Chinese, Hindus, Japanese, and, perhaps, the Egyptians. These early peoples observed certain definite rules in connection with the art of music, although their knowledge did not reach the level of a science.

Stringed musical instruments probably originated with the hunter's bow, a weapon favored by the armies of ancient Egypt. One of the most primitive stringed instruments, the *nanga*, resembled a harp with three or four strings, each yielding only one note. An example dating back to 1500 B.C. can be seen in the British Museum. The Museum also exhibits an 11-stringed harp with a gold-decorated, bull-headed sounding box, found at Ur in a royal tomb dating from about 2600 B.C. As early as 3000 B.C., stringed instruments such as harps were depicted on walls of Egyptian tombs.

Our present system of music is based on ancient Greek civilization. The Greek philosopher and mathematician Pythagoras (582–507 B.C.) is considered to be the first person to investigate musical sounds on a scientific basis (Fig. 1.1). Among other things, Pythagoras

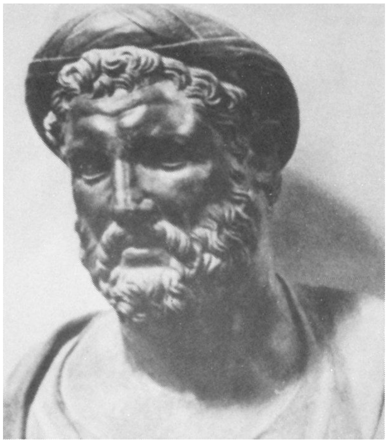


FIGURE 1.1 Pythagoras. (Reprinted with permission from L. E. Navia, *Pythagoras: An Annotated Bibliography*, Garland Publishing, Inc., New York, 1990).

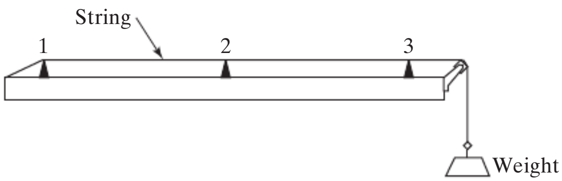


FIGURE 1.2 Monochord.

conducted experiments on a vibrating string by using a simple apparatus called a monochord. In the monochord shown in Fig. 1.2 the wooden bridges labeled 1 and 3 are fixed. Bridge 2 is made movable while the tension in the string is held constant by the hanging weight. Pythagoras observed that if two like strings of different lengths are subject to the same tension, the shorter one emits a higher note; in addition, if the shorter string is half the length of the longer one, the shorter one will emit a note an octave above the other. Pythagoras left no written account of his work (Fig. 1.3), but it has been described by others. Although the concept of pitch was developed by the time of Pythagoras, the relation between the pitch and the frequency was not understood until the time of Galileo in the sixteenth century.

Around 350 B.C., Aristotle wrote treatises on music and sound, making observations such as "the voice is sweeter than the sound of instruments," and "the sound of the flute is sweeter than that of the lyre." In 320 B.C., Aristoxenus, a pupil of Aristotle and a musician,

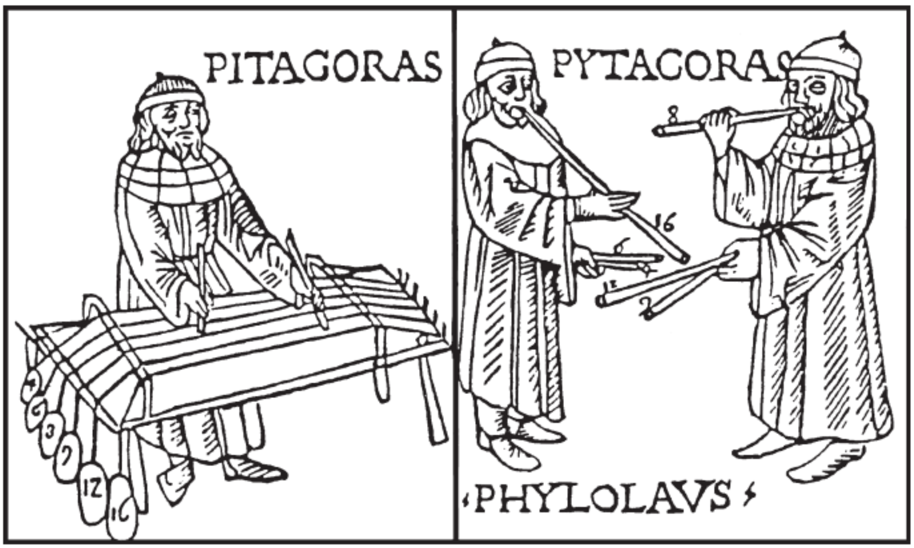


FIGURE 1.3 Pythagoras as a musician. (Reprinted with permission from D. E. Smith, *History of Mathematics*, Vol. I, Dover Publications, Inc., New York, 1958.)

wrote a three-volume work entitled *Elements of Harmony*. These books are perhaps the oldest ones available on the subject of music written by the investigators themselves. In about 300 B.C., in a treatise called *Introduction to Harmonics*, Euclid, wrote briefly about music without any reference to the physical nature of sound. No further advances in scientific knowledge of sound were made by the Greeks.

It appears that the Romans derived their knowledge of music completely from the Greeks, except that Vitruvius, a famous Roman architect, wrote in about 20 B.C. on the acoustic properties of theaters. His treatise, entitled *De Architectura Libri Decem*, was lost for many years, to be rediscovered only in the fifteenth century. There appears to have been no development in the theories of sound and vibration for nearly 16 centuries after the work of Vitruvius.

China experienced many earthquakes in ancient times. Zhang Heng, who served as a historian and astronomer in the second century, perceived a need to develop an instrument to measure earthquakes precisely. In A.D. 132 he invented the world's first seismograph [1.3, 1.4]. It was made of fine cast bronze, had a diameter of eight chi (a chi is equal to 0.237 meter), and was shaped like a wine jar (Fig. 1.4). Inside the jar was a mechanism consisting of pendulums surrounded by a group of eight levers pointing in eight directions. Eight dragon figures, with a bronze ball in the mouth of each, were arranged on the outside of the seismograph. Below each dragon was a toad with mouth open upward. A strong earthquake in any direction would tilt the pendulum in that direction, triggering the lever in the dragon head. This opened the mouth of the dragon, thereby releasing its bronze ball, which fell in the mouth of the toad with a clanging sound. Thus the seismograph enabled the monitoring personnel to know both the time and direction of occurrence of the earthquake.

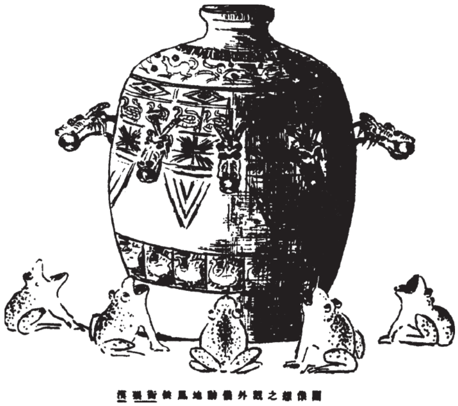


FIGURE 1.4 The world's first seismograph, invented in China in A.D. 132. (Reprinted with permission from R. Taton (ed.), *History of Science*, Basic Books, Inc., New York, 1957.)

1.2.2 From Galileo to Rayleigh

Galileo Galilei (1564–1642) is considered to be the founder of modern experimental science. In fact, the seventeenth century is often considered the "century of genius" since the foundations of modern philosophy and science were laid during that period. Galileo was inspired to study the behavior of a simple pendulum by observing the pendulum movements of a lamp in a church in Pisa. One day, while feeling bored during a sermon, Galileo was staring at the ceiling of the church. A swinging lamp caught his attention. He started measuring the period of the pendulum movements of the lamp with his pulse and found to his amazement that the time period was independent of the amplitude of swings. This led him to conduct more experiments on the simple pendulum. In *Discourses Concerning Two New Sciences*, published in 1638, Galileo discussed vibrating bodies. He described the dependence of the frequency of vibration on the length of a simple pendulum, along with the phenomenon of sympathetic vibrations (resonance). Galileo's writings also indicate that he had a clear understanding of the relationship between the frequency, length, tension, and density of a vibrating stretched string [1.5]. However, the first correct published account of the vibration of strings was given by the French mathematician and theologian, Marin Mersenne (1588–1648) in his book *Harmonicorum Liber*, published in 1636. Mersenne also measured, for the first time, the frequency of vibration of a long string and from that predicted the frequency of a shorter string having the same density and tension. Mersenne is considered by many the father of acoustics. He is often credited with the discovery of the laws of vibrating strings because he published the results in 1636, two years before Galileo. However, the credit belongs to Galileo, since the laws were written many years earlier but their publication was prohibited by the orders of the Inquisitor of Rome until 1638.

Inspired by the work of Galileo, the Academia del Cimento was founded in Florence in 1657; this was followed by the formations of the Royal Society of London in 1662 and the Paris Academie des Sciences in 1666. Later, Robert Hooke (1635–1703) also conducted experiments to find a relation between the pitch and frequency of vibration of a string. However, it was Joseph Sauveur (1653–1716) who investigated these experiments thoroughly and coined the word "acoustics" for the science of sound [1.6]. Sauveur in France and John Wallis (1616–1703) in England observed, independently, the phenomenon of mode shapes, and they found that a vibrating stretched string can have no motion at certain points and violent motion at intermediate points. Sauveur called the former points *nodes* and the latter ones *loops*. It was found that such vibrations had higher frequencies than that associated with the simple vibration of the string with no nodes. In fact, the higher frequencies were found to be integral multiples of the frequency of simple vibration, and Sauveur called the higher frequencies harmonics and the frequency of simple vibration the fundamental frequency. Sauveur also found that a string can vibrate with several of its harmonics present at the same time. In addition, he observed the phenomenon of beats when two organ pipes of slightly different pitches are sounded together. In 1700 Sauveur calculated, by a somewhat dubious method, the frequency of a stretched string from the measured sag of its middle point.

Sir Isaac Newton (1642–1727) published his monumental work, *Philosophiae Naturalis Principia Mathematica*, in 1686, describing the law of universal gravitation as well as the three laws of motion and other discoveries. Newton's second law of motion is routinely used in modern books on vibrations to derive the equations of motion of a vibrating body. The theoretical (dynamical) solution of the problem of the vibrating string was found in 1713 by the English mathematician Brook Taylor (1685–1731), who also presented the famous Taylor's theorem on infinite series. The natural frequency of vibration obtained from the equation of motion derived by Taylor agreed with the experimental values observed by Galileo and Mersenne. The procedure adopted by Taylor was perfected through the introduction of partial derivatives in the equations of motion by Daniel Bernoulli (1700–1782), Jean D'Alembert (1717–1783), and Leonard Euler (1707–1783).

The possibility of a string vibrating with several of its harmonics present at the same time (with displacement of any point at any instant being equal to the algebraic sum of displacements for each harmonic) was proved through the dynamic equations of Daniel Bernoulli in his memoir, published by the Berlin Academy in 1755 [1.7]. This characteristic was referred to as the principle of the coexistence of small oscillations, which, in present-day terminology, is the principle of superposition. This principle was proved to be most valuable in the development of the theory of vibrations and led to the possibility of expressing any arbitrary function (i.e., any initial shape of the string) using an infinite series of sines and cosines. Because of this implication, D'Alembert and Euler doubted the validity of this principle. However, the validity of this type of expansion was proved by J. B. J. Fourier (1768–1830) in his *Analytical Theory of Heat* in 1822.

The analytical solution of the vibrating string was presented by Joseph Lagrange (1736–1813) in his memoir published by the Turin Academy in 1759. In his study, Lagrange assumed that the string was made up of a finite number of equally spaced identical mass particles, and he established the existence of a number of independent frequencies equal to the number of mass particles. When the number of particles was allowed to be infinite, the resulting frequencies were found to be the same as the harmonic frequencies of the stretched string. The method of setting up the differential equation of the motion of a string (called the wave equation), presented in most modern books on vibration theory, was first developed by D'Alembert in his memoir published by the Berlin Academy in 1750. The vibration of thin beams supported and clamped in different ways was first studied by Euler in 1744 and Daniel Bernoulli in 1751. Their approach has become known as the Euler-Bernoulli or thin beam theory.

Charles Coulomb did both theoretical and experimental studies in 1784 on the torsional oscillations of a metal cylinder suspended by a wire (Fig. 1.5). By assuming that the resisting torque of the twisted wire is proportional to the angle of twist, he derived the equation of motion for the torsional vibration of the suspended cylinder. By integrating the equation of motion, he found that the period of oscillation is independent of the angle of twist.

There is an interesting story related to the development of the theory of vibration of plates [1.8]. In 1802 the German scientist, E. F. F. Chladni (1756–1824) developed the method of placing sand on a vibrating plate to find its mode shapes and observed the beauty and intricacy of the modal patterns of the vibrating plates. In 1809 the French Academy invited Chladni to give a demonstration of his experiments. Napoléon Bonaparte, who attended the meeting, was very impressed and presented a sum of 3,000 francs to the academy, to be awarded to the first person to give a satisfactory mathematical theory of the vibration of plates. By the closing date of the competition in October

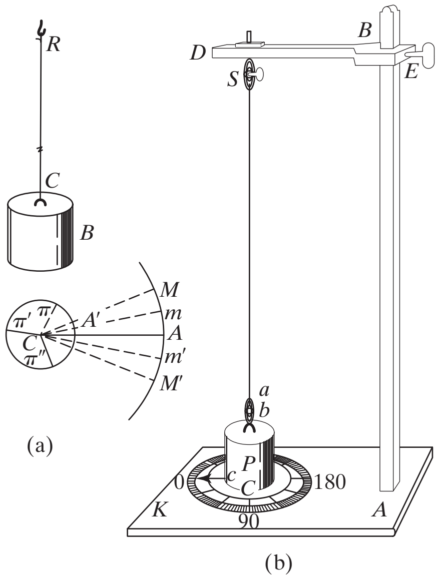


FIGURE 1.5 Coulomb's device for torsional vibration tests. (Reprinted with permission from S. P. Timoshenko, *History of Strength of Materials*, McGraw-Hill Book Company, Inc., New York, 1953.)

1811, only one candidate, Sophie Germain, had entered the contest. But Lagrange, who was one of the judges, noticed an error in the derivation of her differential equation of motion. The academy opened the competition again, with a new closing date of October 1813. Sophie Germain again entered the contest, presenting the correct form of the differential equation. However, the academy did not award the prize to her because the judges wanted physical justification of the assumptions made in her derivation. The competition was opened once more. In her third attempt, Sophie Germain was finally awarded the prize in 1815, although the judges were not completely satisfied with her theory. In fact, it was later found that her differential equation was correct but the boundary conditions were erroneous. The correct boundary conditions for the vibration of plates were given in 1850 by G. R. Kirchhoff (1824–1887).

In the meantime, the problem of vibration of a rectangular flexible membrane, which is important for the understanding of the sound emitted by drums, was solved for the first time by Simeon Poisson (1781–1840). The vibration of a circular membrane was studied by R. F. A. Clebsch (1833–1872) in 1862. After this, vibration studies were done on a number of practical mechanical and structural systems. In 1877 Lord Baron Rayleigh published his book on the theory of sound [1.9]; it is considered a classic on the subject of sound and vibration even today. Notable among the many contributions of Rayleigh is the method of finding the fundamental frequency of vibration of a conservative system by making use of the principle of conservation of energy—now known as Rayleigh's method.

This method proved to be a helpful technique for the solution of difficult vibration problems. An extension of the method, which can be used to find multiple natural frequencies, is known as the Rayleigh-Ritz method.

1.2.3 Recent Contributions

In 1902 Frahm investigated the importance of torsional vibration study in the design of the propeller shafts of steamships. The dynamic vibration absorber, which involves the addition of a secondary spring-mass system to eliminate the vibrations of a main system, was also proposed by Frahm in 1909. Among the modern contributers to the theory of vibrations, the names of Stodola, De Laval, Timoshenko, and Mindlin are notable. Aurel Stodola (1859–1943) contributed to the study of vibration of beams, plates, and membranes. He developed a method for analyzing vibrating beams that is also applicable to turbine blades. Noting that every major type of prime mover gives rise to vibration problems, C. G. P. De Laval (1845–1913) presented a practical solution to the problem of vibration of an unbalanced rotating disk. After noticing failures of steel shafts in high-speed turbines, he used a bamboo fishing rod as a shaft to mount the rotor. He observed that this system not only eliminated the vibration of the unbalanced rotor but also survived up to speeds as high as 100,000 rpm [1.10].

Stephen Timoshenko (1878–1972), by considering the effects of rotary inertia and shear deformation, presented an improved theory of vibration of beams, which has become known as the Timoshenko or thick beam theory. A similar theory was presented by R. D. Mindlin for the vibration analysis of thick plates by including the effects of rotary inertia and shear deformation.

It has long been recognized that many basic problems of mechanics, including those of vibrations, are nonlinear. Although the linear treatments commonly adopted are quite satisfactory for most purposes, they are not adequate in all cases. In nonlinear systems, phenonmena may occur that are theoretically impossible in linear systems. The mathematical theory of nonlinear vibrations began to develop in the works of Poincaré and Lyapunov at the end of the nineteenth century. Poincaré developed the perturbation method in 1892 in connection with the approximate solution of nonlinear celestial mechanics problems. In 1892, Lyapunov laid the foundations of modern stability theory, which is applicable to all types of dynamical systems. After 1920, the studies undertaken by Duffing and van der Pol brought the first definite solutions into the theory of nonlinear vibrations and drew attention to its importance in engineering. In the last 40 years, authors like Minorsky and Stoker have endeavored to collect in monographs the main results concerning nonlinear vibrations. Most practical applications of nonlinear vibration involved the use of some type of a perturbation-theory approach. The modern methods of perturbation theory were surveyed by Nayfeh [1.11].

Random characteristics are present in diverse phenomena such as earthquakes, winds, transportation of goods on wheeled vehicles, and rocket and jet engine noise. It became necessary to devise concepts and methods of vibration analysis for these random effects. Although Einstein considered Brownian movement, a particular type of random vibration, as long ago as 1905, no applications were investigated until 1930. The introduction of the correlation function by Taylor in 1920 and of the spectral density by Wiener and Khinchin in the early 1930s opened new prospects for progress in the theory of random vibrations. Papers by Lin and Rice, published between 1943 and 1945, paved

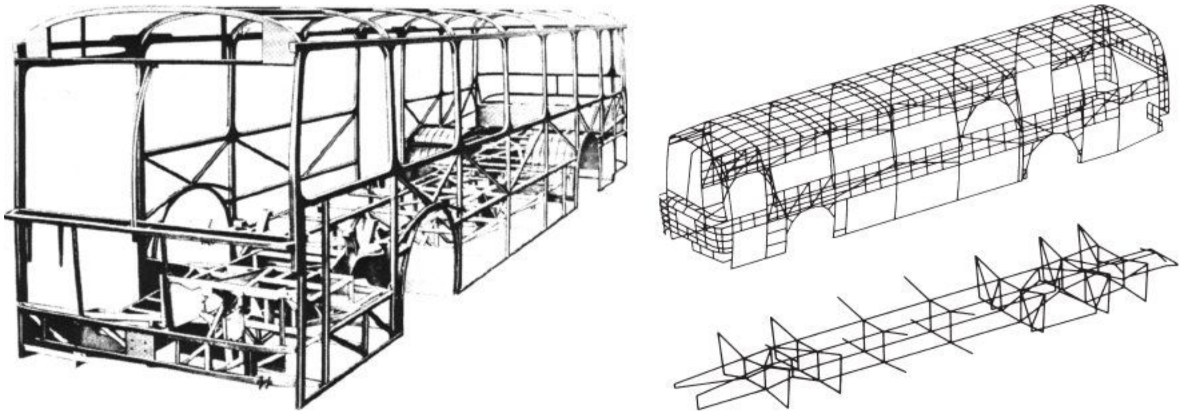


FIGURE 1.6 Finite element idealization of the body of a bus [1.16]. (Reprinted with permission © 1974 Society of Automotive Engineers, Inc.)

the way for the application of random vibrations to practical engineering problems. The monographs of Crandall and Mark and of Robson systematized the existing knowledge in the theory of random vibrations [1.12, 1.13].

Until about 40 years ago, vibration studies, even those dealing with complex engineering systems, were done by using gross models, with only a few degrees of freedom. However, the advent of high-speed digital computers in the 1950s made it possible to treat moderately complex systems and to generate approximate solutions in semidefinite form, relying on classical solution methods but using numerical evaluation of certain terms that cannot be expressed in closed form. The simultaneous development of the finite element method enabled engineers to use digital computers to conduct numerically detailed vibration analysis of complex mechanical, vehicular, and structural systems displaying thousands of degrees of freedom [1.14]. Although the finite element method was not so named until recently, the concept was used centuries ago. For example, ancient mathematicians found the circumference of a circle by approximating it as a polygon, where each side of the polygon, in present-day notation, can be called a finite element. The finite element method as known today was presented by Turner, Clough, Martin, and Topp in connection with the analysis of aircraft structures [1.15]. Figure 1.6 shows the finite element idealization of the body of a bus [1.16].

1.3 Importance of the Study of Vibration

Most human activities involve vibration in one form or other. For example, we hear because our eardrums vibrate and see because light waves undergo vibration. Breathing is associated with the vibration of lungs and walking involves (periodic) oscillatory motion of legs and hands. Human speech requires the oscillatory motion of larynges (and tongues) [1.17]. Early scholars in the field of vibration concentrated their efforts on understanding the natural phenomena and developing mathematical theories to describe the vibration of physical systems. In recent times, many investigations have been motivated by the engineering applications of vibration, such as the design of machines, foundations, structures, engines, turbines, and control systems.

Most prime movers have vibrational problems due to the inherent unbalance in the engines. The unbalance may be due to faulty design or poor manufacture. Imbalance in diesel engines, for example, can cause ground waves sufficiently powerful to create a nuisance in urban areas. The wheels of some locomotives can rise more than a centimeter off the track at high speeds due to imbalance. In turbines, vibrations cause spectacular mechanical failures. Engineers have not yet been able to prevent the failures that result from blade and disk vibrations in turbines. Naturally, the structures designed to support heavy centrifugal machines, like motors and turbines, or reciprocating machines, like steam and gas engines and reciprocating pumps, are also subjected to vibration. In all these situations, the structure or machine component subjected to vibration can fail because of material fatigue resulting from the cyclic variation of the induced stress. Furthermore, the vibration causes more rapid wear of machine parts such as bearings and gears and also creates excessive noise. In machines, vibration can loosen fasteners such as nuts. In metal cutting processes, vibration can cause chatter, which leads to a poor surface finish.

Whenever the natural frequency of vibration of a machine or structure coincides with the frequency of the external excitation, there occurs a phenomenon known as *resonance*, which leads to excessive deflections and failure. The literature is full of accounts of system failures brought about by resonance and excessive vibration of components and systems (see Fig. 1.7). Because of the devastating effects that vibrations can have on machines

FIGURE 1.7 Tacoma Narrows bridge during wind-induced vibration. The bridge opened on July 1, 1940, and collapsed on November 7, 1940. (Farquharson photo, Historical Photography Collection, University of Washington Libraries.)

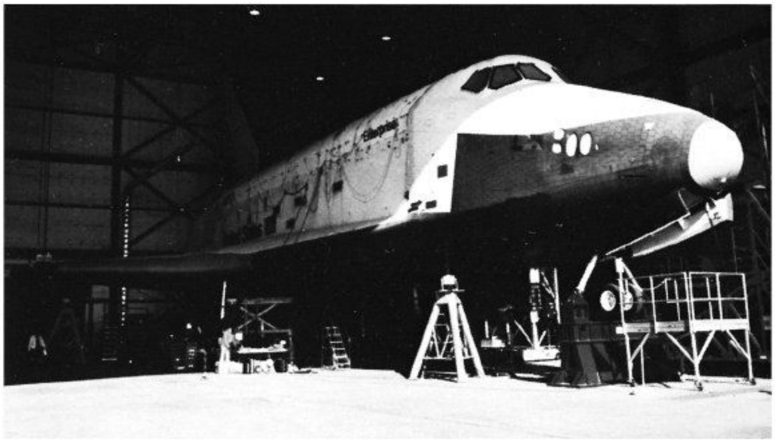


FIGURE 1.8 Vibration testing of the space shuttle *Enterprise*. (Courtesy of NASA.)

and structures, vibration testing [1.18] has become a standard procedure in the design and development of most engineering systems (see Fig. 1.8).

In many engineering systems, a human being acts as an integral part of the system. The transmission of vibration to human beings results in discomfort and loss of efficiency. The vibration and noise generated by engines causes annoyance to people and, sometimes, damage to property. Vibration of instrument panels can cause their malfunction or difficulty in reading the meters [1.19]. Thus one of the important purposes of vibration study is to reduce vibration through proper design of machines and their mountings. In this

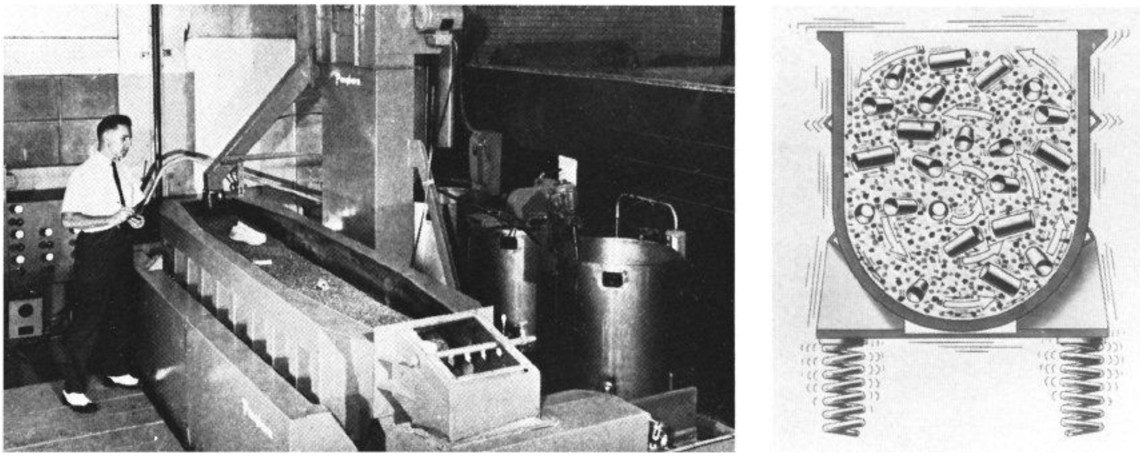


FIGURE 1.9 Vibratory finishing process. (Reprinted courtesy of the Society of Manufacturing Engineers, © 1964 The Tool and Manufacturing Engineer.)

connection, the mechanical engineer tries to design the engine or machine so as to minimize imbalance, while the structural engineer tries to design the supporting structure so as to ensure that the effect of the imbalance will not be harmful [1.20].

In spite of its detrimental effects, vibration can be utilized profitably in several consumer and industrial applications. In fact, the applications of vibratory equipment have increased considerably in recent years [1.21]. For example, vibration is put to work in vibratory conveyors, hoppers, sieves, compactors, washing machines, electric toothbrushes, dentist's drills, clocks, and electric massaging units. Vibration is also used in pile driving, vibratory testing of materials, vibratory finishing processes, and electronic circuits to filter out the unwanted frequencies (see Fig. 1.9). Vibration has been found to improve the efficiency of certain machining, casting, forging, and welding processes. It is employed to simulate earthquakes for geological research and also to conduct studies in the design of nuclear reactors.

1.4 Basic Concepts of Vibration

1.4.1 Vibration

Any motion that repeats itself after an interval of time is called *vibration* or *oscillation*. The swinging of a pendulum and the motion of a plucked string are typical examples of vibration. The theory of vibration deals with the study of oscillatory motions of bodies and the forces associated with them.

1.4.2 Elementary Parts of Vibrating Systems

A vibratory system, in general, includes a means for storing potential energy (spring or elasticity), a means for storing kinetic energy (mass or inertia), and a means by which energy is gradually lost (damper).

The vibration of a system involves the transfer of its potential energy to kinetic energy and of kinetic energy to potential energy, alternately. If the system is damped, some energy is dissipated in each cycle of vibration and must be replaced by an external source if a state of steady vibration is to be maintained.

As an example, consider the vibration of the simple pendulum shown in Fig. 1.10. Let the bob of mass m be released after being given an angular displacement θ. At position 1 the velocity of the bob and hence its kinetic energy is zero. But it has a potential energy of magnitude $mgl(1 - \cos\theta)$ with respect to the datum position 2. Since the gravitational force mg induces a torque $mgl \sin\theta$ about the point O, the bob starts swinging to the left from position 1. This gives the bob certain angular acceleration in the clockwise direction, and by the time it reaches position 2, all of its potential energy will be converted into kinetic energy. Hence the bob will not stop in position 2 but will continue to swing to position 3. However, as it passes the mean position 2, a counterclockwise torque due to gravity starts acting on the bob and causes the bob to decelerate. The velocity of the bob reduces to zero at the left extreme position. By this time, all the kinetic energy of the bob will be converted to potential energy. Again due to the gravity torque, the bob continues to attain a counterclockwise velocity. Hence the bob starts swinging back with progressively increasing velocity and passes the mean position again. This process keeps repeating, and the pendulum will have oscillatory motion. However, in practice, the magnitude of oscillation (θ) gradually decreases and the pendulum ultimately stops due to the resistance (damping) offered by the surrounding medium (air). This means that some energy is dissipated in each cycle of vibration due to damping by the air.

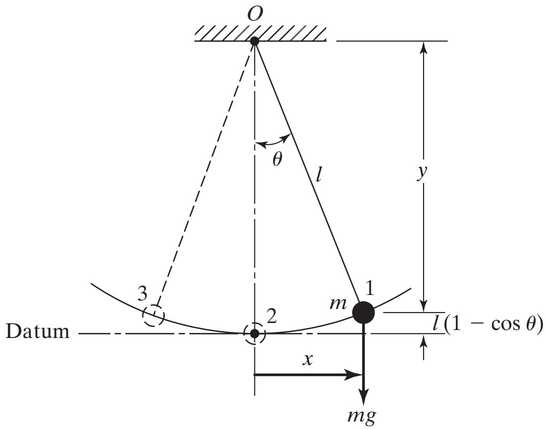


FIGURE 1.10 A simple pendulum.

1.4.3 Number of Degrees of Freedom

The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines the number of degrees of freedom of the system. The simple pendulum shown in Fig. 1.10, as well as each of the systems shown in Fig. 1.11, represents a single-degree-of-freedom system. For example, the motion of the simple pendulum (Fig. 1.10) can be stated either in terms of the angle θ or in terms of the Cartesian coordinates x and y. If the coordinates x and y are used to describe the motion, it must be recognized that these coordinates are not independent. They are related to each other through the relation $x^2 + y^2 = l^2$, where l is the constant length of the pendulum. Thus any one coordinate can describe the motion of the pendulum. In this example, we find that the choice of θ as the independent coordinate will be more convenient than the choice of x or y. For the slider shown in Fig. 1.11(a), either the angular coordinate θ or the coordinate x can be used to describe the motion. In Fig. 1.11(b), the linear coordinate x can

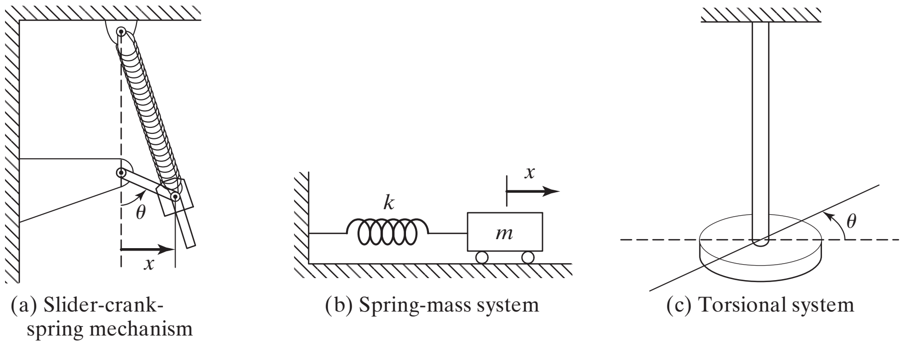


FIGURE 1.11 Single-degree-of-freedom systems.

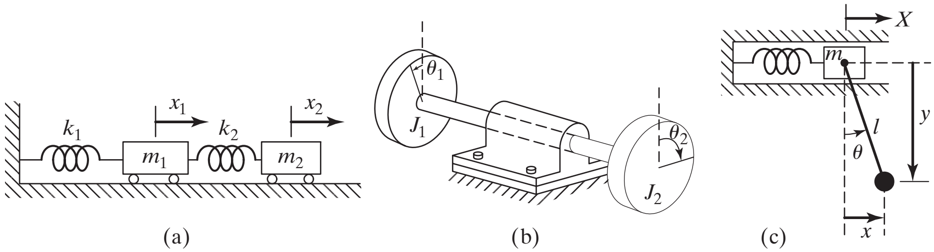


FIGURE 1.12 Two-degree-of-freedom systems.

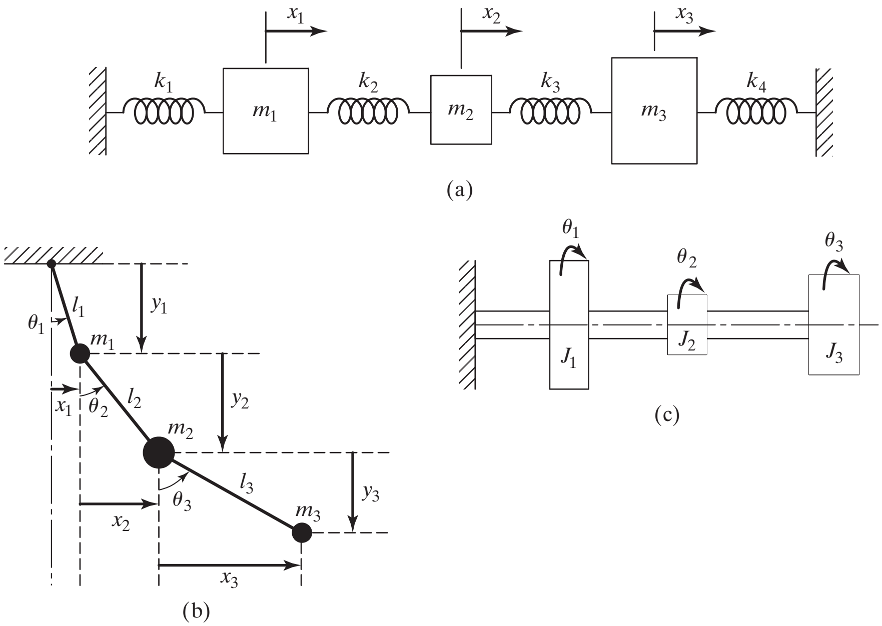


FIGURE 1.13 Three–degree-of-freedom systems.

be used to specify the motion. For the torsional system (long bar with a heavy disk at the end) shown in Fig. 1.11(c), the angular coordinate θ can be used to describe the motion.

Some examples of two- and three-degree-of-freedom systems are shown in Figs. 1.12 and 1.13, respectively. Figure 1.12(a) shows a two-mass, two-spring system that is described by the two linear coordinates x_1 and x_2. Figure 1.12(b) denotes a two-rotor system whose motion can be specified in terms of θ_1 and θ_2. The motion of the system shown in Fig. 1.12(c) can be described completely either by X and θ or by x, y, and X. In the latter case, x and y are constrained as $x^2 + y^2 = l^2$ where l is a constant.

For the systems shown in Figs. 1.13(a) and 1.13(c), the coordinates $x_i\,(i = 1, 2, 3)$ and $\theta_i\,(i = 1, 2, 3)$ can be used, respectively, to describe the motion. In the case of the

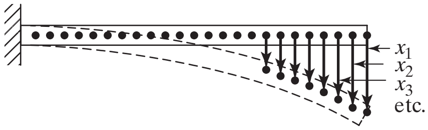


FIGURE 1.14 A cantilever beam (an infinite-number-of-degrees-of-freedom system).

system shown in Fig. 1.13(b), $\theta_i (i = 1, 2, 3)$ specifies the positions of the masses $m_i (i = 1, 2, 3)$. An alternate method of describing this system is in terms of x_i and $y_i (i = 1, 2, 3)$; but in this case the constraints $x_i^2 + y_i^2 = l_i^2 (i = 1, 2, 3)$ have to be considered.

The coordinates necessary to describe the motion of a system constitute a set of *generalized coordinates*. These are usually denoted as $q_1, q_2, \ldots$ and may represent Cartesian and/or non-Cartesian coordinates.

1.4.4 Discrete and Continuous Systems

A large number of practical systems can be described using a finite number of degrees of freedom, such as the simple systems shown in Figs. 1.10 to 1.13. Some systems, especially those involving continuous elastic members, have an infinite number of degrees of freedom. As a simple example, consider the cantilever beam shown in Fig. 1.14. Since the beam has an infinite number of mass points, we need an infinite number of coordinates to specify its deflected configuration. The infinite number of coordinates defines its elastic deflection curve. Thus the cantilever beam has an infinite number of degrees of freedom. Most structural and machine systems have deformable (elastic) members and therefore have an infinite number of degrees of freedom.

Systems with a finite number of degrees of freedom are called *discrete* or *lumped parameter* systems, and those with an infinite number of degrees of freedom are called *continuous* or *distributed* systems.

Most of the time, continuous systems are approximated as discrete systems, and solutions are obtained in a simpler manner. Although treatment of a system as continuous gives exact results, the analytical methods available for dealing with continuous systems are limited to a narrow selection of problems, such as uniform beams, slender rods, and thin plates. Hence most of the practical systems are studied by treating them as finite lumped masses, springs, and dampers. In general, more accurate results are obtained by increasing the number of masses, springs, and dampers—that is, by increasing the number of degrees of freedom.

1.5 Classification of Vibration

Vibration can be classified in several ways. Some of the important classifications are as follows.

1.5.1 Free and Forced Vibration

Free Vibration. If a system, after an initial disturbance, is left to vibrate on its own, the ensuing vibration is known as *free vibration.* No external force acts on the system. The oscillation of a simple pendulum is an example of free vibration.

Forced Vibration. If a system is subjected to an external force (often, a repeating type of force), the resulting vibration is known as *forced vibration.* The oscillation that arises in machines such as diesel engines is an example of forced vibration.

If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as *resonance* occurs, and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines, and airplane wings have been associated with the occurrence of resonance.

1.5.2 Undamped and Damped Vibration

If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as *undamped vibration.* If any energy is lost in this way, however, it is called *damped vibration.* In many physical systems, the amount of damping is so small that it can be disregarded for most engineering purposes. However, consideration of damping becomes extremely important in analyzing vibratory systems near resonance.

1.5.3 Linear and Nonlinear Vibration

If all the basic components of a vibratory system—the spring, the mass, and the damper—behave linearly, the resulting vibration is known as *linear vibration.* If, however, any of the basic components behave nonlinearly, the vibration is called *nonlinear vibration.* The differential equations that govern the behavior of linear and nonlinear vibratory systems are linear and nonlinear, respectively. If the vibration is linear, the principle of superposition holds, and the mathematical techniques of analysis are well developed. For nonlinear vibration, the superposition principle is not valid, and techniques of analysis are less well known. Since all vibratory systems tend to behave nonlinearly with increasing amplitude of oscillation, a knowledge of nonlinear vibration is desirable in dealing with practical vibratory systems.

1.5.4 Deterministic and Random Vibration

If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called *deterministic.* The resulting vibration is known as *deterministic vibration.*

In some cases, the excitation is *nondeterministic* or *random;* the value of the excitation at a given time cannot be predicted. In these cases, a large collection of records of the excitation may exhibit some statistical regularity. It is possible to estimate averages such as the mean and mean square values of the excitation. Examples of random excitations are wind velocity, road roughness, and ground motion during earthquakes. If the excitation is random, the resulting vibration is called *random vibration.* In this case the vibratory response of the system is also random; it can be described only in terms of statistical quantities. Figure 1.15 shows examples of deterministic and random excitations.

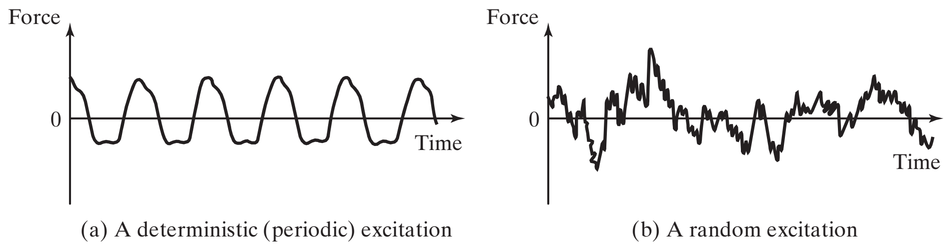


FIGURE 1.15 Deterministic and random excitations.

1.6 Vibration Analysis Procedure

A vibratory system is a dynamic one for which the variables such as the excitations (inputs) and responses (outputs) are time dependent. The response of a vibrating system generally depends on the initial conditions as well as the external excitations. Most practical vibrating systems are very complex, and it is impossible to consider all the details for a mathematical analysis. Only the most important features are considered in the analysis to predict the behavior of the system under specified input conditions. Often the overall behavior of the system can be determined by considering even a simple model of the complex physical system. Thus the analysis of a vibrating system usually involves mathematical modeling, derivation of the governing equations, solution of the equations, and interpretation of the results.

Step 1: Mathematical Modeling. The purpose of mathematical modeling is to represent all the important features of the system for the purpose of deriving the mathematical (or analytical) equations governing the system's behavior. The mathematical model should include enough details to allow describing the system in terms of equations without making it too complex. The mathematical model may be linear or nonlinear, depending on the behavior of the system's components. Linear models permit quick solutions and are simple to handle; however, nonlinear models sometimes reveal certain characteristics of the system that cannot be predicted using linear models. Thus a great deal of engineering judgment is needed to come up with a suitable mathematical model of a vibrating system.

Sometimes the mathematical model is gradually improved to obtain more accurate results. In this approach, first a very crude or elementary model is used to get a quick insight into the overall behavior of the system. Subsequently, the model is refined by including more components and/or details so that the behavior of the system can be observed more closely. To illustrate the procedure of refinement used in mathematical modeling, consider the forging hammer shown in Fig. 1.16(a). It consists of a frame, a falling weight known as the tup, an anvil, and a foundation block. The anvil is a massive steel block on which material is forged into desired shape by the repeated blows of the tup. The anvil is usually mounted on an elastic pad to reduce the transmission of vibration to the foundation block and the frame [1.22]. For a first approximation, the frame, anvil, elastic pad, foundation block, and soil are modeled as a single–degree of freedom system as shown in Fig. 1.16(b). For a refined approximation, the weights of the frame and anvil and

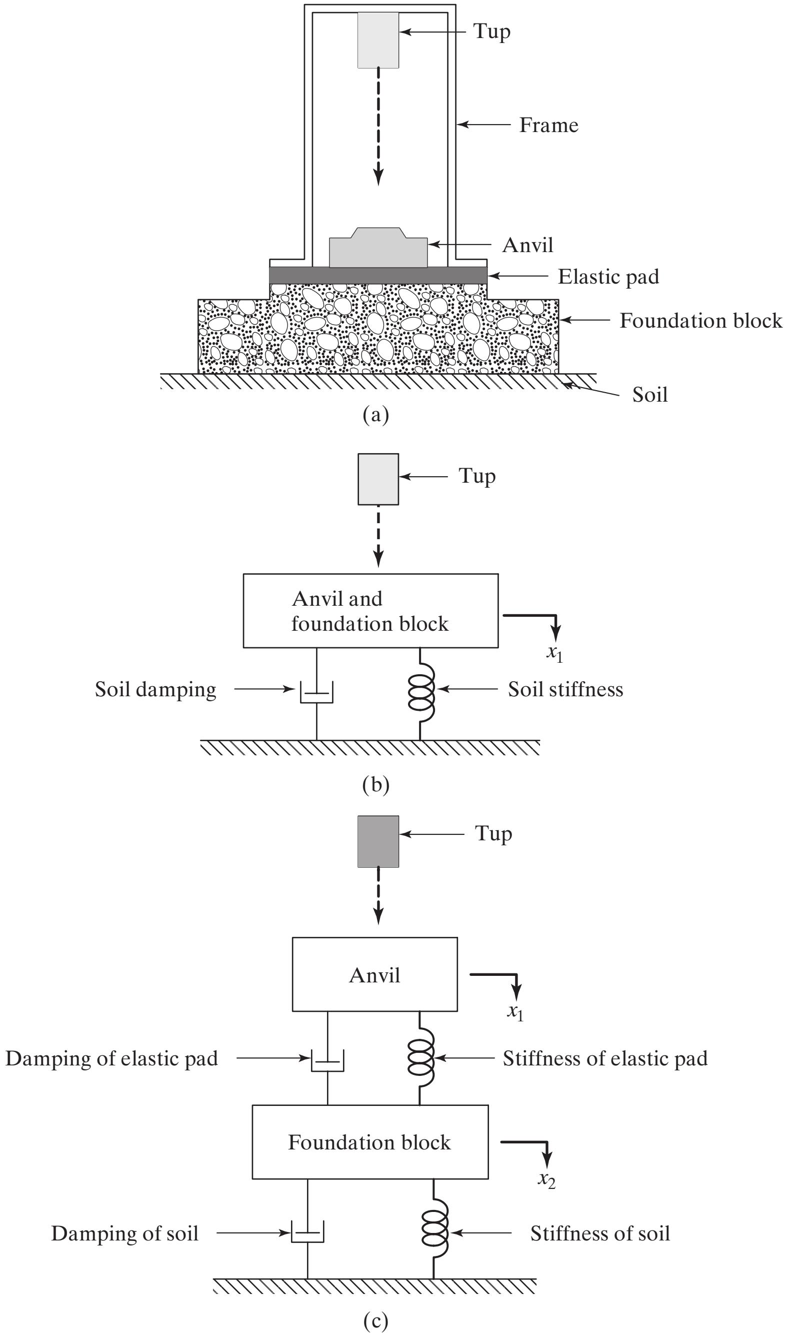


FIGURE 1.16 Modeling of a forging hammer.

the foundation block are represented separately with a two-degree-of-freedom model as shown in Fig. 1.16(c). Further refinement of the model can be made by considering eccentric impacts of the tup, which cause each of the masses shown in Fig. 1.16(c) to have both vertical and rocking (rotation) motions in the plane of the paper.

Step 2: Derivation of Governing Equations. Once the mathematical model is available, we use the principles of dynamics and derive the equations that describe the vibration of the system. The equations of motion can be derived conveniently by drawing the free-body diagrams of all the masses involved. The free-body diagram of a mass can be obtained by isolating the mass and indicating all externally applied forces, the reactive forces, and the inertia forces. The equations of motion of a vibrating system are usually in the form of a set of ordinary differential equations for a discrete system and partial differential equations for a continuous system. The equations may be linear or nonlinear, depending on the behavior of the components of the system. Several approaches are commonly used to derive the governing equations. Among them are Newton's second law of motion, D'Alembert's principle, and the principle of conservation of energy.

Step 3: Solution of the Governing Equations. The equations of motion must be solved to find the response of the vibrating system. Depending on the nature of the problem, we can use one of the following techniques for finding the solution: standard methods of solving differential equations, Laplace transform methods, matrix methods,[1] and numerical methods. If the governing equations are nonlinear, they can seldom be solved in closed form. Furthermore, the solution of partial differential equations is far more involved than that of ordinary differential equations. Numerical methods involving computers can be used to solve the equations. However, it will be difficult to draw general conclusions about the behavior of the system using computer results.

Step 4: Interpretation of the Results. The solution of the governing equations gives the displacements, velocities, and accelerations of the various masses of the system. These results must be interpreted with a clear view of the purpose of the analysis and the possible design implications of the results.

EXAMPLE 1.1 Mathematical Model of a Motorcycle

Figure 1.17(a) shows a motorcycle with a rider. Develop a sequence of three mathematical models of the system for investigating vibration in the vertical direction. Consider the elasticity of the tires, elasticity and damping of the struts (in the vertical direction), masses of the wheels, and elasticity, damping, and mass of the rider.

Solution: We start with the simplest model and refine it gradually. When the equivalent values of the mass, stiffness, and damping of the system are used, we obtain a single-degree-of-freedom model

[1]The basic definitions and operations of matrix theory are given in Appendix A.

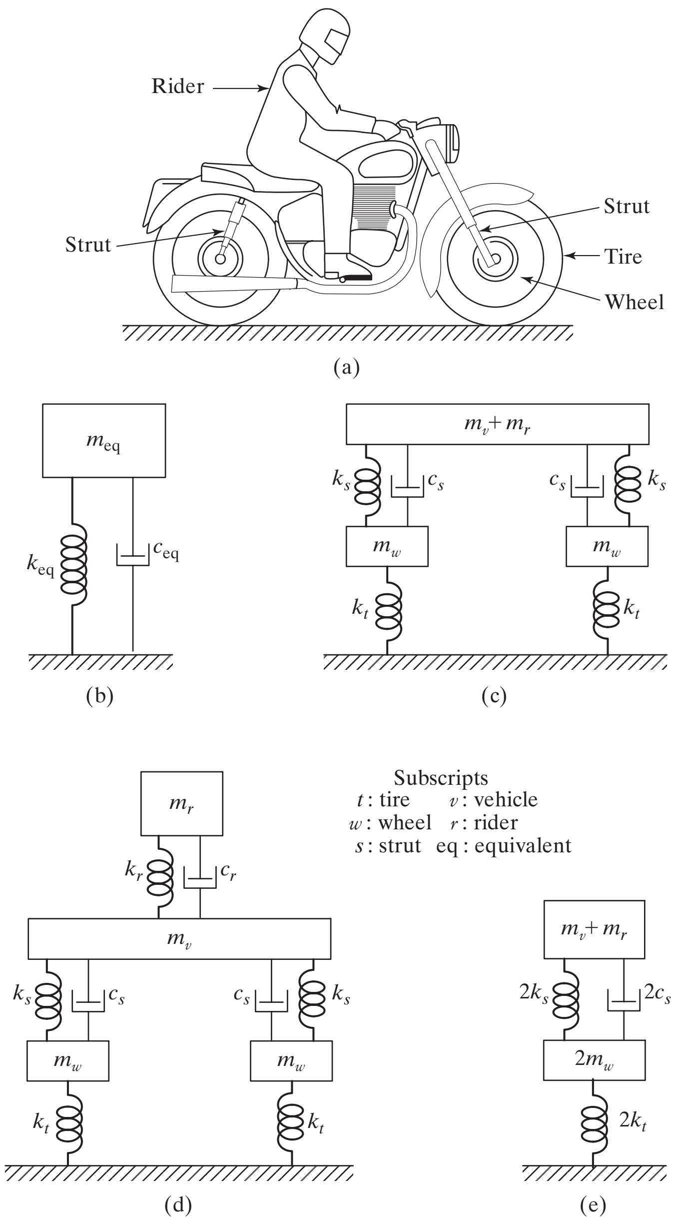


FIGURE 1.17 Motorcycle with a rider—a physical system and mathematical model.

of the motorcycle with a rider as indicated in Fig. 1.17(b). In this model, the equivalent stiffness (k_{eq}) includes the stiffnesses of the tires, struts, and rider. The equivalent damping constant (c_{eq}) includes the damping of the struts and the rider. The equivalent mass includes the masses of the wheels, vehicle body, and the rider. This model can be refined by representing the masses of wheels, elasticity of the tires, and elasticity and damping of the struts separately, as shown in Fig. 1.17(c). In this model, the mass of the vehicle body (m_v) and the mass of the rider (m_r) are shown as a single mass, $m_v + m_r$. When the elasticity (as spring constant k_r) and damping (as damping constant c_r) of the rider are considered, the refined model shown in Fig. 1.17(d) can be obtained.

Note that the models shown in Figs. 1.17(b) to (d) are not unique. For example, by combining the spring constants of both tires, the masses of both wheels, and the spring and damping constants of both struts as single quantities, the model shown in Fig. 1.17(e) can be obtained instead of Fig. 1.17(c).

■

1.7 Spring Elements

A spring is a type of mechanical link, which in most applications is assumed to have negligible mass and damping. The most common type of spring is the helical-coil spring used in retractable pens and pencils, staplers, and suspensions of freight trucks and other vehicles. Several other types of springs can be identified in engineering applications. In fact, any elastic or deformable body or member, such as a cable, bar, beam, shaft or plate, can be considered as a spring. A spring is commonly represented as shown in Fig. 1.18(a). If the free length of the spring, with no forces acting, is denoted l, it undergoes a change in length when an axial force is applied. For example, when a tensile force F is applied at its free end 2, the spring undergoes an elongation x as shown in Fig. 1.18(b), while a compressive force F applied at the free end 2 causes a reduction in length x as shown in Fig. 1.18(c).

A spring is said to be linear if the elongation or reduction in length x is related to the applied force F as

$$F = kx \tag{1.1}$$

where k is a constant, known as the *spring constant* or *spring stiffness* or *spring rate*. The spring constant k is always positive and denotes the force (positive or negative) required to

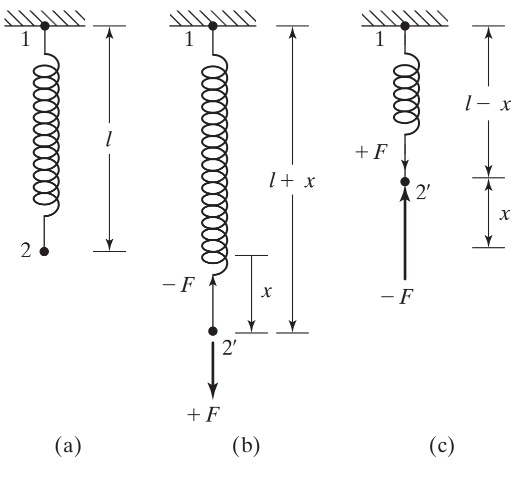


FIGURE 1.18 Deformation of a spring.

cause a unit deflection (elongation or reduction in length) in the spring. When the spring is stretched (or compressed) under a tensile (or compressive) force F, according to Newton's third law of motion, a restoring force or reaction of magnitude $-F$(or $+F$) is developed opposite to the applied force. This restoring force tries to bring the stretched (or compressed) spring back to its original unstretched or free length as shown in Fig. 1.18(b) (or 1.18(c)). If we plot a graph between F and x, the result is a straight line according to Eq. (1.1). The work done (U) in deforming a spring is stored as strain or potential energy in the spring, and it is given by

$$U = \frac{1}{2}kx^2 \tag{1.2}$$

1.7.1 Nonlinear Springs

Most springs used in practical systems exhibit a nonlinear force-deflection relation, particularly when the deflections are large. If a nonlinear spring undergoes small deflections, it can be replaced by a linear spring by using the procedure discussed in Section 1.7.2. In vibration analysis, nonlinear springs whose force-deflection relations are given by

$$F = ax + bx^3; \qquad a > 0 \tag{1.3}$$

are commonly used. In Eq. (1.3), a denotes the constant associated with the linear part and b indicates the constant associated with the (cubic) nonlinearity. The spring is said to be hard if $b > 0$, linear if $b = 0$, and soft if $b < 0$. The force-deflection relations for various values of b are shown in Fig. 1.19.

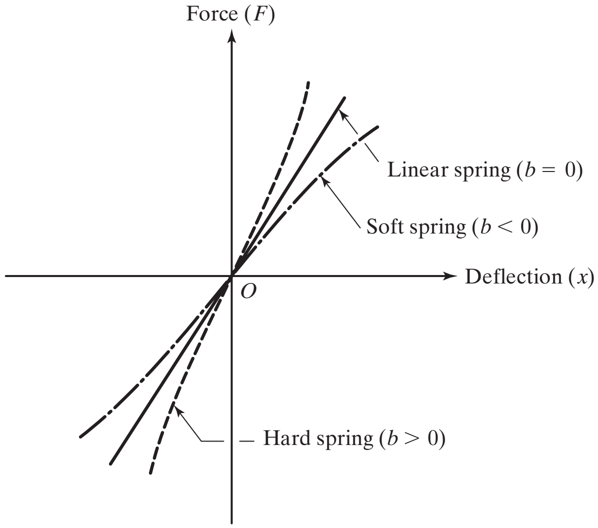


FIGURE 1.19 Nonlinear and linear springs.

Some systems, involving two or more springs, may exhibit a nonlinear force-displacement relationship although the individual springs are linear. Some examples of such systems are shown in Figs. 1.20 and 1.21. In Fig. 1.20(a), the weight (or force) W travels

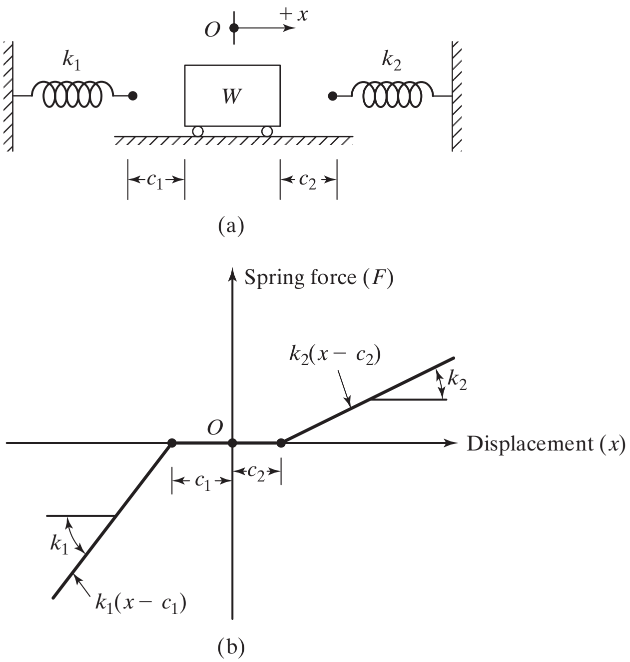


FIGURE 1.20 Nonlinear spring force-displacement relation.

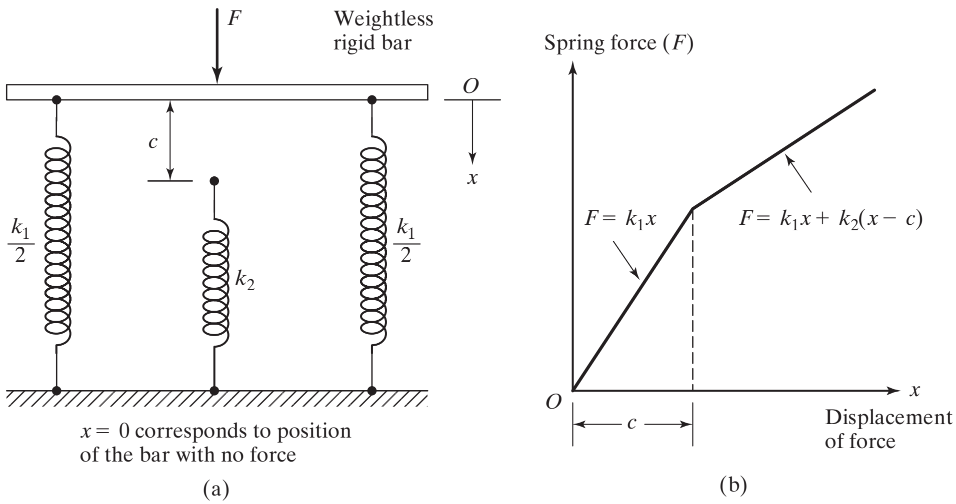


FIGURE 1.21 Nonlinear spring force-displacement relation.

freely through the clearances c_1 and c_2 present in the system. Once the weight comes into contact with a particular spring, after passing through the corresponding clearance, the spring force increases in proportion to the spring constant of the particular spring (see Fig. 1.20(b)). It can be seen that the resulting force-displacement relation, although piecewise linear, denotes a nonlinear relationship.

In Fig. 1.21 (a), the two springs, with stiffnesses k_1 and k_2, have different lengths. Note that the spring with stiffness k_1 is shown, for simplicity, in the form of two parallel springs, each with a stiffness of $k_1/2$. Spring arrangement models of this type can be used in the vibration analysis of packages and suspensions used in aircraft landing gears.

When the spring k_1 deflects by an amount $x = c$, the second spring starts providing an additional stiffness k_2 to the system. The resulting nonlinear force-displacement relationship is shown in Fig. 1.21 (b).

1.7.2 Linearization of a Nonlinear Spring

Actual springs are nonlinear and follow Eq. (1.1) only up to a certain deformation. Beyond a certain value of deformation (after point A in Fig. 1.22), the stress exceeds the yield point of the material and the force-deformation relation becomes nonlinear [1.23, 1.24]. In many practical applications we assume that the deflections are small and make use of the linear relation in Eq. (1.1). Even, if the force-deflection relation of a spring is nonlinear, as shown in Fig. 1.23, we often approximate it as a linear one by using a linearization process [1.24, 1.25]. To illustrate the linearization process, let the static equilibrium load F acting on the spring cause a deflection of x^*. If an incremental force ΔF is added to F, the spring deflects by an additional quantity Δx. The new spring force $F + \Delta F$ can be expressed using Taylor's series expansion about the static equilibrium position x^* as

$$
\begin{aligned}
F + \Delta F &= F(x^* + \Delta x) \\
&= F(x^*) + \left.\frac{dF}{dx}\right|_{x^*} (\Delta x) + \frac{1}{2!} \left.\frac{d^2F}{dx^2}\right|_{x^*} (\Delta x)^2 + \dots
\end{aligned}
\tag{1.4}
$$

For small values of Δx, the higher-order derivative terms can be neglected to obtain

$$
F + \Delta F = F(x^*) + \left.\frac{dF}{dx}\right|_{x^*} (\Delta x) \tag{1.5}
$$

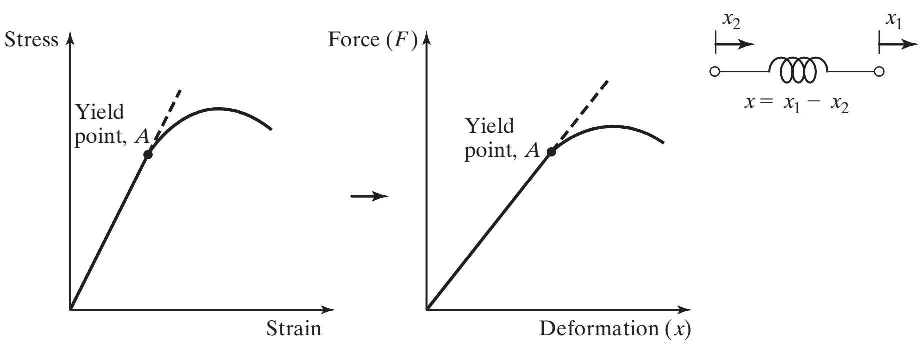


FIGURE 1.22 Nonlinearity beyond proportionality limit.

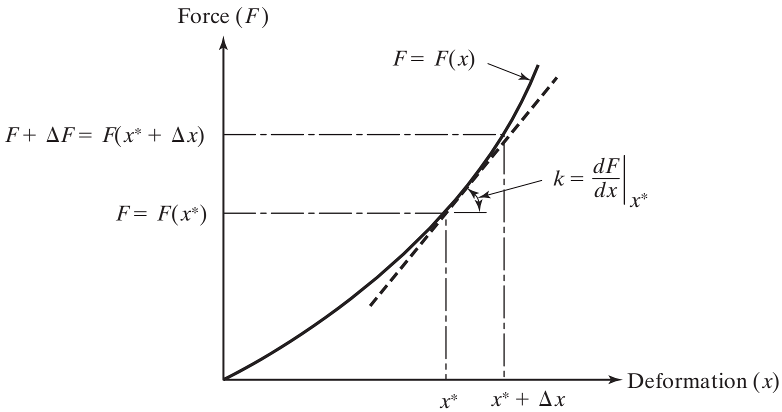


FIGURE 1.23 Linearization process.

Since $F = F(x^*)$, we can express ΔF as

$$\Delta F = k\,\Delta x \tag{1.6}$$

where k is the linearized spring constant at x^* given by

$$k = \left.\frac{dF}{dx}\right|_{x^*} \tag{1.7}$$

We may use Eq. (1.6) for simplicity, but sometimes the error involved in the approximation may be very large.

EXAMPLE 1.2 Equivalent Linearized Spring Constant

A precision milling machine, weighing 1000 lb, is supported on a rubber mount. The force-deflection relationship of the rubber mount is given by

$$F = 2000x + 200x^3 \tag{E.1}$$

where the force (F) and the deflection (x) are measured in pounds and inches, respectively. Determine the equivalent linearized spring constant of the rubber mount at its static equilibrium position.

Solution: The static equilibrium position of the rubber mount (x^*), under the weight of the milling machine, can be determined from Eq. (E.1):

$$1000 = 2000x^* + 200(x^*)^3$$

or

$$200(x^*)^3 + 2000x^* - 1000 = 0 \tag{E.2}$$

The roots of the cubic equation, (E.2), can be found (for example, using the function *roots* in MATLAB) as

$$x^* = 0.4884, \qquad -0.2442 + 3.1904i, \quad \text{and} \quad -0.2442 - 3.1904i$$

The static equilibrium position of the rubber mount is given by the real root of Eq. (E.2): $x^* = 0.4884$ in. The equivalent linear spring constant of the rubber mount at its static equilibrium position can be determined using Eq. (1.7):

$$k_{\text{eq}} = \left.\frac{dF}{dx}\right|_{x^*} = 2000 + 600(x^*)^2 = 2000 + 600(0.4884^2) = 2143.1207 \text{ lb/in.}$$

Note: The equivalent linear spring constant, $k_{\text{eq}} = 2143.1207$ lb/in., predicts the static deflection of the milling machine as

$$x = \frac{F}{k_{\text{eq}}} = \frac{1000}{2143.1207} = 0.4666 \text{ in.}$$

which is slightly different from the true value of 0.4884 in. The error is due to the truncation of the higher-order derivative terms in Eq. (1.4).

■

1.7.3 Spring Constants of Elastic Elements

As stated earlier, any elastic or deformable member (or element) can be considered as a spring. The equivalent spring constants of simple elastic members such as rods, beams, and hollow shafts are given on the inside front cover of the book. The procedure of finding the equivalent spring constant of elastic members is illustrated through the following examples.

EXAMPLE 1.3 Spring Constant of a Rod

Find the equivalent spring constant of a uniform rod of length l, cross-sectional area A, and Young's modulus E subjected to an axial tensile (or compressive) force F as shown in Fig. 1.24(a).

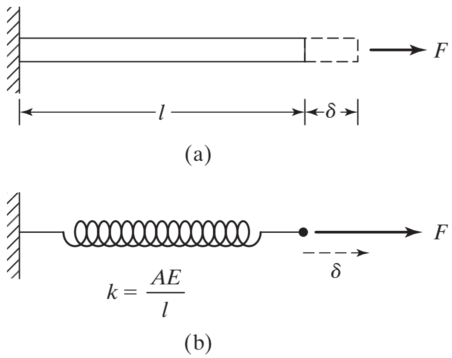


FIGURE 1.24 Spring constant of a rod.

Solution: The elongation (or shortening) δ of the rod under the axial tensile (or compressive) force F can be expressed as

$$\delta = \frac{\delta}{l} l = \varepsilon l = \frac{\sigma}{E} l = \frac{Fl}{AE} \tag{E.1}$$

where $\varepsilon = \dfrac{\textit{change in length}}{\textit{original length}} = \dfrac{\delta}{l}$ is the strain and $\sigma = \dfrac{\textit{force}}{\textit{area}} = \dfrac{F}{A}$ is the stress induced in the rod. Using the definition of the spring constant k, we obtain from Eq. (E.1):

$$k = \frac{\textit{force applied}}{\textit{resulting deflection}} = \frac{F}{\delta} = \frac{AE}{l} \tag{E.2}$$

The significance of the equivalent spring constant of the rod is shown in Fig. 1.24(b).

■

EXAMPLE 1.4 Spring Constant of a Cantilever Beam

Find the equivalent spring constant of a cantilever beam subjected to a concentrated load F at its end as shown in Fig. 1.25(a).

Solution: We assume, for simplicity, that the self weight (or mass) of the beam is negligible and the concentrated load F is due to the weight of a point mass ($W = mg$). From strength of materials [1.26], we know that the end deflection of the beam due to a concentrated load $F = W$ is given by

$$\delta = \frac{Wl^3}{3EI} \tag{E.1}$$

where E is the Young's modulus and I is the moment of inertia of the cross section of the beam about the bending or z-axis (i.e., axis perpendicular to the page). Hence the spring constant of the beam is (Fig. 1.25(b)):

$$k = \frac{W}{\delta} = \frac{3EI}{l^3} \tag{E.2}$$

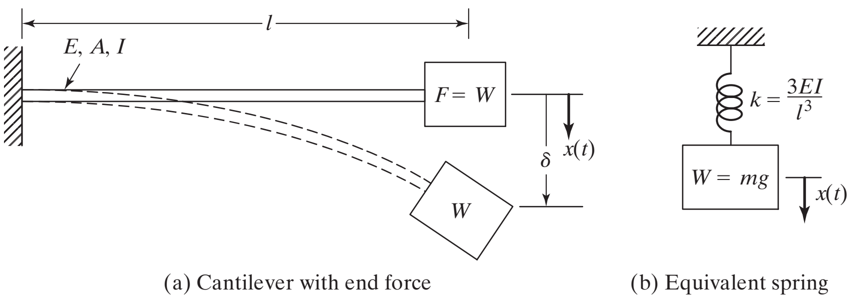


FIGURE 1.25 Spring constant of a cantilever beam.

Notes:

1. It is possible for a cantilever beam to be subjected to concentrated loads in two directions at its end—one in the y direction (F_y) and the other in the z direction (F_z)—as shown in Fig. 1.26(a). When the load is applied along the y direction, the beam bends about the z-axis (Fig. 1.26(b)) and hence the equivalent spring constant will be equal to

$$k = \frac{3EI_{zz}}{l^3} \tag{E.3}$$

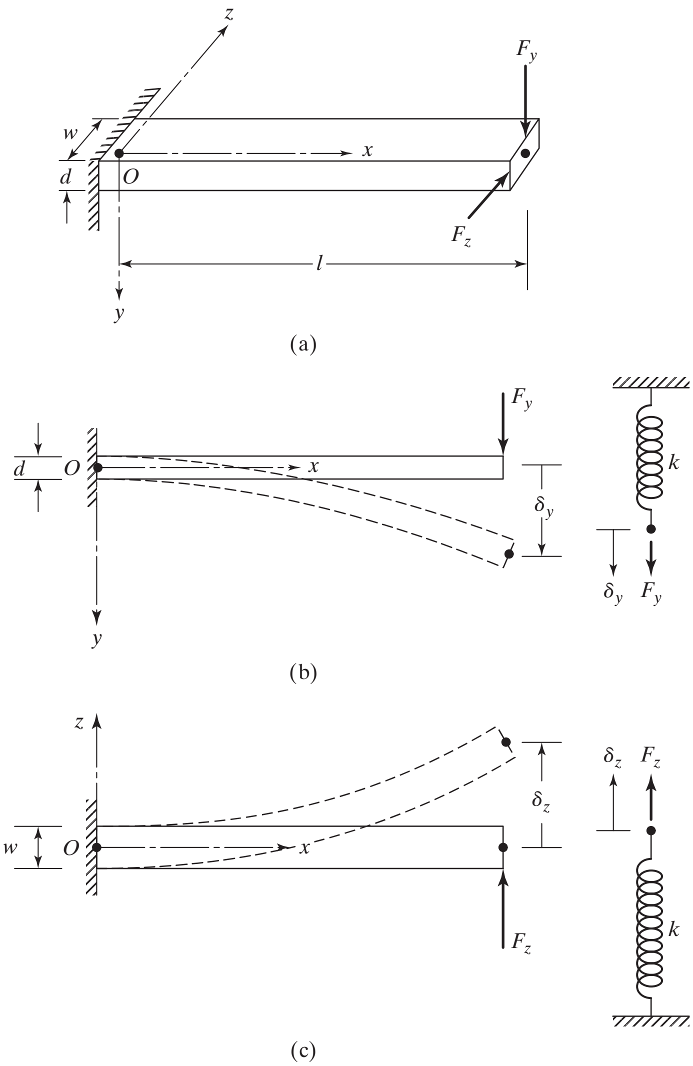


FIGURE 1.26 Spring constants of a beam in two directions.

When the load is applied along the z direction, the beam bends about the y-axis (Fig. 1.26(c)) and hence the equivalent spring constant will be equal to

$$k = \frac{3EI_{yy}}{l^3} \tag{E.4}$$

2. The spring constants of beams with different end conditions can be found in a similar manner using results from strength of materials. The representative formulas given in Appendix B can be used to find the spring constants of the indicated beams and plates. For example, to find the spring constant of a fixed-fixed beam subjected to a concentrated force P at $x = a$ (Case 3 in Appendix B), first we express the deflection of the beam at the load point $(x = a)$, using $b = l - a$, as

$$y = \frac{P(l-a)^2a^2}{6EIl^3}[3al - 3a^2 - a(l-a)] = \frac{Pa^2(l-a)^2(al-a^2)}{3EIl^3} \tag{E.5}$$

and then find the spring constant (k) as

$$k = \frac{P}{y} = \frac{3EIl^3}{a^2(l-a)^2(al-a^2)} \tag{E.6}$$

where $I = I_{zz}$.

3. The effect of the self weight (or mass) of the beam can also be included in finding the spring constant of the beam (see Example 2.9 in Chapter 2).

■

1.7.4 Combination of Springs

In many practical applications, several linear springs are used in combination. These springs can be combined into a single equivalent spring as indicated below.

Case 1: Springs in Parallel. To derive an expression for the equivalent spring constant of springs connected in parallel, consider the two springs shown in Fig. 1.27(a). When a load W is applied, the system undergoes a static deflection δ_{st} as shown in Fig. 1.27(b). Then the free-body diagram, shown in Fig. 1.27(c), gives the equilibrium equation

$$W = k_1\delta_{st} + k_2\delta_{st} \tag{1.8}$$

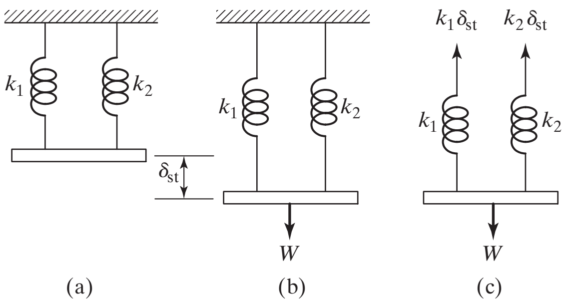


FIGURE 1.27 Springs in parallel.

If k_{eq} denotes the equivalent spring constant of the combination of the two springs, then for the same static deflection δ_{st}, we have

$$W = k_{eq}\delta_{st} \tag{1.9}$$

Equations (1.8) and (1.9) give

$$k_{eq} = k_1 + k_2 \tag{1.10}$$

In general, if we have n springs with spring constants $k_1, k_2, \ldots, k_n$ in parallel, then the equivalent spring constant k_{eq} can be obtained:

$$k_{eq} = k_1 + k_2 + \cdots + k_n \tag{1.11}$$

Case 2: Springs in Series. Next we derive an expression for the equivalent spring constant of springs connected in series by considering the two springs shown in Fig. 1.28(a). Under the action of a load W, springs 1 and 2 undergo elongations δ_1 and δ_2, respectively, as shown in Fig. 1.28(b). The total elongation (or static deflection) of the system, δ_{st}, is given by

$$\delta_{st} = \delta_1 + \delta_2 \tag{1.12}$$

Since both springs are subjected to the same force W, we have the equilibrium shown in Fig. 1.28(c):

$$\begin{aligned} W &= k_1\delta_1 \\ W &= k_2\delta_2 \end{aligned} \tag{1.13}$$

If k_{eq} denotes the equivalent spring constant, then for the same static deflection,

$$W = k_{eq}\delta_{st} \tag{1.14}$$

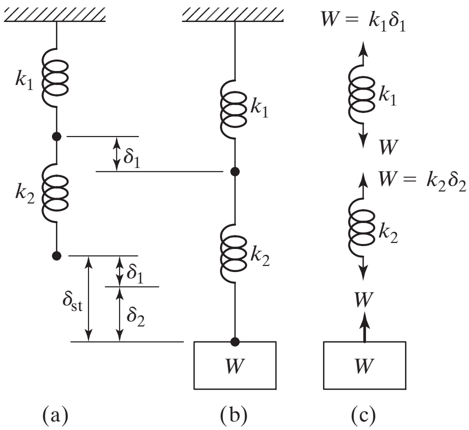


FIGURE 1.28 Springs in series.

Equations (1.13) and (1.14) give

$$k_1\delta_1 = k_2\delta_2 = k_{eq}\delta_{st}$$

or

$$\delta_1 = \frac{k_{eq}\delta_{st}}{k_1} \quad \text{and} \quad \delta_2 = \frac{k_{eq}\delta_{st}}{k_2} \tag{1.15}$$

Substituting these values of δ_1 and δ_2 into Eq. (1.12), we obtain

$$\frac{k_{eq}\delta_{st}}{k_1} + \frac{k_{eq}\delta_{st}}{k_2} = \delta_{st}$$

—that is,

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \tag{1.16}$$

Equation (1.16) can be generalized to the case of n springs in series:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} \tag{1.17}$$

In certain applications, springs are connected to rigid components such as pulleys, levers, and gears. In such cases, an equivalent spring constant can be found using energy equivalence, as illustrated in Examples 1.8 and 1.9.

EXAMPLE 1.5 Equivalent k of a Suspension System

Figure 1.29 shows the suspension system of a freight truck with a parallel-spring arrangement. Find the equivalent spring constant of the suspension if each of the three helical springs is made of steel with a shear modulus $G = 80 \times 10^9$ N/m^2 and has five effective turns, mean coil diameter $D = 20$ cm, and wire diameter $d = 2$ cm.

Solution: The stiffness of each helical spring is given by

$$k = \frac{d^4G}{8D^3n} = \frac{(0.02)^4(80 \times 10^9)}{8(0.2)^3(5)} = 40{,}000.0 \text{ N/m}$$

(See inside front cover for the formula.)

Since the three springs are identical and parallel, the equivalent spring constant of the suspension system is given by

$$k_{eq} = 3k = 3(40{,}000.0) = 120{,}000.0 \text{ N/m}$$

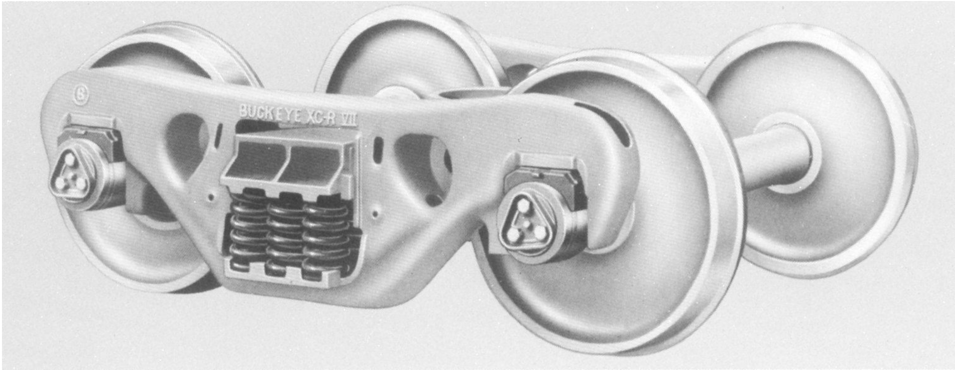


FIGURE 1.29 Parallel arrangement of springs in a freight truck. (Courtesy of Buckeye Steel Castings Company.)

■

EXAMPLE 1.6

Torsional Spring Constant of a Propeller Shaft

Determine the torsional spring constant of the steel propeller shaft shown in Fig. 1.30.

Solution: We need to consider the segments 12 and 23 of the shaft as springs in combination. From Fig. 1.30 the torque induced at any cross section of the shaft (such as *AA* or *BB*) can be seen to be

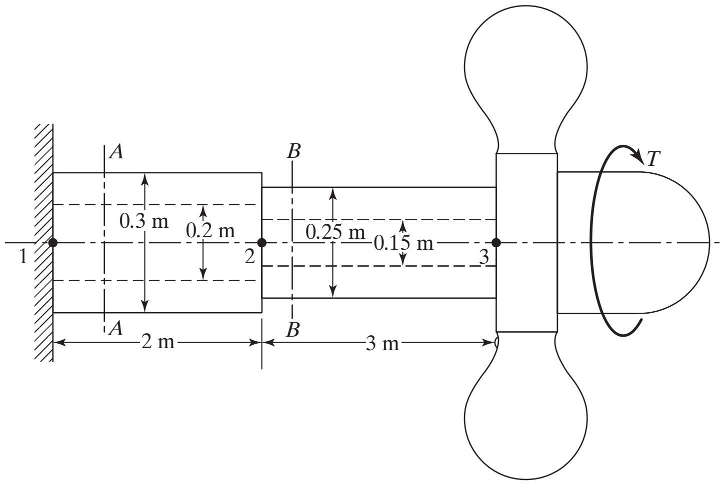


FIGURE 1.30 Propeller shaft.

equal to the torque applied at the propeller, T. Hence the elasticities (springs) corresponding to the two segments 12 and 23 are to be considered as series springs. The spring constants of segments 12 and 23 of the shaft ($k_{t_{12}}$ and $k_{t_{23}}$) are given by

$$k_{t_{12}} = \frac{GJ_{12}}{l_{12}} = \frac{G\pi(D_{12}^4 - d_{12}^4)}{32 l_{12}} = \frac{(80 \times 10^9)\pi(0.3^4 - 0.2^4)}{32(2)}$$

$$= 25.5255 \times 10^6 \text{ N-m/rad}$$

$$k_{t_{23}} = \frac{GJ_{23}}{l_{23}} = \frac{G\pi(D_{23}^4 - d_{23}^4)}{32 l_{23}} = \frac{(80 \times 10^9)\pi(0.25^4 - 0.15^4)}{32(3)}$$

$$= 8.9012 \times 10^6 \text{ N-m/rad}$$

Since the springs are in series, Eq. (1.16) gives

$$k_{t_{eq}} = \frac{k_{t_{12}} k_{t_{23}}}{k_{t_{12}} + k_{t_{23}}} = \frac{(25.5255 \times 10^6)(8.9012 \times 10^6)}{(25.5255 \times 10^6 + 8.9012 \times 10^6)} = 6.5997 \times 10^6 \text{ N-m/rad}$$

■

EXAMPLE 1.7 Equivalent k of Hoisting Drum

A hoisting drum, carrying a steel wire rope, is mounted at the end of a cantilever beam as shown in Fig. 1.31(a). Determine the equivalent spring constant of the system when the suspended length of the wire rope is l. Assume that the net cross-sectional diameter of the wire rope is d and the Young's modulus of the beam and the wire rope is E.

Solution: The spring constant of the cantilever beam is given by

$$k_b = \frac{3EI}{b^3} = \frac{3E}{b^3}\left(\frac{1}{12}at^3\right) = \frac{Eat^3}{4b^3} \tag{E.1}$$

The stiffness of the wire rope subjected to axial loading is

$$k_r = \frac{AE}{l} = \frac{\pi d^2 E}{4l} \tag{E.2}$$

Since both the wire rope and the cantilever beam experience the same load W, as shown in Fig. 1.31(b), they can be modeled as springs in series, as shown in Fig. 1.31(c). The equivalent spring constant k_{eq} is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_b} + \frac{1}{k_r} = \frac{4b^3}{Eat^3} + \frac{4l}{\pi d^2 E}$$

or

$$k_{eq} = \frac{E}{4}\left(\frac{\pi a t^3 d^2}{\pi d^2 b^3 + l a t^3}\right) \tag{E.3}$$

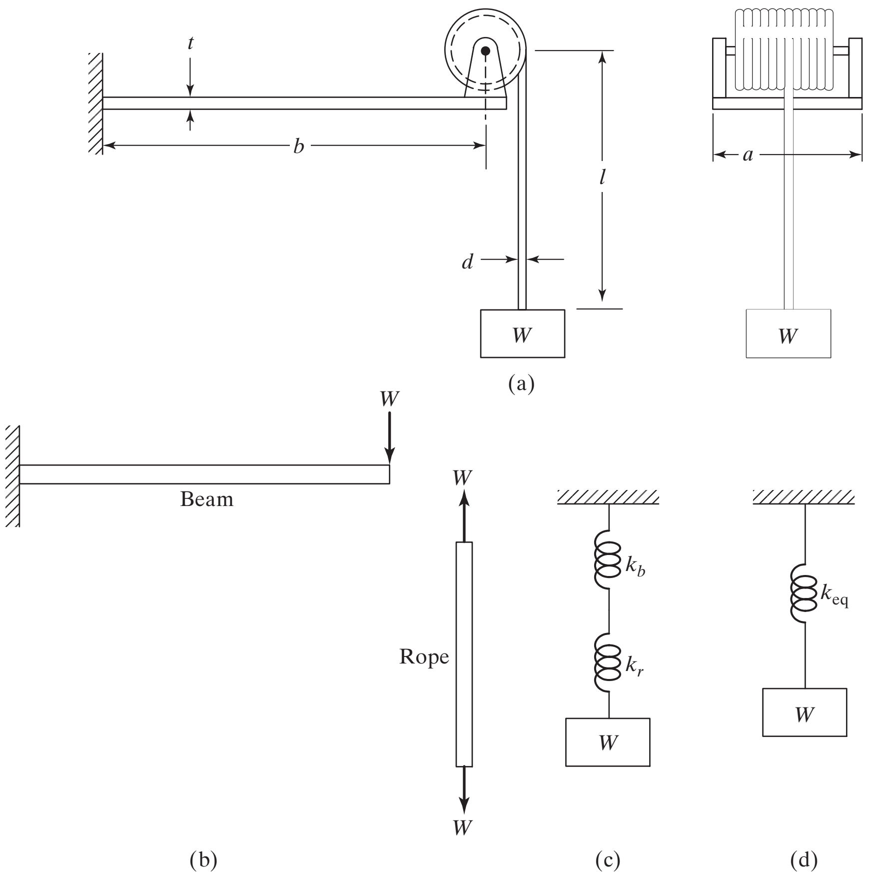


FIGURE 1.31 Hoisting drum.

■

EXAMPLE 1.8 Equivalent *k* of a Crane

The boom *AB* of the crane shown in Fig. 1.32(a) is a uniform steel bar of length 10 m and area of cross section 2,500 mm^2. A weight *W* is suspended while the crane is stationary. The cable *CDEBF* is made of steel and has a cross-sectional area of 100 mm^2. Neglecting the effect of the cable *CDEB*, find the equivalent spring constant of the system in the vertical direction.

Solution: The equivalent spring constant can be found using the equivalence of potential energies of the two systems. Since the base of the crane is rigid, the cable and the boom can be considered to be fixed at points *F* and *A*, respectively. Also, the effect of the cable *CDEB* is negligible; hence the weight *W* can be assumed to act through point *B* as shown in Fig. 1.32(b).

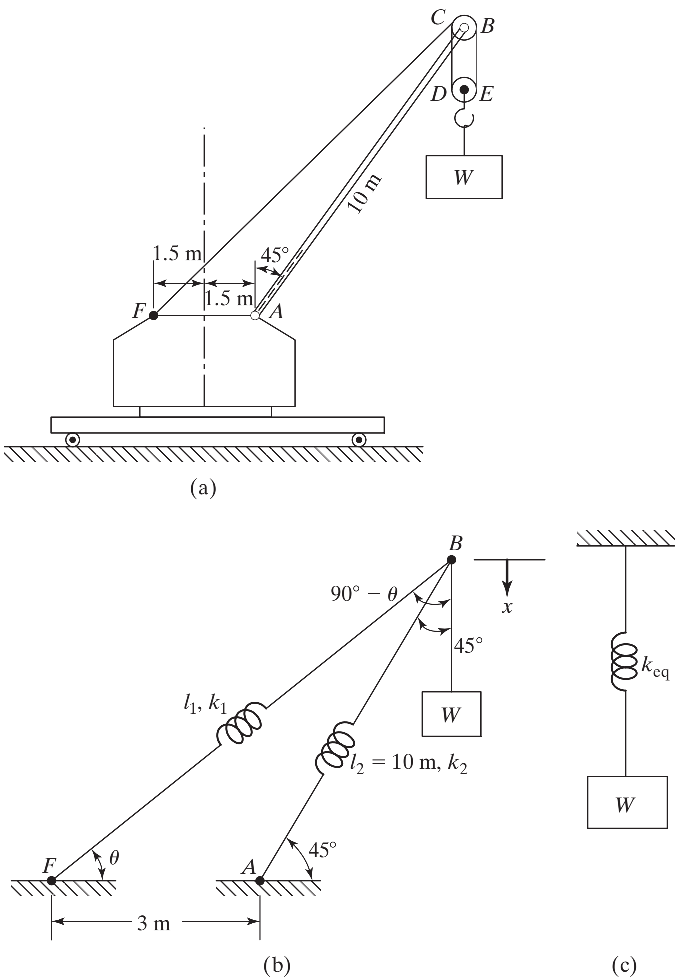


FIGURE 1.32 Crane lifting a load.

A vertical displacement x of point B will cause the spring (boom) to deform by an amount and the spring (cable) to deform by an amount The length of the cable FB, l_1, is given by Fig. 1.32(b):

$$l_1^2 = 3^2 + 10^2 - 2(3)(10)\cos 135^\circ = 151.426, \quad l_1 = 12.3055 \text{ m}$$

The angle θ satisfies the relation

$$l_1^2 + 3^2 - 2(l_1)(3)\cos\theta = 10^2, \quad \cos\theta = 0.8184, \quad \theta = 35.0736^\circ$$

The total potential energy (U) stored in the springs k_1 and k_2 can be expressed, using Eq. (1.2) as

$$U = \frac{1}{2}k_1[x\cos(90^\circ - \theta)]^2 + \frac{1}{2}k_2[x\cos(90^\circ - 45^\circ)]^2 \tag{E.1}$$

where

$$k_1 = \frac{A_1E_1}{l_1} = \frac{(100 \times 10^{-6})(207 \times 10^9)}{12.3055} = 1.6822 \times 10^6 \text{ N/m}$$

and

$$k_2 = \frac{A_2E_2}{l_2} = \frac{(2500 \times 10^{-6})(207 \times 10^9)}{10} = 5.1750 \times 10^7 \text{ N/m}$$

Since the equivalent spring in the vertical direction undergoes a deformation x, the potential energy of the equivalent spring (U_{eq}) is given by

$$U_{eq} = \tfrac{1}{2}k_{eq}x^2 \tag{E.2}$$

By setting $U = U_{eq}$, we obtain the equivalent spring constant of the system as

$$k_{eq} = k_1 \sin^2 \theta + k_2 \sin^2 45° = k_1 \sin^2 35.0736° + k_2 \sin^2 45° = 26.4304 \times 10^6 \text{ N/m}$$

■

EXAMPLE 1.9 Equivalent k of a Rigid Bar Connected by Springs

A hinged rigid bar of length l is connected by two springs of stiffnesses k_1 and k_2 and is subjected to a force F as shown in Fig. 1.33(a). Assuming that the angular displacement of the bar (θ) is small, find the equivalent spring constant of the system that relates the applied force F to the resulting displacement x.

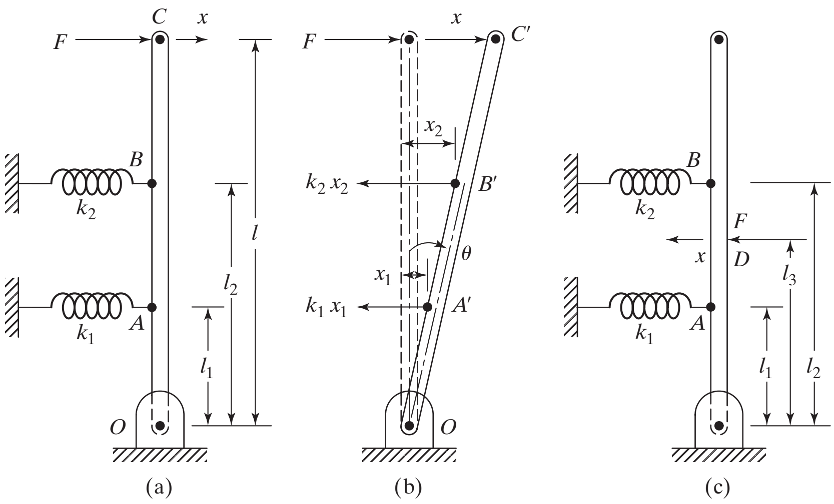


FIGURE 1.33 Rigid bar connected by springs.

Solution: For a small angular displacement of the rigid bar (θ), the points of attachment of springs k_1 and k_2 (A and B) and the point of application (C) of the force F undergo the linear or horizontal displacements $l_1 \sin\theta$, $l_2 \sin\theta$, and $l \sin\theta$, respectively. Since θ is small, the horizontal displacements of points A, B, and C can be approximated as $x_1 = l_1\theta$, $x_2 = l_2\theta$ and $x = l\theta$, respectively. The reactions of the springs, k_1x_1 and k_2x_2, will be as indicated in Fig. 1.33(b). The equivalent spring constant of the system (k_{eq}) referred to the point of application of the force F can be determined by considering the moment equilibrium of the forces about the hinge point O:

$$k_1x_1(l_1) + k_2x_2(l_2) = F(l)$$

or

$$F = k_1\left(\frac{x_1l_1}{l}\right) + k_2\left(\frac{x_2l_2}{l}\right) \tag{E.1}$$

By expressing F as $k_{eq}x$, Eq. (E.1) can be written as

$$F = k_{eq}x = k_1\left(\frac{x_1l_1}{l}\right) + k_2\left(\frac{x_2l_2}{l}\right) \tag{E.2}$$

Using $x_1 = l_1\theta$, $x_2 = l_2\theta$, and $x = l\theta$, Eq. (E.2) yields the desired result:

$$k_{eq} = k_1\left(\frac{l_1}{l}\right)^2 + k_2\left(\frac{l_2}{l}\right)^2 \tag{E.3}$$

Notes:

1. If the force F is applied at another point D of the rigid bar as shown in Fig. 1.33(c), the equivalent spring constant referred to point D can be found as

$$k_{eq} = k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2 \tag{E.4}$$

2. The equivalent spring constant, k_{eq}, of the system can also be found by using the relation:

$$\text{Work done by the applied force } F = \text{Strain energy stored in springs } k_1 \text{ and } k_2 \tag{E.5}$$

For the system shown in Fig. 1.33(a), Eq. (E.5) gives

$$\frac{1}{2}Fx = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 \tag{E.6}$$

from which Eq. (E.3) can readily be obtained.

3. Although the two springs appear to be connected to the rigid bar in parallel, the formula of parallel springs (Eq. 1.12) cannot be used because the displacements of the two springs are not the same.

■

1.7.5 Spring Constant Associated with the Restoring Force due to Gravity

In some applications, a restoring force or moment due to gravity is developed when a mass undergoes a displacement. In such cases, an equivalent spring constant can be associated with the restoring force or moment of gravity. The following example illustrates the procedure.

EXAMPLE 1.10 Spring Constant Associated with Restoring Force due to Gravity

Figure 1.34 shows a simple pendulum of length l with a bob of mass m. Considering an angular displacement θ of the pendulum, determine the equivalent spring constant associated with the restoring force (or moment).

Solution: When the pendulum undergoes an angular displacement θ, the mass m moves by a distance $l \sin \theta$ along the horizontal (x) direction. The restoring moment or torque (T) created by the weight of the mass (mg) about the pivot point O is given by

$$T = mg(l \sin \theta) \tag{E.1}$$

For small angular displacements θ, $\sin \theta$ can be approximated as $\sin \theta \approx \theta$ (see Appendix A) and Eq. (E.1) becomes

$$\mathrm{T} = mgl\theta \tag{E.2}$$

By expressing Eq. (E.2) as

$$T = k_t\theta \tag{E.3}$$

the desired equivalent torsional spring constant k_t can be identified as

$$k_t = mgl \tag{E.4}$$

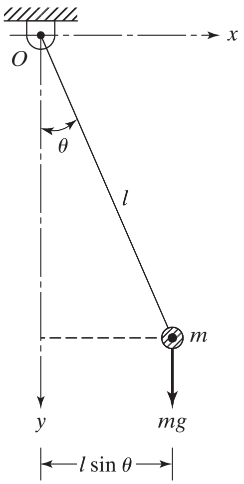


FIGURE 1.34 Simple pendulum. ■

1.8 Mass or Inertia Elements

The mass or inertia element is assumed to be a rigid body; it can gain or lose kinetic energy whenever the velocity of the body changes. From Newton's second law of motion, the product of the mass and its acceleration is equal to the force applied to the mass. Work is equal to the force multiplied by the displacement in the direction of the force, and the work done on a mass is stored in the form of the mass's kinetic energy.

In most cases, we must use a mathematical model to represent the actual vibrating system, and there are often several possible models. The purpose of the analysis often determines which mathematical model is appropriate. Once the model is chosen, the mass or inertia elements of the system can be easily identified. For example, consider again the cantilever beam with an end mass shown in Fig. 1.25(a). For a quick and reasonably accurate analysis, the mass and damping of the beam can be disregarded; the system can be modeled as a spring-mass system, as shown in Fig. 1.25(b). The tip mass m represents the mass element, and the elasticity of the beam denotes the stiffness of the spring. Next, consider a multistory building subjected to an earthquake. Assuming that the mass of the frame is negligible compared to the masses of the floors, the building can be modeled as a multi-degree-of-freedom system, as shown in Fig. 1.35. The masses at the various floor levels represent the mass elements, and the elasticities of the vertical members denote the spring elements.

1.8.1 Combination of Masses

In many practical applications, several masses appear in combination. For a simple analysis, we can replace these masses by a single equivalent mass, as indicated below [1.27].

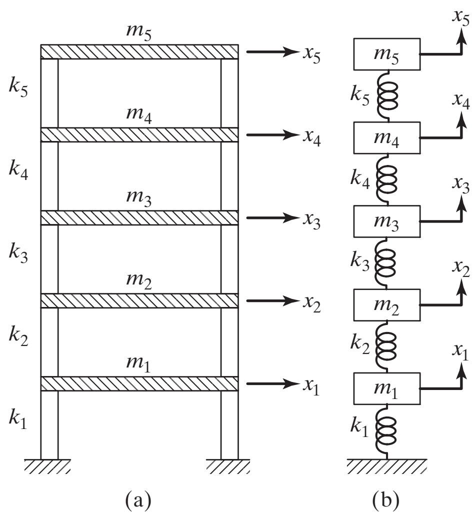


FIGURE 1.35 Idealization of a multistory building as a multi-degree-of-freedom system.

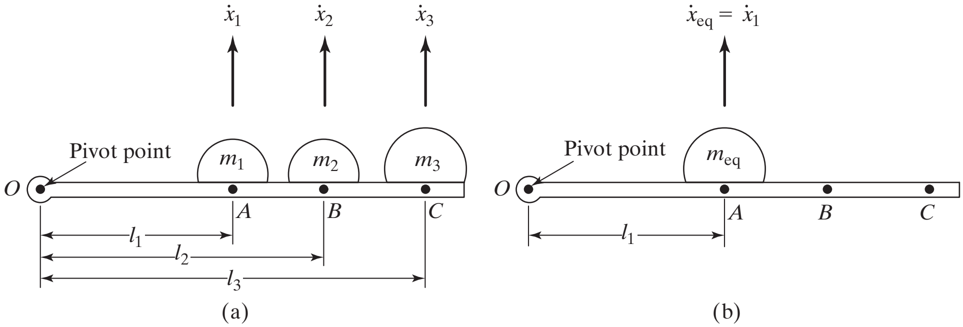


FIGURE 1.36 Translational masses connected by a rigid bar.

Case 1: Translational Masses Connected by a Rigid Bar. Let the masses be attached to a rigid bar that is pivoted at one end, as shown in Fig. 1.36(a). The equivalent mass can be assumed to be located at any point along the bar. To be specific, we assume the location of the equivalent mass to be that of mass m_1. The velocities of masses $m_2(\dot{x}_2)$ and $m_3(\dot{x}_3)$ can be expressed in terms of the velocity of mass $m_1(\dot{x}_1)$, by assuming small angular displacements for the bar, as

$$\dot{x}_2 = \frac{l_2}{l_1}\dot{x}_1, \quad \dot{x}_3 = \frac{l_3}{l_1}\dot{x}_1 \tag{1.18}$$

and

$$\dot{x}_{eq} = \dot{x}_1 \tag{1.19}$$

By equating the kinetic energy of the three-mass system to that of the equivalent mass system, we obtain

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 = \frac{1}{2}m_{eq}\dot{x}_{eq}^2 \tag{1.20}$$

This equation gives, in view of Eqs. (1.18) and (1.19):

$$m_{eq} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3 \tag{1.21}$$

It can be seen that the equivalent mass of a system composed of several masses (each moving at a different velocity) can be thought of as the imaginary mass which, while moving with a specified velocity v, will have the same kinetic energy as that of the system.

Case 2: Translational and Rotational Masses Coupled Together. Let a mass m having a translational velocity $\dot{x}$ be coupled to another mass (of mass moment of inertia J_0) having a rotational velocity $\dot{\theta}$, as in the rack-and-pinion arrangement shown in Fig. 1.37.

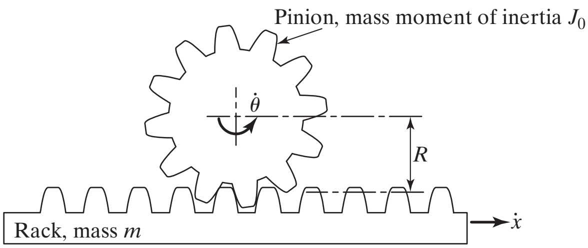


FIGURE 1.37 Translational and rotational masses in a rack-and-pinion arrangement.

These two masses can be combined to obtain either (1) a single equivalent translational mass m_{eq} or (2) a single equivalent rotational mass J_{eq}, as shown below.

1. *Equivalent translational mass.* The kinetic energy of the two masses is given by

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 \tag{1.22}$$

and the kinetic energy of the equivalent mass can be expressed as

$$T_{eq} = \frac{1}{2}m_{eq}\dot{x}_{eq}^2 \tag{1.23}$$

Since $\dot{x}_{eq} = \dot{x}$ and $\dot{\theta} = \dot{x}/R$, the equivalence of T and T_{eq} gives

$$\frac{1}{2}m_{eq}\dot{x}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\left(\frac{\dot{x}}{R}\right)^2$$

—that is,

$$m_{eq} = m + \frac{J_0}{R^2} \tag{1.24}$$

2. *Equivalent rotational mass.* Here $\dot{\theta}_{eq} = \dot{\theta}$ and $\dot{x} = \dot{\theta}R$, and the equivalence of T and T_{eq} leads to

$$\frac{1}{2}J_{eq}\dot{\theta}^2 = \frac{1}{2}m(\dot{\theta}R)^2 + \frac{1}{2}J_0\dot{\theta}^2$$

or

$$J_{eq} = J_0 + mR^2 \tag{1.25}$$

EXAMPLE 1.11 Equivalent Mass of a System

Find the equivalent mass of the system shown in Fig. 1.38, where the rigid link 1 is attached to the pulley and rotates with it.

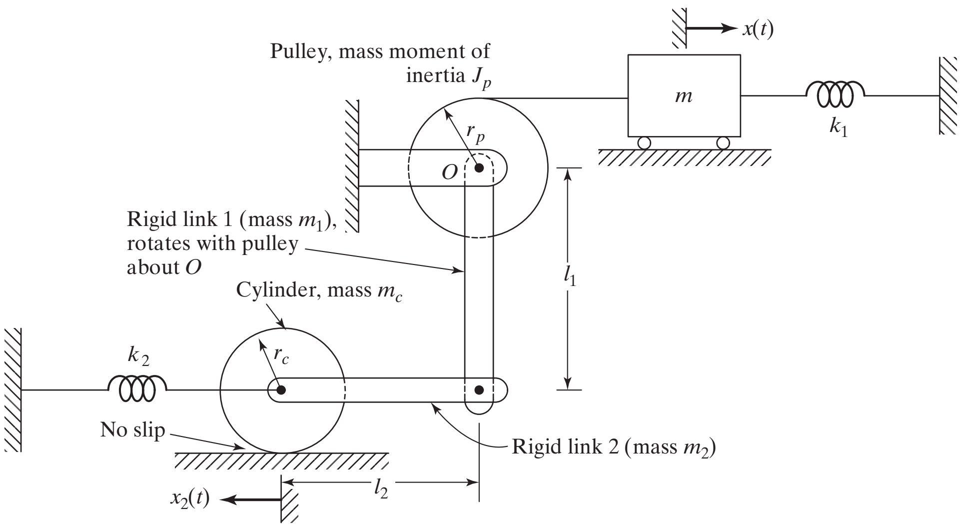


FIGURE 1.38 System considered for finding equivalent mass.

Solution: Assuming small displacements, the equivalent mass (m_{eq}) can be determined using the equivalence of the kinetic energies of the two systems. When the mass m is displaced by a distance x, the pulley and the rigid link 1 rotate by an angle $\theta_p = \theta_1 = x/r_p$. This causes the rigid link 2 and the cylinder to be displaced by a distance $x_2 = \theta_p l_1 = x l_1/r_p$. Since the cylinder rolls without slippage, it rotates by an angle $\theta_c = x_2/r_c = x l_1/r_p r_c$. The kinetic energy of the system (T) can be expressed (for small displacements) as:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_p\dot{\theta}_p^2 + \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}J_c\dot{\theta}_c^2 + \frac{1}{2}m_c\dot{x}_2^2 \tag{E.1}$$

where J_p, J_1, and J_c denote the mass moments of inertia of the pulley, link 1 (about O), and cylinder, respectively, $\dot{\theta}_p$, $\dot{\theta}_1$, and $\dot{\theta}_c$ indicate the angular velocities of the pulley, link 1 (about O), and cylinder, respectively, and $\dot{x}$ and $\dot{x}_2$ represent the linear velocities of the mass m and link 2, respectively. Noting that $J_c = m_c r_c^2/2$ and $J_1 = m_1 l_1^2/3$, Eq. (E.1) can be rewritten as

$$\begin{aligned} T = {} & \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_p\left(\frac{\dot{x}}{r_p}\right)^2 + \frac{1}{2}\left(\frac{m_1 l_1^2}{3}\right)\left(\frac{\dot{x}}{r_p}\right)^2 + \frac{1}{2}m_2\left(\frac{\dot{x}l_1}{r_p}\right)^2 \\ & + \frac{1}{2}\left(\frac{m_c r_c^2}{2}\right)\left(\frac{\dot{x}l_1}{r_p r_c}\right)^2 + \frac{1}{2}m_c\left(\frac{\dot{x}l_1}{r_p}\right)^2 \end{aligned} \tag{E.2}$$

By equating Eq. (E.2) to the kinetic energy of the equivalent system

$$T = \frac{1}{2}m_{\text{eq}}\dot{x}^2 \tag{E.3}$$

we obtain the equivalent mass of the system as

$$m_{\text{eq}} = m + \frac{J_p}{r_p^2} + \frac{1}{3}\frac{m_1 l_1^2}{r_p^2} + \frac{m_2 l_1^2}{r_p^2} + \frac{1}{2}\frac{m_c l_1^2}{r_p^2} + m_c\frac{l_1^2}{r_p^2} \tag{E.4}$$

■

EXAMPLE 1.12 Cam-Follower Mechanism

A cam-follower mechanism (Fig. 1.39) is used to convert the rotary motion of a shaft into the oscillating or reciprocating motion of a valve. The follower system consists of a pushrod of mass m_p, a rocker arm of mass m_r, and mass moment of inertia J_r about its C.G., a valve of mass m_r, and a valve spring of negligible mass [1.28–1.30]. Find the equivalent mass (m_{eq}) of this cam-follower system by assuming the location of m_{eq} as (i) point A and (ii) point C.

Solution: The equivalent mass of the cam-follower system can be determined using the equivalence of the kinetic energies of the two systems. Due to a vertical displacement x of the pushrod, the rocker arm rotates by an angle $\theta_r = x/l_1$ about the pivot point, the valve moves downward by $x_v = \theta_r l_2 = x l_2/l_1$, and the C.G. of the rocker arm moves downward by $x_r = \theta_r l_3 = x l_3/l_1$. The kinetic energy of the system (T) can be expressed as[2]

$$T = \frac{1}{2} m_p \dot{x}_p^2 + \frac{1}{2} m_v \dot{x}_v^2 + \frac{1}{2} J_r \dot{\theta}_r^2 + \frac{1}{2} m_r \dot{x}_r^2 \tag{E.1}$$

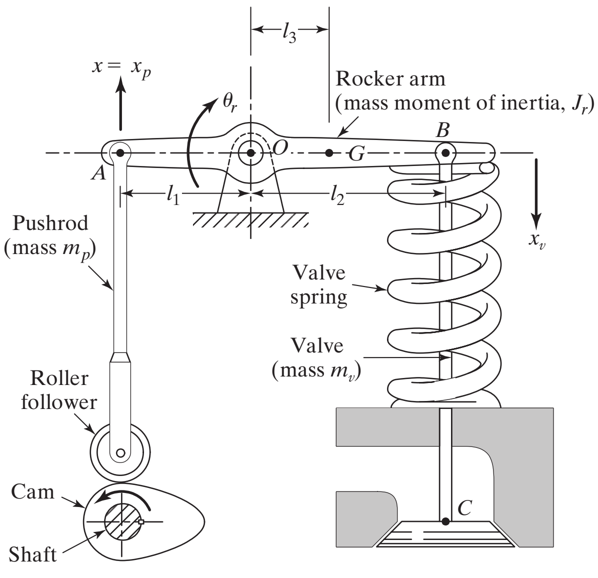


FIGURE 1.39 Cam-follower system.

[2]If the valve spring has a mass m_s, then its equivalent mass will be $\frac{1}{3} m_s$ (see Example 2.8). Thus the kinetic energy of the valve spring will be $\frac{1}{2}(\frac{1}{3} m_s)\dot{x}_v^2$.

where $\dot{x}_p$, $\dot{x}_r$, and $\dot{x}_v$ are the linear velocities of the pushrod, C.G. of the rocker arm, and the valve, respectively, and $\dot{\theta}_r$ is the angular velocity of the rocker arm.

(i) If m_{eq} denotes the equivalent mass placed at point A, with $\dot{x}_{\text{eq}} = \dot{x}$, the kinetic energy of the equivalent mass system T_{eq} is given by

$$T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{x}_{\text{eq}}^2 \tag{E.2}$$

By equating T and T_{eq}, and noting that

$$\dot{x}_p = \dot{x}, \quad \dot{x}_v = \frac{\dot{x} l_2}{l_1}, \quad \dot{x}_r = \frac{\dot{x} l_3}{l_1}, \quad \text{and} \quad \dot{\theta}_r = \frac{\dot{x}}{l_1}$$

we obtain

$$m_{\text{eq}} = m_p + \frac{J_r}{l_1^2} + m_v \frac{l_2^2}{l_1^2} + m_r \frac{l_3^2}{l_1^2} \tag{E.3}$$

(ii) Similarly, if the equivalent mass is located at point C, $\dot{x}_{\text{eq}} = \dot{x}_v$ and

$$T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{x}_{\text{eq}}^2 = \frac{1}{2} m_{\text{eq}} \dot{x}_v^2 \tag{E.4}$$

Equating (E.4) and (E.1) gives

$$m_{\text{eq}} = m_v + \frac{J_r}{l_2^2} + m_p \left(\frac{l_1}{l_2}\right)^2 + m_r \left(\frac{l_3}{l_2}\right)^2 \tag{E.5}$$

■

1.9 Damping Elements

In many practical systems, the vibrational energy is gradually converted to heat or sound. Due to the reduction in the energy, the response, such as the displacement of the system, gradually decreases. The mechanism by which the vibrational energy is gradually converted into heat or sound is known as *damping*. Although the amount of energy converted into heat or sound is relatively small, the consideration of damping becomes important for an accurate prediction of the vibration response of a system. A damper is assumed to have neither mass nor elasticity, and damping force exists only if there is relative velocity between the two ends of the damper. It is difficult to determine the causes of damping in practical systems. Hence damping is modeled as one or more of the following types.

Viscous Damping. Viscous damping is the most commonly used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water, or oil, the resistance offered by the fluid to the moving body causes energy to be dissipated. In this case, the amount of dissipated energy depends on many factors, such as the size and shape of the vibrating body, the viscosity of the fluid, the frequency of vibration, and the velocity of the vibrating body. In viscous damping, the damping force is proportional to the velocity of the vibrating body. Typical examples of viscous damping include (1) fluid film between sliding surfaces, (2) fluid flow around a piston in a cylinder, (3) fluid flow through an orifice, and (4) fluid film around a journal in a bearing.

Coulomb or Dry-Friction Damping. Here the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused by friction between rubbing surfaces that either are dry or have insufficient lubrication.

Material or Solid or Hysteretic Damping. When a material is deformed, energy is absorbed and dissipated by the material [1.31]. The effect is due to friction between the internal planes, which slip or slide as the deformations take place. When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop as indicated in Fig. 1.40(a). The area of this loop denotes the energy lost per unit volume of the body per cycle due to damping.[3]

1.9.1 Construction of Viscous Dampers

Viscous dampers can be constructed in several ways. For instance, when a plate moves relative to another parallel plate with a viscous fluid in between the plates, a viscous damper can be obtained. The following examples illustrate the various methods of constructing viscous dampers used in different applications.

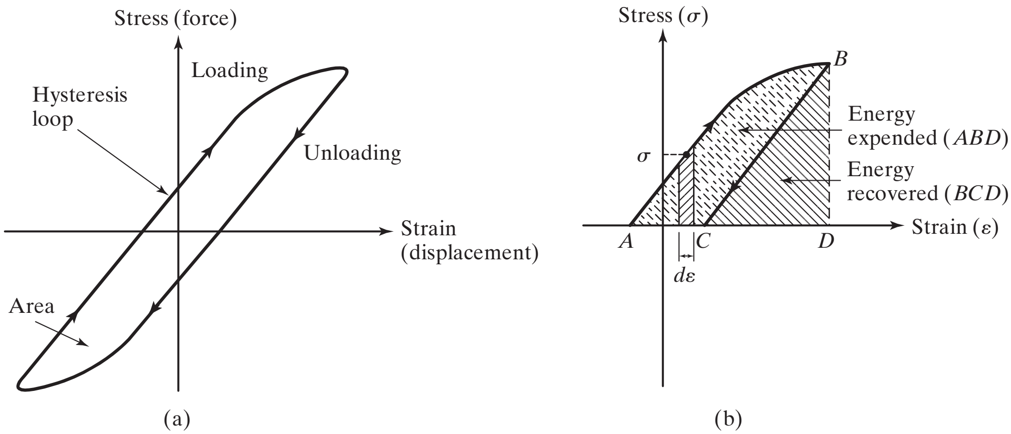


FIGURE 1.40 Hysteresis loop for elastic materials.

[3]When the load applied to an elastic body is increased, the stress (σ) and the strain (ε) in the body also increase. The area under the σ-ε curve, given by

$$u = \int \sigma \, d\varepsilon$$

denotes the energy expended (work done) per unit volume of the body. When the load on the body is decreased, energy will be recovered. When the unloading path is different from the loading path, the area ABC in Fig. 1.40(b)—the area of the hysteresis loop in Fig. 1.40(a)—denotes the energy lost per unit volume of the body.

EXAMPLE 1.13 Damping Constant of Parallel Plates Separated by Viscous Fluid

Consider two parallel plates separated by a distance h, with a fluid of viscosity μ between the plates. Derive an expression for the damping constant when one plate moves with a velocity v relative to the other as shown in Fig. 1.41.

Solution: Let one plate be fixed and let the other plate be moved with a velocity v in its own plane. The fluid layers in contact with the moving plate move with a velocity v, while those in contact with the fixed plate do not move. The velocities of intermediate fluid layers are assumed to vary linearly between 0 and v, as shown in Fig. 1.41. According to Newton's law of viscous flow, the shear stress (τ) developed in the fluid layer at a distance y from the fixed plate is given by

$$\tau = \mu \frac{du}{dy} \tag{E.1}$$

where $du/dy = v/h$ is the velocity gradient. The shear or resisting force (F) developed at the bottom surface of the moving plate is

$$F = \tau A = \frac{\mu A v}{h} \tag{E.2}$$

where A is the surface area of the moving plate. By expressing F as

$$F = cv \tag{E.3}$$

the damping constant c can be found as

$$c = \frac{\mu A}{h} \tag{E.4}$$

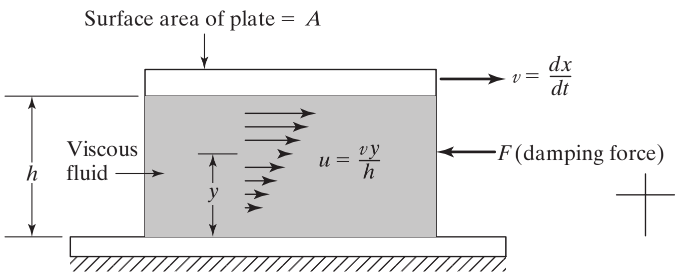


FIGURE 1.41 Parallel plates with a viscous fluid in between. ■

EXAMPLE 1.14 Clearance in a Bearing

A bearing, which can be approximated as two flat plates separated by a thin film of lubricant (Fig. 1.42), is found to offer a resistance of 400 N when SAE 30 oil is used as the lubricant and the relative velocity between the plates is 10 m/s. If the area of the plates (A) is 0.1 m^2, determine the clearance between the plates. Assume the absolute viscosity of SAE 30 oil as 50 μreyn or 0.3445 Pa-s.

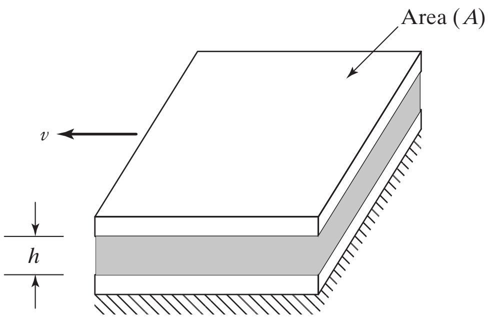


FIGURE 1.42 Flat plates separated by thin film of lubricant.

Solution: Since the resisting force (F) can be expressed as $F = cv$, where c is the damping constant and v is the velocity, we have

$$c = \frac{F}{v} = \frac{400}{10} = 40 \text{ N-s/m} \tag{E.1}$$

By modeling the bearing as a flat-plate-type damper, the damping constant is given by Eq. (E.4) of Example 1.13:

$$c = \frac{\mu A}{h} \tag{E.2}$$

Using the known data, Eq. (E.2) gives

$$c = 40 = \frac{(0.3445)(0.1)}{h} \quad \text{or} \quad h = 0.86125 \text{ mm} \tag{E.3}$$

■

EXAMPLE 1.15 Damping Constant of a Journal Bearing

A journal bearing is used to provide lateral support to a rotating shaft as shown in Fig. 1.43. If the radius of the shaft is R, angular velocity of the shaft is ω, radial clearance between the shaft and the bearing is d, viscosity of the fluid (lubricant) is μ, and the length of the bearing is l, derive an expression for the rotational damping constant of the journal bearing. Assume that the leakage of the fluid is negligible.

Solution: The damping constant of the journal bearing can be determined using the equation for the shear stress in viscous fluid. The fluid in contact with the rotating shaft will have a linear velocity (in tangential direction) of $v = R\omega$, while the fluid in contact with the stationary bearing will have zero velocity. Assuming a linear variation for the velocity of the fluid in the radial direction, we have

$$v(r) = \frac{vr}{d} = \frac{rR\omega}{d} \tag{E.1}$$

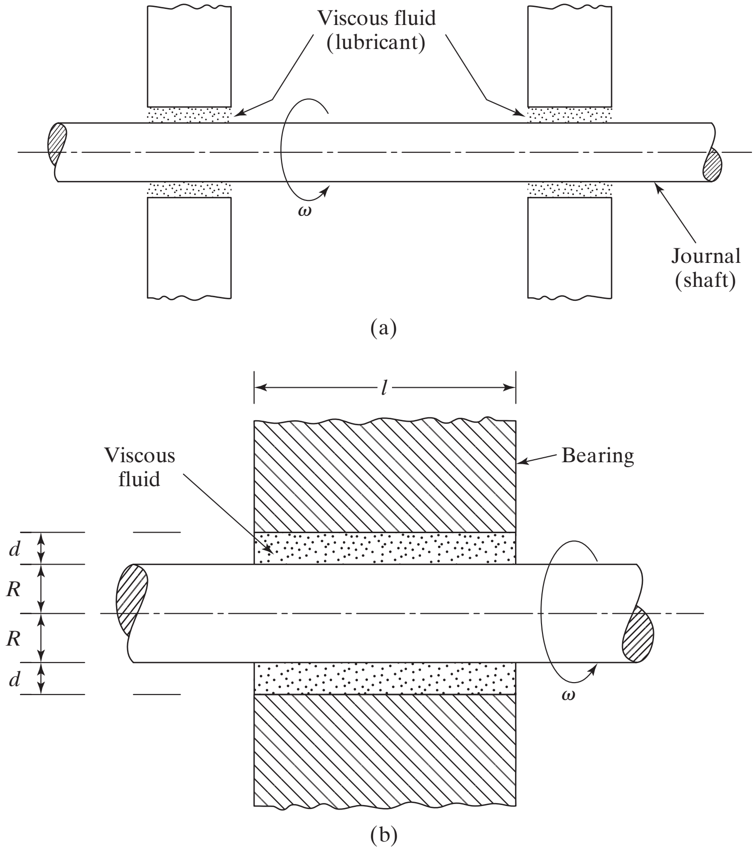


FIGURE 1.43 A journal bearing.

The shearing stress (τ) in the lubricant is given by the product of the radial velocity gradient and the viscosity of the lubricant:

$$\tau = \mu \frac{dv}{dr} = \frac{\mu R \omega}{d} \tag{E.2}$$

The force required to shear the fluid film is equal to stress times the area. The torque on the shaft (T) is equal to the force times the lever arm, so that

$$T = (\tau A) R \tag{E.3}$$

where $A = 2\pi R l$ is the surface area of the shaft exposed to the lubricant. Thus Eq. (E.3) can be rewritten as

$$T = \left(\frac{\mu R \omega}{d}\right)(2\pi R l) R = \frac{2\pi \mu R^3 l \omega}{d} \tag{E.4}$$

From the definition of the rotational damping constant of the bearing (c_t):

$$c_t = \frac{T}{\omega} \tag{E.5}$$

we obtain the desired expression for the rotational damping constant as

$$c_t = \frac{2\pi\mu R^3 l}{d} \tag{E.6}$$

Note: Equation (E.4) is called Petroff's law and was published originally in 1883. This equation is widely used in the design of journal bearings [1.43].

■

EXAMPLE 1.16 Piston-Cylinder Dashpot

Develop an expression for the damping constant of the dashpot shown in Fig. 1.44(a).

Solution: The damping constant of the dashpot can be determined using the shear-stress equation for viscous fluid flow and the rate-of-fluid-flow equation. As shown in Fig. 1.44(a), the dashpot consists of a piston of diameter D and length l, moving with velocity v_0 in a cylinder filled with a liquid of viscosity μ [1.24, 1.32]. Let the clearance between the piston and the cylinder wall be d. At a distance y from the moving surface, let the velocity and shear stress be v and τ, and at a distance $(y + dy)$ let the velocity and shear stress be $(v - dv)$ and $(\tau + d\tau)$, respectively (see Fig. 1.44(b)). The negative sign for dv shows that the velocity decreases as we move toward the cylinder wall. The viscous force on this annular ring is equal to

$$F = \pi Dl\, d\tau = \pi Dl \frac{d\tau}{dy} dy \tag{E.1}$$

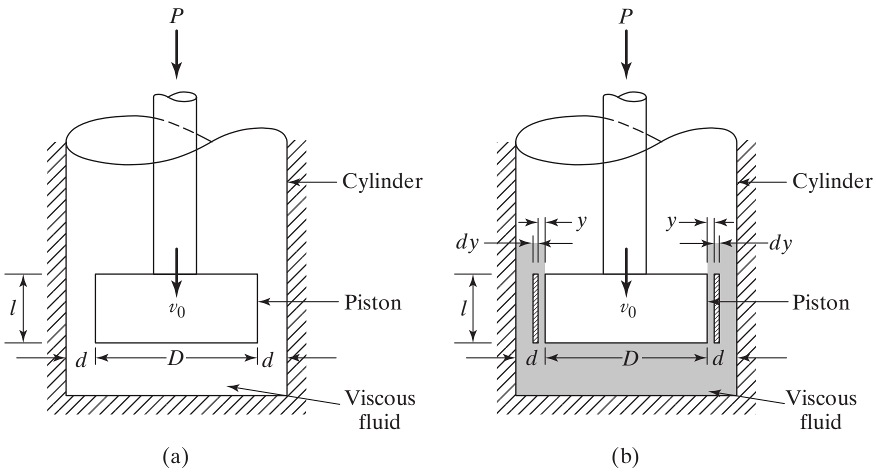


FIGURE 1.44 A dashpot.

But the shear stress is given by

$$\tau = -\mu \frac{dv}{dy} \tag{E.2}$$

where the negative sign is consistent with a decreasing velocity gradient [1.33]. Using Eq. (E.2) in Eq. (E.1), we obtain

$$F = -\pi Dl\, dy\, \mu \frac{d^2v}{dy^2} \tag{E.3}$$

The force on the piston will cause a pressure difference on the ends of the element, given by

$$p = \frac{P}{\left(\frac{\pi D^2}{4}\right)} = \frac{4P}{\pi D^2} \tag{E.4}$$

Thus the pressure force on the end of the element is

$$p(\pi D\, dy) = \frac{4P}{D}\, dy \tag{E.5}$$

where $(\pi D\, dy)$ denotes the annular area between y and $(y + dy)$. If we assume uniform mean velocity in the direction of motion of the fluid, the forces given in Eqs. (E.3) and (E.5) must be equal. Thus we get

$$\frac{4P}{D}\, dy = -\pi Dl\, dy\, \mu \frac{d^2v}{dy^2}$$

or

$$\frac{d^2v}{dy^2} = -\frac{4P}{\pi D^2 l \mu} \tag{E.6}$$

Integrating this equation twice and using the boundary conditions $v = -v_0$ at $y = 0$ and $v = 0$ at $y = d$, we obtain

$$v = \frac{2P}{\pi D^2 l \mu}(yd - y^2) - v_0\left(1 - \frac{y}{d}\right) \tag{E.7}$$

The rate of flow through the clearance space can be obtained by integrating the rate of flow through an element between the limits $y = 0$ and $y = d$:

$$Q = \int_0^d v\pi D\, dy = \pi D\left[\frac{2Pd^3}{6\pi D^2 l \mu} - \frac{1}{2} v_0 d\right] \tag{E.8}$$

The volume of the liquid flowing through the clearance space per second must be equal to the volume per second displaced by the piston. Hence the velocity of the piston will be equal to this rate of flow divided by the piston area. This gives

$$v_0 = \frac{Q}{\left(\frac{\pi}{4} D^2\right)} \tag{E.9}$$

Equations (E.9) and (E.8) lead to

$$P = \left[\frac{3\pi D^3 l \left(1 + \frac{2d}{D}\right)}{4d^3} \right] \mu v_0 \tag{E.10}$$

By writing the force as $P = cv_0$, the damping constant c can be found as

$$c = \mu \left[\frac{3\pi D^3 l}{4d^3} \left(1 + \frac{2d}{D}\right) \right] \tag{E.11}$$

■

1.9.2 Linearization of a Nonlinear Damper

If the force (F)-velocity (v) relationship of a damper is nonlinear:

$$F = F(v) \tag{1.26}$$

a linearization process can be used about the operating velocity (v^*), as in the case of a nonlinear spring. The linearization process gives the equivalent damping constant as

$$c = \left. \frac{dF}{dv} \right|_{v^*} \tag{1.27}$$

1.9.3 Combination of Dampers

In some dynamic systems, multiple dampers are used. In such cases, all the dampers are replaced by a single equivalent damper. When dampers appear in combination, we can use procedures similar to those used in finding the equivalent spring constant of multiple springs to find a single equivalent damper. For example, when two translational dampers, with damping constants c_1 and c_2, appear in combination, the equivalent damping constant (c_{eq}) can be found as (see Problem 1.55):

Parallel dampers: $$c_{eq} = c_1 + c_2 \tag{1.28}$$

Series dampers: $$\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} \tag{1.29}$$

EXAMPLE 1.17 Equivalent Spring and Damping Constants of a Machine Tool Support

A precision milling machine is supported on four shock mounts, as shown in Fig. 1.45(a). The elasticity and damping of each shock mount can be modeled as a spring and a viscous damper, as shown in Fig. 1.45(b). Find the equivalent spring constant, k_{eq}, and the equivalent damping constant, c_{eq}, of the machine tool support in terms of the spring constants (k_i) and damping constants (c_i) of the mounts.

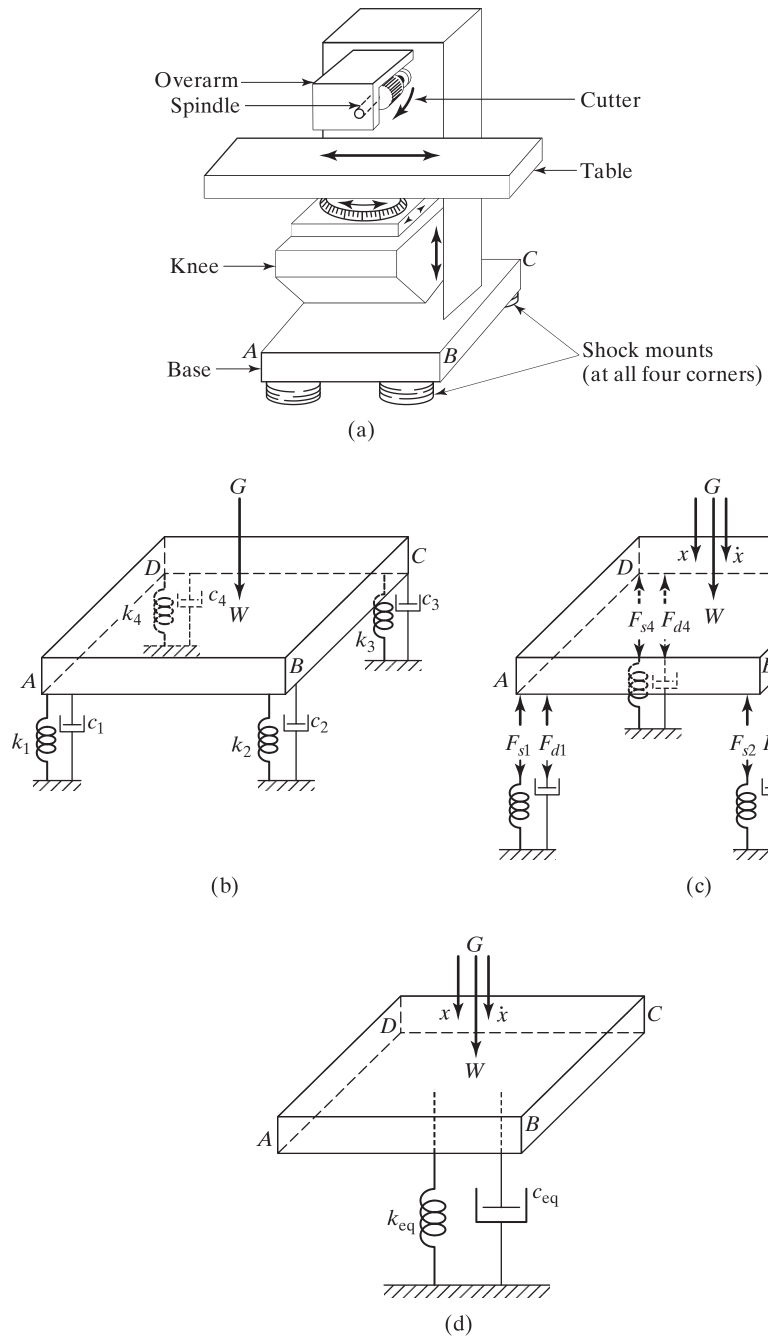


FIGURE 1.45 Horizontal milling machine.

Solution: The free-body diagrams of the four springs and four dampers are shown in Fig. 1.45(c). Assuming that the center of mass, G, is located symmetrically with respect to the four springs and dampers, we notice that all the springs will be subjected to the same displacement, x, and all the dampers will be subject to the same relative velocity $\dot{x}$, where x and $\dot{x}$ denote the displacement and velocity, respectively, of the center of mass, G. Hence the forces acting on the springs (F_{si}) and the dampers (F_{di}) can be expressed as

$$\begin{aligned} F_{si} &= k_i x; \quad i = 1, 2, 3, 4 \\ F_{di} &= c_i \dot{x}; \quad i = 1, 2, 3, 4 \end{aligned} \tag{E.1}$$

Let the total forces acting on all the springs and all the dampers be F_s and F_d, respectively (see Fig. 1.45,(d)). The force equilibrium equations can thus be expressed as

$$\begin{aligned} F_s &= F_{s1} + F_{s2} + F_{s3} + F_{s4} \\ F_d &= F_{d1} + F_{d2} + F_{d3} + F_{d4} \end{aligned} \tag{E.2}$$

where $F_s + F_d = W$, with W denoting the total vertical force (including the inertia force) acting on the milling machine. From Fig. 1.45(d), we have

$$\begin{aligned} F_s &= k_{\text{eq}} x \\ F_d &= c_{\text{eq}} \dot{x} \end{aligned} \tag{E.3}$$

Equation (E.2), along with Eqs. (E.1) and (E.3), yields

$$\begin{aligned} k_{\text{eq}} &= k_1 + k_2 + k_3 + k_4 = 4k \\ c_{\text{eq}} &= c_1 + c_2 + c_3 + c_4 = 4c \end{aligned} \tag{E.4}$$

when $k_i = k$ and $c_i = c$ for $i = 1, 2, 3, 4$.

Note: If the center of mass, G, is not located symmetrically with respect to the four springs and dampers, the ith spring experiences a displacement of x_i and the ith damper experiences a velocity of $\dot{x}_i$, where x_i and $\dot{x}_i$ can be related to the displacement x and velocity $\dot{x}$ of the center of mass of the milling machine, G. In such a case, Eqs. (E.1) and (E.4) need to be modified suitably.

■

1.10 Harmonic Motion

Oscillatory motion may repeat itself regularly, as in the case of a simple pendulum, or it may display considerable irregularity, as in the case of ground motion during an earthquake. If the motion is repeated after equal intervals of time, it is called *periodic motion.* The simplest type of periodic motion is *harmonic motion.* The motion imparted to the mass m due to the Scotch yoke mechanism shown in Fig. 1.46 is an example of simple harmonic motion [1.24, 1.34, 1.35]. In this system, a crank of radius A rotates about the point O. The other end of the crank, P, slides in a slotted rod, which reciprocates in the vertical guide R. When the crank rotates at an angular velocity ω, the end point S of the slotted link and

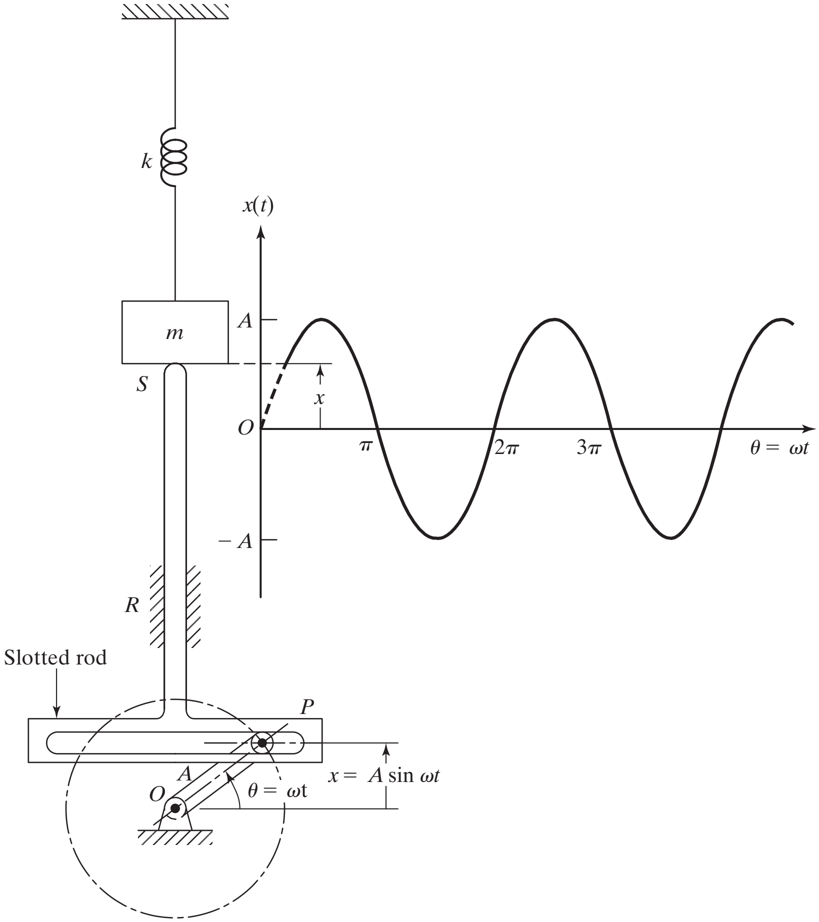


FIGURE 1.46 Scotch yoke mechanism.

hence the mass m of the spring-mass system are displaced from their middle positions by an amount x (in time t) given by

$$x = A \sin \theta = A \sin \omega t \tag{1.30}$$

This motion is shown by the sinusoidal curve in Fig. 1.46. The velocity of the mass m at time t is given by

$$\frac{dx}{dt} = \omega A \cos \omega t \tag{1.31}$$

and the acceleration by

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin \omega t = -\omega^2 x \tag{1.32}$$

It can be seen that the acceleration is directly proportional to the displacement. Such a vibration, with the acceleration proportional to the displacement and directed toward the mean position, is known as *simple harmonic motion*. The motion given by $x = A \cos \omega t$ is another example of a simple harmonic motion. Figure 1.46 clearly shows the similarity between cyclic (harmonic) motion and sinusoidal motion.

1.10.1 Vectorial Representation of Harmonic Motion

Harmonic motion can be represented conveniently by means of a vector $\overrightarrow{OP}$ of magnitude A rotating at a constant angular velocity ω. In Fig. 1.47, the projection of the tip of the vector $\vec{X} = \overrightarrow{OP}$ on the vertical axis is given by

$$y = A \sin \omega t \tag{1.33}$$

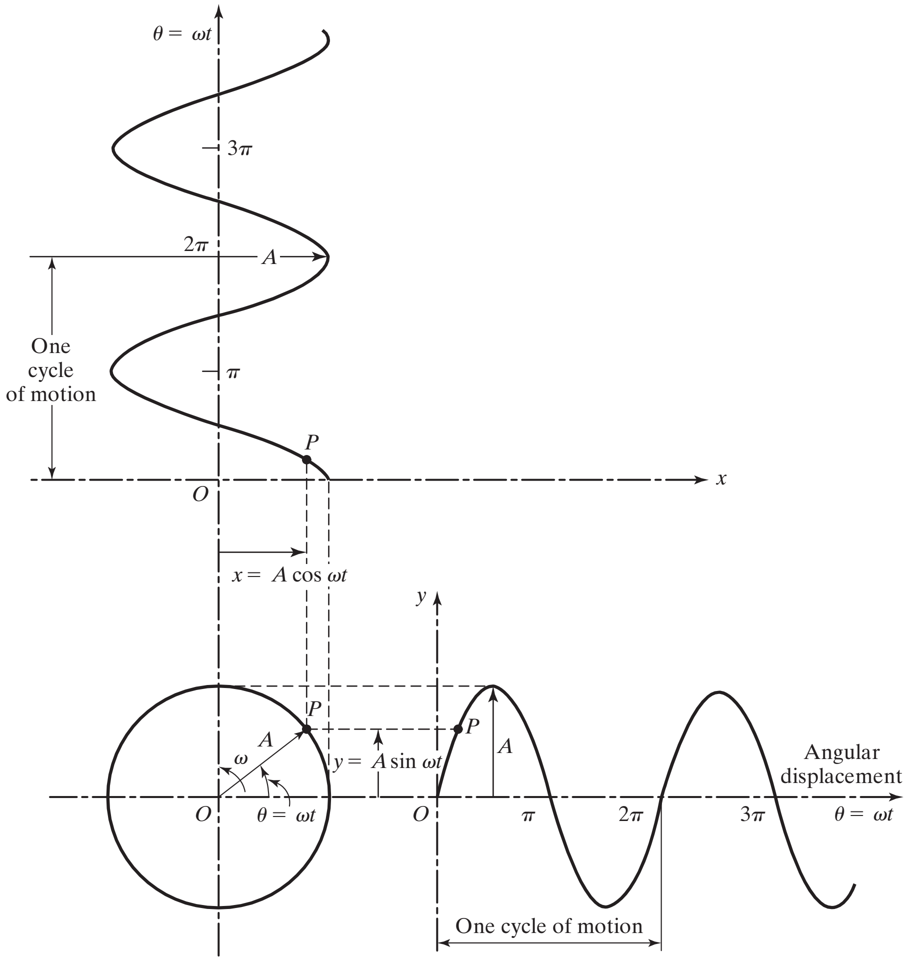


FIGURE 1.47 Harmonic motion as the projection of the end of a rotating vector.

and its projection on the horizontal axis by

$$x = A \cos \omega t \tag{1.34}$$

1.10.2 Complex-Number Representation of Harmonic Motion

As seen above, the vectorial method of representing harmonic motion requires the description of both the horizontal and vertical components. It is more convenient to represent harmonic motion using a complex-number representation. Any vector $\vec{X}$ in the xy-plane can be represented as a complex number:

$$\vec{X} = a + ib \tag{1.35}$$

where $i = \sqrt{-1}$ and a and b denote the x and y components of $\vec{X}$, respectively (see Fig. 1.48). Components a and b are also called the *real* and *imaginary* parts of the vector $\vec{X}$. If A denotes the modulus or absolute value of the vector $\vec{X}$, and θ represents the argument or the angle between the vector and the x-axis, then $\vec{X}$ can also be expressed as

$$\vec{X} = A \cos \theta + iA \sin \theta \tag{1.36}$$

with

$$A = (a^2 + b^2)^{1/2} \tag{1.37}$$

and

$$\theta = \tan^{-1} \frac{b}{a} \tag{1.38}$$

Noting that $i^2 = -1$, $i^3 = -i$, $i^4 = 1, \ldots,$ $\cos \theta$ and $i \sin \theta$ can be expanded in a series as

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \cdots \tag{1.39}$$

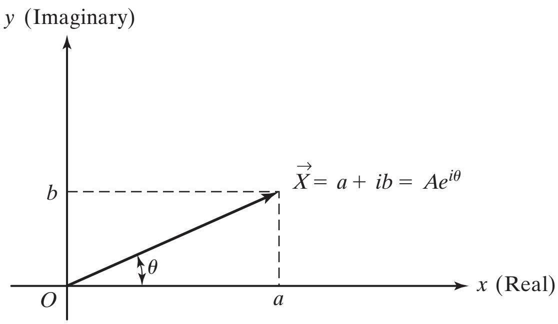


FIGURE 1.48 Representation of a complex number.

$$i \sin\theta = i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right] = i\theta + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots \tag{1.40}$$

Equations (1.39) and (1.40) yield

$$(\cos\theta + i\sin\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = e^{i\theta} \tag{1.41}$$

and

$$(\cos\theta - i\sin\theta) = 1 - i\theta + \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \cdots = e^{-i\theta} \tag{1.42}$$

Thus Eq. (1.36) can be expressed as

$$\vec{X} = A(\cos\theta + i\sin\theta) = Ae^{i\theta} \tag{1.43}$$

1.10.3 Complex Algebra

Complex numbers are often represented without using a vector notation as

$$z = a + ib \tag{1.44}$$

where a and b denote the real and imaginary parts of z. The addition, subtraction, multiplication, and division of complex numbers can be achieved by using the usual rules of algebra. Let

$$z_1 = a_1 + ib_1 = A_1 e^{i\theta_1} \tag{1.45}$$

$$z_2 = a_2 + ib_2 = A_2 e^{i\theta_2} \tag{1.46}$$

where

$$A_j = \sqrt{a_j^2 + b_j^2}; \qquad j = 1, 2 \tag{1.47}$$

and

$$\theta_j = \tan^{-1}\left(\frac{b_j}{a_j}\right); \qquad j = 1, 2 \tag{1.48}$$

The sum and difference of z_1 and z_2 can be found as

$$\begin{aligned} z_1 + z_2 &= A_1 e^{i\theta_1} + A_2 e^{i\theta_2} = (a_1 + ib_1) + (a_2 + ib_2) \\ &= (a_1 + a_2) + i(b_1 + b_2) \end{aligned} \tag{1.49}$$

$$\begin{aligned} z_1 - z_2 &= A_1 e^{i\theta_1} - A_2 e^{i\theta_2} = (a_1 + ib_1) - (a_2 + ib_2) \\ &= (a_1 - a_2) + i(b_1 - b_2) \end{aligned} \tag{1.50}$$

1.10.4 Operations on Harmonic Functions

Using complex-number representation, the rotating vector $\vec{X}$ of Fig. 1.47 can be written as

$$\vec{X} = Ae^{i\omega t} \tag{1.51}$$

where ω denotes the circular frequency (rad/sec) of rotation of the vector $\vec{X}$ in counterclockwise direction. The differentiation of the harmonic motion given by Eq. (1.51) with respect to time gives

$$\frac{d\vec{X}}{dt} = \frac{d}{dt}(Ae^{i\omega t}) = i\omega Ae^{i\omega t} = i\omega\vec{X} \tag{1.52}$$

$$\frac{d^2\vec{X}}{dt^2} = \frac{d}{dt}(i\omega Ae^{i\omega t}) = -\omega^2 Ae^{i\omega t} = -\omega^2\vec{X} \tag{1.53}$$

Thus the displacement, velocity, and acceleration can be expressed as[4]

$$\text{displacement} = \text{Re}[Ae^{i\omega t}] = A\cos\omega t \tag{1.54}$$

$$\text{velocity} = \text{Re}[i\omega Ae^{i\omega t}] = -\omega A\sin\omega t$$
$$= \omega A\cos(\omega t + 90^\circ) \tag{1.55}$$

$$\text{acceleration} = \text{Re}[-\omega^2 Ae^{i\omega t}] = -\omega^2 A\cos\omega t$$
$$= \omega^2 A\cos(\omega t + 180^\circ) \tag{1.56}$$

where Re denotes the real part. These quantities are shown as rotating vectors in Fig. 1.49. It can be seen that the acceleration vector leads the velocity vector by 90°, and the latter leads the displacement vector by 90°.

Harmonic functions can be added vectorially, as shown in Fig. 1.50. If $\text{Re}(\vec{X}_1) = A_1\cos\omega t$ and $\text{Re}(\vec{X}_2) = A_2\cos(\omega t + \theta)$, then the magnitude of the resultant vector $\vec{X}$ is given by

$$A = \sqrt{(A_1 + A_2\cos\theta)^2 + (A_2\sin\theta)^2} \tag{1.57}$$

and the angle α by

$$\alpha = \tan^{-1}\left(\frac{A_2\sin\theta}{A_1 + A_2\cos\theta}\right) \tag{1.58}$$

[4]If the harmonic displacement is originally given as $x(t) = A\sin\omega t$, then we have

$$\begin{aligned}\text{displacement} &= \text{Im}[Ae^{i\omega t}] = A\sin\omega t\\ \text{velocity} &= \text{Im}[i\omega Ae^{i\omega t}] = \omega A\sin(\omega t + 90^\circ)\\ \text{acceleration} &= \text{Im}[-\omega^2 Ae^{i\omega t}] = \omega^2 A\sin(\omega t + 180^\circ)\end{aligned}$$

where Im denotes the imaginary part.

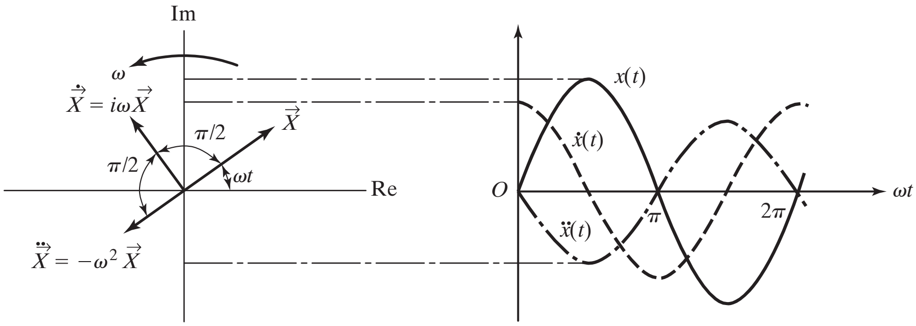


FIGURE 1.49 Displacement, velocity, and accelerations as rotating vectors.

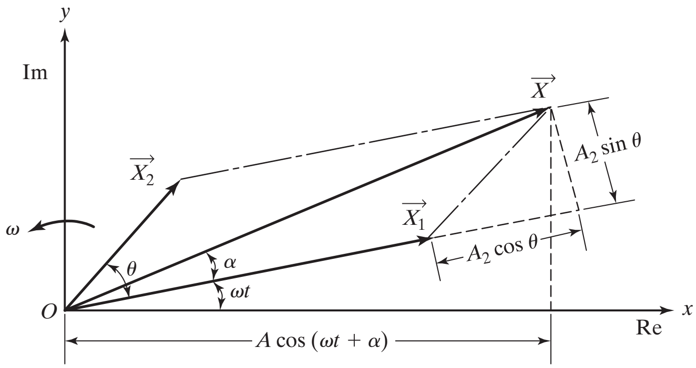


FIGURE 1.50 Vectorial addition of harmonic functions.

Since the original functions are given as real components, the sum $\vec{X}_1 + \vec{X}_2$ is given by $\mathrm{Re}(\vec{X}) = A\cos(\omega t + \alpha)$.

EXAMPLE 1.18 Addition of Harmonic Motions

Find the sum of the two harmonic motions $x_1(t) = 10\cos\omega t$ and $x_2(t) = 15\cos(\omega t + 2)$.

Solution: *Method 1*: *By using trigonometric relations*: Since the circular frequency is the same for both $x_1(t)$ and $x_2(t)$, we express the sum as

$$x(t) = A\cos(\omega t + \alpha) = x_1(t) + x_2(t) \tag{E.1}$$

That is,

$$\begin{aligned} A(\cos\omega t\cos\alpha - \sin\omega t\sin\alpha) &= 10\cos\omega t + 15\cos(\omega t + 2) \\ &= 10\cos\omega t + 15(\cos\omega t\cos 2 - \sin\omega t\sin 2) \end{aligned} \tag{E.2}$$

That is,

$$\cos \omega t(A \cos \alpha) - \sin \omega t(A \sin \alpha) = \cos \omega t(10 + 15 \cos 2) - \sin \omega t(15 \sin 2) \quad \text{(E.3)}$$

By equating the corresponding coefficients of $\cos \omega t$ and $\sin \omega t$ on both sides, we obtain

$$\begin{aligned} A \cos \alpha &= 10 + 15 \cos 2 \\ A \sin \alpha &= 15 \sin 2 \\ A &= \sqrt{(10 + 15 \cos 2)^2 + (15 \sin 2)^2} \\ &= 14.1477 \end{aligned} \quad \text{(E.4)}$$

and

$$\alpha = \tan^{-1}\left(\frac{15 \sin 2}{10 + 15 \cos 2}\right) = 74.5963° \quad \text{(E.5)}$$

Method 2: By using vectors: For an arbitrary value of ωt, the harmonic motions $x_1(t)$ and $x_2(t)$ can be denoted graphically as shown in Fig. 1.51. By adding them vectorially, the resultant vector $x(t)$ can be found to be

$$x(t) = 14.1477 \cos(\omega t + 74.5963°) \quad \text{(E.6)}$$

Method 3: By using complex-number representation: The two harmonic motions can be denoted in terms of complex numbers:

$$\begin{aligned} x_1(t) &= \text{Re}[A_1 e^{i\omega t}] \equiv \text{Re}[10 e^{i\omega t}] \\ x_2(t) &= \text{Re}[A_2 e^{i(\omega t + 2)}] \equiv \text{Re}[15 e^{i(\omega t + 2)}] \end{aligned} \quad \text{(E.7)}$$

The sum of $x_1(t)$ and $x_2(t)$ can be expressed as

$$x(t) = \text{Re}[A e^{i(\omega t + \alpha)}] \quad \text{(E.8)}$$

where A and α can be determined using Eqs. (1.47) and (1.48) as $A = 14.1477$ and $\alpha = 74.5963°$.

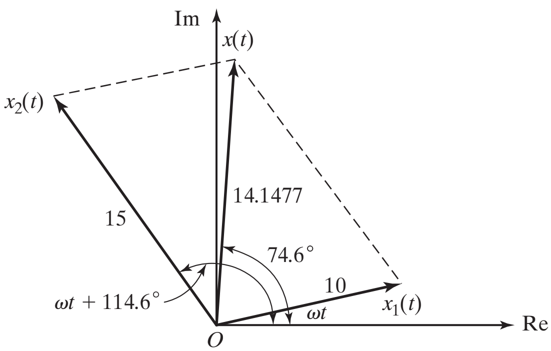


FIGURE 1.51 Addition of harmonic motions.

■

1.10.5 Definitions and Terminology

The following definitions and terminology are useful in dealing with harmonic motion and other periodic functions.

Cycle. The movement of a vibrating body from its undisturbed or equilibrium position to its extreme position in one direction, then to the equilibrium position, then to its extreme position in the other direction, and back to equilibrium position is called a *cycle* of vibration. One revolution (i.e., angular displacement of 2π radians) of the pin P in Fig. 1.46 or one revolution of the vector $\overrightarrow{OP}$ in Fig. 1.47 constitutes a cycle.

Amplitude. The maximum displacement of a vibrating body from its equilibrium position is called the *amplitude* of vibration. In Figs. 1.46 and 1.47 the amplitude of vibration is equal to A.

Period of oscillation. The time taken to complete one cycle of motion is known as the *period of oscillation* or *time period* and is denoted by τ. It is equal to the time required for the vector $\overrightarrow{OP}$ in Fig. 1.47 to rotate through an angle of 2π and hence

$$\tau = \frac{2\pi}{\omega} \tag{1.59}$$

where ω is called the circular frequency.

Frequency of oscillation. The number of cycles per unit time is called the *frequency of oscillation* or simply the *frequency* and is denoted by f. Thus

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi} \tag{1.60}$$

Here ω is called the circular frequency to distinguish it from the linear frequency $f = \omega/2\pi$. The variable ω denotes the angular velocity of the cyclic motion; f is measured in cycles per second (hertz) while ω is measured in radians per second.

Phase angle. Consider two vibratory motions denoted by

$$x_1 = A_1 \sin \omega t \tag{1.61}$$

$$x_2 = A_2 \sin(\omega t + \phi) \tag{1.62}$$

The two harmonic motions given by Eqs. (1.61) and (1.62) are called *synchronous* because they have the same frequency or angular velocity, ω. Two synchronous oscillations need not have the same amplitude, and they need not attain their maximum values at the same time. The motions given by Eqs. (1.61) and (1.62) can be represented graphically as shown in Fig. 1.52. In this figure, the second vector $\overrightarrow{OP_2}$ leads the first one $\overrightarrow{OP_1}$ by an angle ϕ, known as the *phase angle*. This means that the maximum of the second vector would occur ϕ radians earlier than that of the first vector. Note that instead of maxima, any other corresponding points can be taken for finding the phase angle. In Eqs. (1.61) and (1.62) or in Fig. 1.52 the two vectors are said to have a *phase difference* of ϕ.

Natural frequency. If a system, after an initial disturbance, is left to vibrate on its own, the frequency with which it oscillates without external forces is known as its *natural frequency*. As will be seen later, a vibratory system having n degrees of freedom will have, in general, n distinct natural frequencies of vibration.

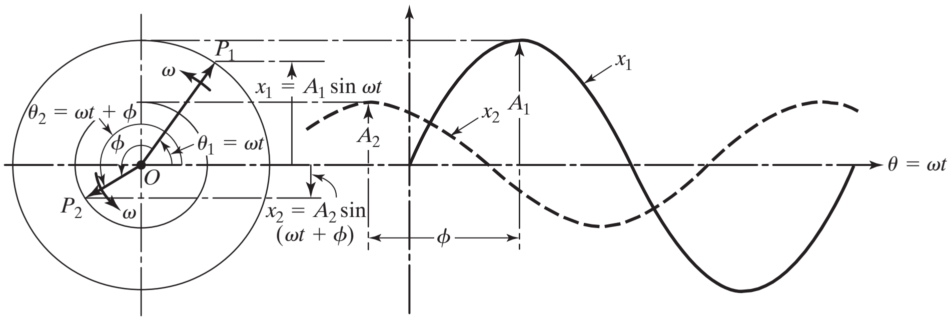


FIGURE 1.52 Phase difference between two vectors.

Beats. When two harmonic motions, with frequencies close to one another, are added, the resulting motion exhibits a phenomenon known as beats. For example, if

$$x_1(t) = X \cos \omega t \tag{1.63}$$

$$x_2(t) = X \cos(\omega + \delta)t \tag{1.64}$$

where δ is a small quantity, the addition of these motions yields

$$x(t) = x_1(t) + x_2(t) = X[\cos \omega t + \cos(\omega + \delta)t] \tag{1.65}$$

Using the relation

$$\cos A + \cos B = 2 \cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) \tag{1.66}$$

Eq. (1.65) can be rewritten as

$$x(t) = 2X \cos \frac{\delta t}{2} \cos\left(\omega + \frac{\delta}{2}\right)t \tag{1.67}$$

This equation is shown graphically in Fig. 1.53. It can be seen that the resulting motion, $x(t)$, represents a cosine wave with frequency $\omega + \delta/2$, which is approximately equal to ω, and with a varying amplitude of $2X \cos \delta t/2$. Whenever the amplitude reaches a maximum, it is called a beat. The frequency (δ) at which the amplitude builds up and dies down between 0 and $2X$ is known as beat frequency. The phenomenon of beats is often observed in machines, structures, and electric power houses. For example, in machines and structures, the beating phenomenon occurs when the forcing frequency is close to the natural frequency of the system (see Section 3.3.2).

Octave. When the maximum value of a range of frequency is twice its minimum value, it is known as an octave band. For example, each of the ranges 75–150 Hz, 150–300 Hz, and 300–600 Hz can be called an octave band. In each case, the maximum and minimum values of frequency, which have a ratio of 2:1, are said to differ by an *octave*.

Decibel. The various quantities encountered in the field of vibration and sound (such as displacement, velocity, acceleration, pressure, and power) are often represented

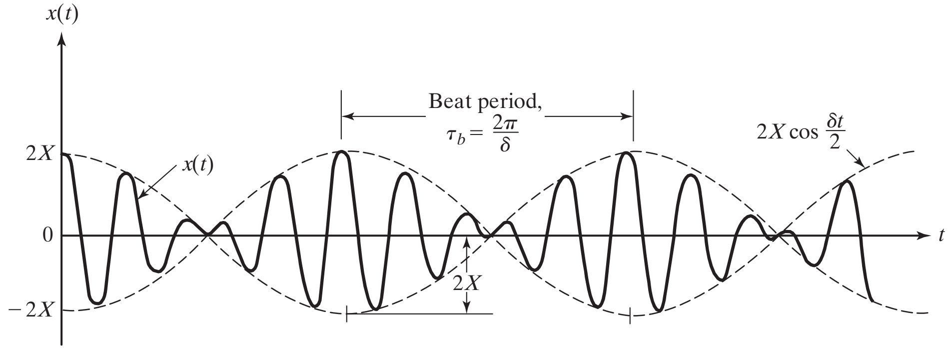


FIGURE 1.53 Phenomenon of beats.

using the notation of *decibel.* A decibel (dB) is originally defined as a ratio of electric powers:

$$\text{dB} = 10 \log\left(\frac{P}{P_0}\right) \tag{1.68}$$

where P_0 is some reference value of power. Since electric power is proportional to the square of the voltage (X), the decibel can also be expressed as

$$\text{dB} = 10 \log\left(\frac{X}{X_0}\right)^2 = 20 \log\left(\frac{X}{X_0}\right) \tag{1.69}$$

where X_0 is a specified reference voltage. In practice, Eq. (1.69) is also used for expressing the ratios of other quantities such as displacements, velocities, accelerations, and pressures. The reference values of X_0 in Eq. (1.69) are usually taken as 2×10^{-5} N/m^2 for pressure and $1\ \mu g = 9.81 \times 10^{-6}$ m/s^2 for acceleration.

1.11 Harmonic Analysis[5]

Although harmonic motion is simplest to handle, the motion of many vibratory systems is not harmonic. However, in many cases the vibrations are periodic—for example, the type shown in Fig. 1.54(a). Fortunately, any periodic function of time can be represented by Fourier series as an infinite sum of sine and cosine terms [1.36].

1.11.1 Fourier Series Expansion

If $x(t)$ is a periodic function with period τ, its Fourier series representation is given by

$$\begin{aligned} x(t) &= \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\,\omega t + \cdots \\ &\quad + b_1 \sin \omega t + b_2 \sin 2\,\omega t + \cdots \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \end{aligned} \tag{1.70}$$

[5]The harmonic analysis forms a basis for Section 4.2.

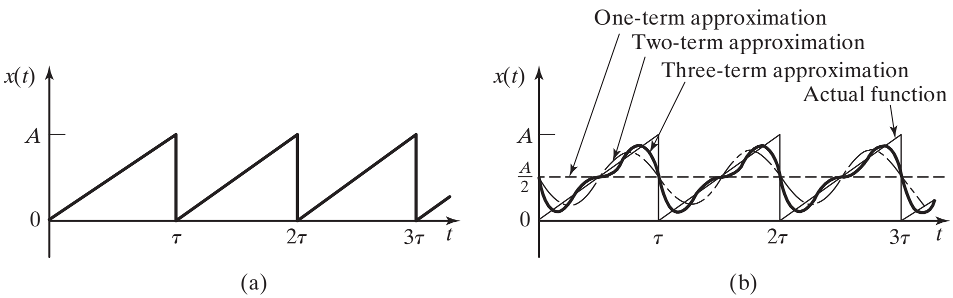


FIGURE 1.54 A periodic function.

where $\omega = 2\pi/\tau$ is the fundamental frequency and $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are constant coefficients. To determine the coefficients a_n and b_n, we multiply Eq. (1.70) by $\cos n\omega t$ and $\sin n\omega t$, respectively, and integrate over one period $\tau = 2\pi/\omega$—for example, from 0 to $2\pi/\omega$. Then we notice that all terms except one on the right-hand side of the equation will be zero, and we obtain

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\,dt = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt \tag{1.71}$$

$$a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\cos n\omega t\,dt = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt \tag{1.72}$$

$$b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt \tag{1.73}$$

The physical interpretation of Eq. (1.70) is that any periodic function can be represented as a sum of harmonic functions. Although the series in Eq. (1.70) is an infinite sum, we can approximate most periodic functions with the help of only a few harmonic functions. For example, the triangular wave of Fig. 1.54(a) can be represented closely by adding only three harmonic functions, as shown in Fig. 1.54(b).

Fourier series can also be represented by the sum of sine terms only or cosine terms only. For example, the series using cosine terms only can be expressed as

$$x(t) = d_0 + d_1\cos(\omega t - \phi_1) + d_2\cos(2\omega t - \phi_2) + \cdots \tag{1.74}$$

where

$$d_0 = a_0/2 \tag{1.75}$$

$$d_n = (a_n^2 + b_n^2)^{1/2} \tag{1.76}$$

and

$$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \tag{1.77}$$

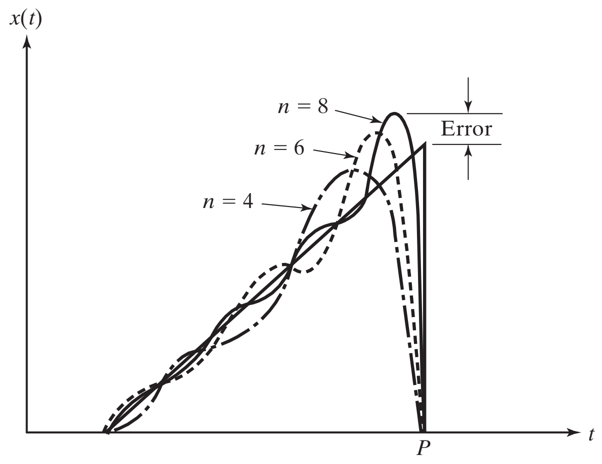


FIGURE 1.55 Gibbs' phenomenon.

Gibbs' Phenomenon. When a periodic function is represented by a Fourier series, an anomalous behavior can be observed. For example, Fig. 1.55 shows a triangular wave and its Fourier series representation using a different number of terms. As the number of terms (n) increases, the approximation can be seen to improve everywhere except in the vicinity of the discontinuity (point P in Fig. 1.55). Here the deviation from the true waveform becomes narrower but not any smaller in amplitude. It has been observed that the error in amplitude remains at approximately 9 percent, even when $k \to \infty$. This behavior is known as Gibbs' phenomenon, after its discoverer.

1.11.2 Complex Fourier Series

The Fourier series can also be represented in terms of complex numbers. By noting, from Eqs. (1.41) and (1.42), that

$$e^{i\omega t} = \cos \omega t + i \sin \omega t \tag{1.78}$$

and

$$e^{-i\omega t} = \cos \omega t - i \sin \omega t \tag{1.79}$$

cos ωt and sin ωt can be expressed as

$$\cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2} \tag{1.80}$$

and

$$\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \tag{1.81}$$

Thus Eq. (1.70) can be written as

$$
\begin{aligned}
x(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \left(\frac{e^{in\omega t} + e^{-in\omega t}}{2} \right) + b_n \left(\frac{e^{in\omega t} - e^{-in\omega t}}{2i} \right) \right\} \\
&= e^{i(0)\omega t} \left(\frac{a_0}{2} - \frac{ib_0}{2} \right) \\
&\quad + \sum_{n=1}^{\infty} \left\{ e^{in\omega t} \left(\frac{a_n}{2} - \frac{ib_n}{2} \right) + e^{-in\omega t} \left(\frac{a_n}{2} + \frac{ib_n}{2} \right) \right\}
\end{aligned} \tag{1.82}
$$

where $b_0 = 0$. By defining the complex Fourier coefficients c_n and c_{-n} as

$$c_n = \frac{a_n - ib_n}{2} \tag{1.83}$$

and

$$c_{-n} = \frac{a_n + ib_n}{2} \tag{1.84}$$

Eq. (1.82) can be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \tag{1.85}$$

The Fourier coefficients c_n can be determined, using Eqs. (1.71) to (1.73), as

$$
\begin{aligned}
c_n = \frac{a_n - ib_n}{2} &= \frac{1}{\tau} \int_0^{\tau} x(t)[\cos n\omega t - i \sin n\omega t]\,dt \\
&= \frac{1}{\tau} \int_0^{\tau} x(t) e^{-in\omega t} dt
\end{aligned} \tag{1.86}
$$

1.11.3 Frequency Spectrum

The harmonic functions $a_n \cos n\omega t$ or $b_n \sin n\omega t$ in Eq. (1.70) are called the *harmonics* of order n of the periodic function $x(t)$. The harmonic of order n has a period τ/n. These harmonics can be plotted as vertical lines on a diagram of amplitude (a_n and b_n or d_n and ϕ_n) versus frequency ($n\omega$), called the *frequency spectrum* or *spectral diagram*. Figure 1.56 shows a typical frequency spectrum.

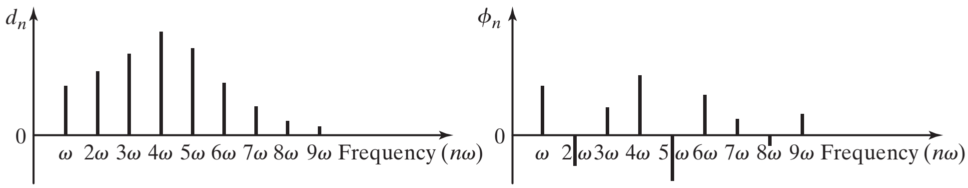


FIGURE 1.56 Frequency spectrum of a typical periodic function of time.

1.11.4 Time- and Frequency-Domain Representations

The Fourier series expansion permits the description of any periodic function using either a time-domain or a frequency-domain representation. For example, a harmonic function given by $x(t) = A \sin \omega t$ in time domain (see Fig. 1.57(a)) can be represented by the amplitude and the frequency ω in the frequency domain (see Fig. 1.57(b)). Similarly, a periodic function, such as a triangular wave, can be represented in time domain, as shown in Fig. 1.57(c), or in frequency domain, as indicated in Fig. 1.57(d). Note that the amplitudes d_n and the phase angles ϕ_n corresponding to the frequencies ω_n can be used in place of the amplitudes a_n and b_n for representation in the frequency domain. Using a Fourier integral (considered in Section 14.9) permits the representation of even nonperiodic functions in

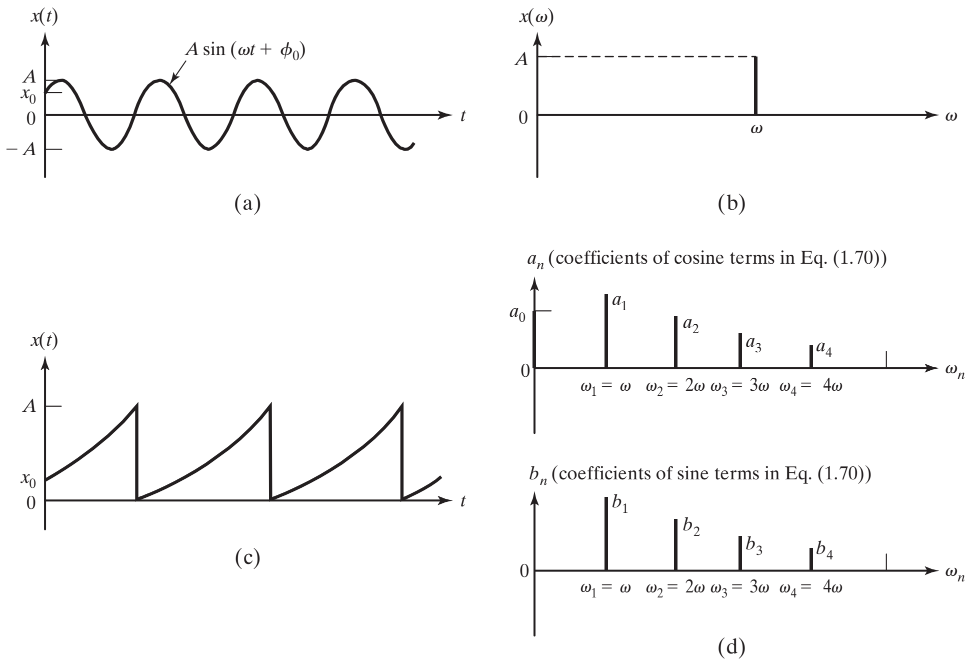


FIGURE 1.57 Representation of a function in time and frequency domains.

either a time domain or a frequency domain. Figure 1.57 shows that the frequency-domain representation does not provide the initial conditions. However, in many practical applications the initial conditions are often considered unnecessary and only the steady-state conditions are of main interest.

1.11.5 Even and Odd Functions

An even function satisfies the relation

$$x(-t) = x(t) \tag{1.87}$$

In this case, the Fourier series expansion of $x(t)$ contains only cosine terms:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t \tag{1.88}$$

where a_0 and a_n are given by Eqs. (1.71) and (1.72), respectively. An odd function satisfies the relation

$$x(-t) = -x(t) \tag{1.89}$$

In this case, the Fourier series expansion of $x(t)$ contains only sine terms:

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t \tag{1.90}$$

where b_n is given by Eq. (1.73). In some cases, a given function may be considered as even or odd depending on the location of the coordinate axes. For example, the shifting of the vertical axis from (a) to (b) or (c) in Fig. 1.58(i) will make it an odd or even function. This means that we need to compute only the coefficients b_n or a_n. Similarly, a shift in the time axis from (d) to (e) amounts to adding a constant equal to the amount of shift. In the case of Fig. 1.58(ii), when the function is considered as an odd function, the Fourier series expansion becomes (see Problem 1.107):

$$x_1(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin \frac{2\pi(2n-1)t}{\tau} \tag{1.91}$$

On the other hand, if the function is considered an even function, as shown in Fig. 1.50(iii), its Fourier series expansion becomes (see Problem 1.107):

$$x_2(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos \frac{2\pi(2n-1)t}{\tau} \tag{1.92}$$

Since the functions $x_1(t)$ and $x_2(t)$ represent the same wave, except for the location of the origin, there exists a relationship between their Fourier series expansions also. Noting that

$$x_1\left(t + \frac{\tau}{4}\right) = x_2(t) \tag{1.93}$$

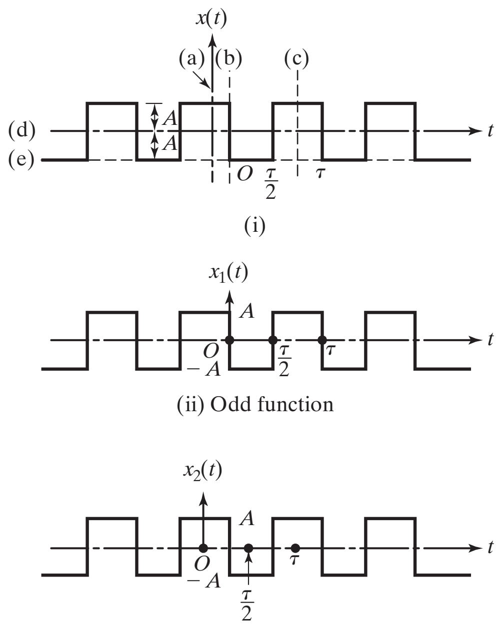


FIGURE 1.58 Even and odd functions.

we find from Eq. (1.91),

$$x_1\left(t+\frac{\tau}{4}\right)=\frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{2\pi(2n-1)}{\tau}\left(t+\frac{\tau}{4}\right)$$

$$=\frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left\{\frac{2\pi(2n-1)t}{\tau}+\frac{2\pi(2n-1)}{4}\right\}\tag{1.94}$$

Using the relation $\sin(A+B)=\sin A\cos B+\cos A\sin B$, Eq. (1.94) can be expressed as

$$x_1\left(t+\frac{\tau}{4}\right)=\frac{4A}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1)}\sin\frac{2\pi(2n-1)t}{\tau}\cos\frac{2\pi(2n-1)}{4}\right.$$

$$\left.+\cos\frac{2\pi(2n-1)t}{\tau}\sin\frac{2\pi(2n-1)}{4}\right\}\tag{1.95}$$

Since $\cos[2\pi(2n-1)/4] = 0$ for $n = 1, 2, 3, \ldots$, and $\sin[2\pi(2n-1)/4] = (-1)^{n+1}$ for $n = 1, 2, 3, \ldots$, Eq. (1.95) reduces to

$$x_1\left(t + \frac{\tau}{4}\right) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos \frac{2\pi(2n-1)t}{\tau} \tag{1.96}$$

which can be identified to be the same as Eq. (1.92).

1.11.6 Half-Range Expansions

In some practical applications, the function $x(t)$ is defined only in the interval 0 to τ as shown in Fig. 1.59(a). In such a case, there is no condition of periodicity of the function, since the function itself is not defined outside the interval 0 to τ. However, we can extend the function arbitrarily to include the interval $-\tau$ to 0 as shown in either Fig. 1.59(b) or Fig. 1.59(c). The extension of the function indicated in Fig. 1.59(b) results in an odd function, $x_1(t)$, while the extension of the function shown in Fig. 1.59(c) results in an even function, $x_2(t)$. Thus the Fourier series expansion of $x_1(t)$ yields only sine terms and that of $x_2(t)$ involves only cosine terms. These Fourier series expansions of $x_1(t)$ and $x_2(t)$ are

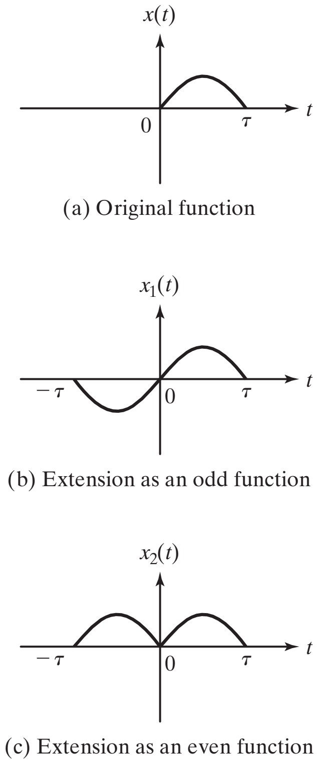


FIGURE 1.59 Extension of a function for half-range expansions.

known as half-range expansions [1.37]. Any of these half-range expansions can be used to find $x(t)$ in the interval 0 to τ.

1.11.7 Numerical Computation of Coefficients

For very simple forms of the function $x(t)$, the integrals of Eqs. (1.71) to (1.73) can be evaluated easily. However, the integration becomes involved if $x(t)$ does not have a simple form. In some practical applications, as in the case of experimental determination of the amplitude of vibration using a vibration transducer, the function $x(t)$ is not available in the form of a mathematical expression; only the values of $x(t)$ at a number of points $t_1, t_2, \ldots, t_N$ are available, as shown in Fig. 1.60. In these cases, the coefficients a_n and b_n of Eqs. (1.71) to (1.73) can be evaluated by using a numerical integration procedure like the trapezoidal or Simpson's rule [1.38].

Let's assume that $t_1, t_2, \ldots, t_N$ are an even number of equidistant points over the period $\tau (N = \text{even})$ with the corresponding values of $x(t)$ given by $x_1 = x(t_1)$, $x_2 = x(t_2)$, $\ldots$, $x_N = x(t_N)$, respectively; then the application of the trapezoidal rule gives the coefficients a_n and b_n (by setting $\tau = N\Delta t$) as:[6]

$$a_0 = \frac{2}{N}\sum_{i=1}^{N} x_i \tag{1.97}$$

$$a_n = \frac{2}{N}\sum_{i=1}^{N} x_i \cos\frac{2n\pi t_i}{\tau} \tag{1.98}$$

$$b_n = \frac{2}{N}\sum_{i=1}^{N} x_i \sin\frac{2n\pi t_i}{\tau} \tag{1.99}$$

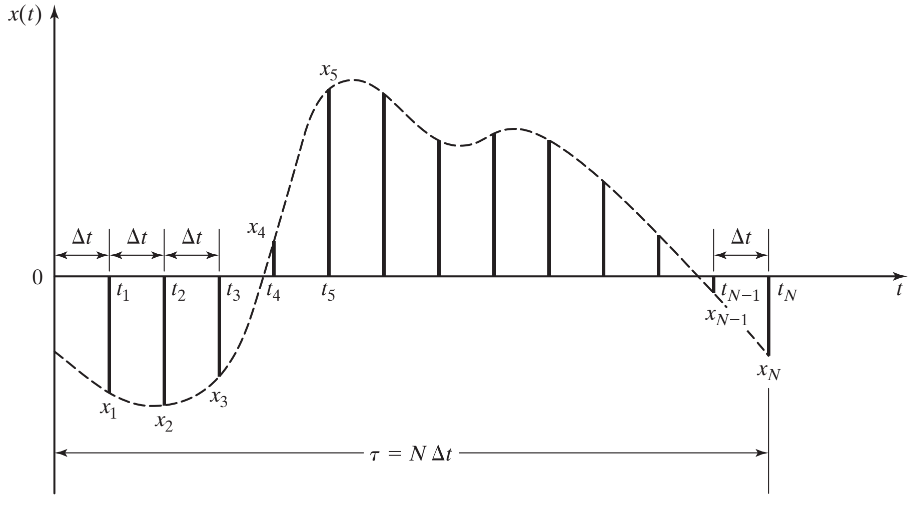


FIGURE 1.60 Values of the periodic function $x(t)$ at discrete points $t_1, t_2, \ldots, t_N$.

[6] N Needs to be an even number for Simpson's rule but not for the trapezoidal rule. Equations (1.97) to (1.99) assume that the periodicity condition, $x_0 = x_N$, holds true.

EXAMPLE 1.19 Fourier Series Expansion

Determine the Fourier series expansion of the motion of the valve in the cam-follower system shown in Fig. 1.61.

Solution: If $y(t)$ denotes the vertical motion of the pushrod, the motion of the valve, $x(t)$, can be determined from the relation:

$$\tan\theta = \frac{y(t)}{l_1} = \frac{x(t)}{l_2}$$

or

$$x(t) = \frac{l_2}{l_1} y(t) \tag{E.1}$$

where

$$y(t) = Y\frac{t}{\tau}; \qquad 0 \le t \le \tau \tag{E.2}$$

and the period is given by $\tau = \frac{2\pi}{\omega}$. By defining

$$A = \frac{Yl_2}{l_1}$$

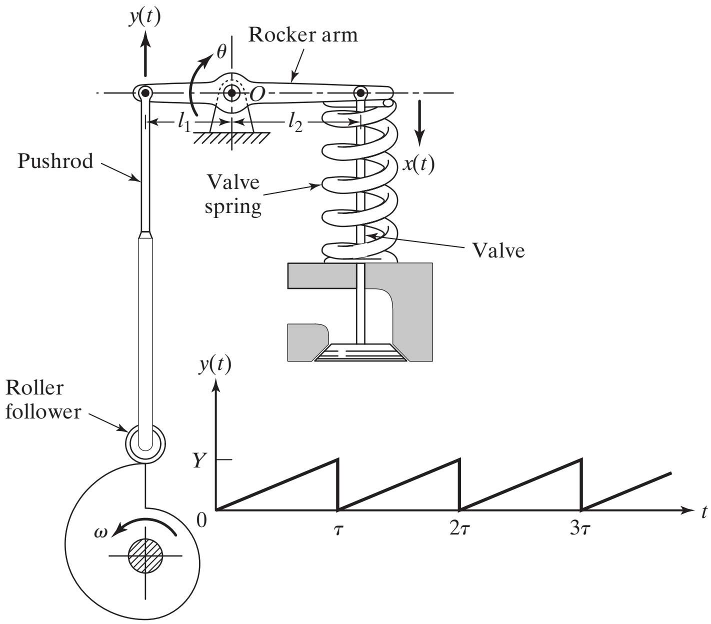


FIGURE 1.61 Cam-follower system.

$x(t)$ can be expressed as

$$x(t) = A\frac{t}{\tau}; \qquad 0 \le t \le \tau \tag{E.3}$$

Equation (E.3) is shown in Fig. 1.54(a). To compute the Fourier coefficients a_n and b_n, we use Eqs. (1.71) to (1.73):

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} A\frac{t}{\tau}\,dt = \frac{\omega}{\pi}\frac{A}{\tau}\left(\frac{t^2}{2}\right)_0^{2\pi/\omega} = A \tag{E.4}$$

$$\begin{aligned} a_n &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\cos n\omega t\cdot dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} A\frac{t}{\tau}\cos n\omega t\cdot dt \\ &= \frac{A\omega}{\pi\tau}\int_0^{2\pi/\omega} t\cos n\omega t\cdot dt = \frac{A}{2\pi^2}\left[\frac{\cos n\omega t}{n^2} + \frac{\omega t\sin n\omega t}{n}\right]_0^{2\pi/\omega} \\ &= 0, \qquad n = 1, 2, \ldots \end{aligned} \tag{E.5}$$

$$\begin{aligned} b_n &= \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\sin n\omega t\cdot dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} A\frac{t}{\tau}\sin n\omega t\cdot dt \\ &= \frac{A\omega}{\pi\tau}\int_0^{2\pi/\omega} t\sin n\omega t\cdot dt = \frac{A}{2\pi^2}\left[\frac{\sin n\omega t}{n^2} - \frac{\omega t\cos n\omega t}{n}\right]_0^{2\pi/\omega} \\ &= -\frac{A}{n\pi}, \qquad n = 1, 2, \ldots \end{aligned} \tag{E.6}$$

Therefore the Fourier series expansion of $x(t)$ is

$$\begin{aligned} x(t) &= \frac{A}{2} - \frac{A}{\pi}\sin\omega t - \frac{A}{2\pi}\sin 2\,\omega t - \ldots \\ &= \frac{A}{\pi}\left[\frac{\pi}{2} - \left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \ldots\right\}\right] \end{aligned} \tag{E.7}$$

The first three terms of the series are shown plotted in Fig. 1.54(b). It can be seen that the approximation reaches the sawtooth shape even with a small number of terms.

■

EXAMPLE 1.20 Numerical Fourier Analysis

The pressure fluctuations of water in a pipe, measured at 0.01-second intervals, are given in Table 1.1. These fluctuations are repetitive in nature. Make a harmonic analysis of the pressure fluctuations and determine the first three harmonics of the Fourier series expansion.

TABLE 1.1

Time Station, i	Time (sec), t_i	Pressure (kN/m^2), p_i
0	0	0
1	0.01	20
2	0.02	34
3	0.03	42
4	0.04	49
5	0.05	53
6	0.06	70
7	0.07	60
8	0.08	36
9	0.09	22
10	0.10	16
11	0.11	7
12	0.12	0

Solution: Since the given pressure fluctuations repeat every 0.12 sec, the period is $\tau = 0.12$ sec and the circular frequency of the first harmonic is 2π radians per 0.12 sec or $\omega = 2\pi/0.12 = 52.36$ rad/sec. As the number of observed values in each wave (N) is 12, we obtain from Eq. (1.97)

$$a_0 = \frac{2}{N}\sum_{i=1}^{N} p_i = \frac{1}{6}\sum_{i=1}^{12} p_i = 68166.7 \tag{E.1}$$

The coefficients a_n and b_n can be determined from Eqs. (1.98) and (1.99):

$$a_n = \frac{2}{N}\sum_{i=1}^{N} p_i \cos\frac{2n\pi t_i}{\tau} = \frac{1}{6}\sum_{i=1}^{12} p_i \cos\frac{2n\pi t_i}{0.12} \tag{E.2}$$

$$b_n = \frac{2}{N}\sum_{i=1}^{N} p_i \sin\frac{2n\pi t_i}{\tau} = \frac{1}{6}\sum_{i=1}^{12} p_i \sin\frac{2n\pi t_i}{0.12} \tag{E.3}$$

The computations involved in Eqs. (E.2) and (E.3) are shown in Table 1.2. From these calculations, the Fourier series expansion of the pressure fluctuations $p(t)$ can be obtained (see Eq. 1.70):

$$\begin{aligned} p(t) &= 34083.3 - 26996.0 \cos 52.36t + 8307.7 \sin 52.36t \\ &\quad + 1416.7 \cos 104.72t + 3608.3 \sin 104.72t - 5833.3 \cos 157.08t \\ &\quad - 2333.3 \sin 157.08t + \cdots \quad \text{N/m}^2 \end{aligned} \tag{E.4}$$

TABLE 1.2

			$n = 1$		$n = 2$		$n = 3$	
i	t_i	p_i	$p_i \cos \frac{2\pi t_i}{0.12}$	$p_i \sin \frac{2\pi t_i}{0.12}$	$p_i \cos \frac{4\pi t_i}{0.12}$	$p_i \sin \frac{4\pi t_i}{0.12}$	$p_i \cos \frac{6\pi t_i}{0.12}$	$p_i \sin \frac{6\pi t_i}{0.12}$
1	0.01	20000	17320	10000	10000	17320	0	20000
2	0.02	34000	17000	29444	−17000	29444	−34000	0
3	0.03	42000	0	42000	−42000	0	0	−42000
4	0.04	49000	−24500	42434	−24500	−42434	49000	0
5	0.05	53000	−45898	26500	26500	−45898	0	53000
6	0.06	70000	−70000	0	70000	0	−70000	0
7	0.07	60000	−51960	−30000	30000	51960	0	−60000
8	0.08	36000	−18000	−31176	−18000	31176	36000	0
9	0.09	22000	0	−22000	−22000	0	0	22000
10	0.10	16000	8000	−13856	−8000	−13856	−16000	0
11	0.11	7000	6062	−3500	3500	−6062	0	−7000
12	0.12	0	0	0	0	0	0	0
$\sum_{i=1}^{12}(\ )$		409000	−161976	49846	8500	21650	−35000	−14000
$\frac{1}{6}\sum_{i=1}^{12}(\ )$		68166.7	−26996.0	8307.7	1416.7	3608.3	−5833.3	−2333.3

■

1.12 Examples Using MATLAB[7]

EXAMPLE 1.21 Graphical Representation of Fourier Series Using MATLAB

Plot the periodic function

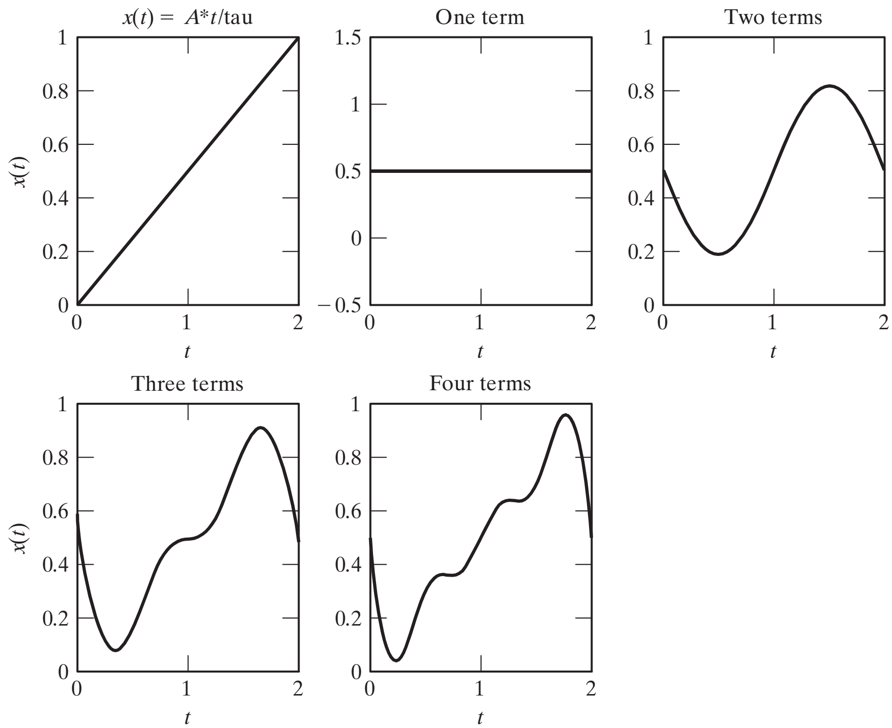
$$x(t) = A\frac{t}{\tau}, \qquad 0 \le t \le \tau \tag{E.1}$$

and its Fourier series representation with four terms

$$\bar{x}(t) = \frac{A}{\pi}\left\{\frac{\pi}{2} - \left(\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t\right)\right\} \tag{E.2}$$

for $0 \le t \le \tau$ with $A = 1$, $\omega = \pi$, and $\tau = \frac{2\pi}{\omega} = 2$.

[7]The source codes of all MATLAB programs are given at the Companion Website.



Equations (E.1) and (E.2) with different numbers of terms.

Solution: A MATLAB program is written to plot Eqs. (E.1) and (E.2) with different numbers of terms as shown below.

```
%ex1_21.m
%plot the function x(t) = A * t / tau
A = 1;
w = pi;
tau = 2;
for i = 1: 101
     t(i) = tau * (i-1)/100;
     x(i) = A * t(i) / tau;
end
subplot(231);
plot(t,x);
ylabel('x(t)');
xlabel('t');
title('x(t) = A*t/tau');
for i = 1: 101
     x1(i) = A / 2;
end
subplot(232);
plot(t,x1);
xlabel('t');
```

```
title('One term');
for i = 1: 101
     x2(i) = A/2 - A * sin(w*t(i)) / pi;
end
subplot(233);
plot(t,x2);
xlabel('t');
title('Two terms');
for i = 1: 101
     x3(i) = A/2 - A * sin(w*t(i)) / pi - A * sin(2*w*t(i)) / (2*pi);
end
subplot(234);
plot(t,x3);
ylabel('x(t)');
xlabel('t');
title('Three terms');
for i = 1: 101
     t(i) = tau * (i-1)/100;
     x4(i) = A/2 - A * sin(w*t(i)) / pi - A * sin(2*w*t(i)) / (2*pi)
     - A * sin(3*w*t(i)) / (3*pi);
end
subplot(235);
plot(t,x4);
xlabel('t');
title('Four terms');
```

■

EXAMPLE 1.22 Graphical Representation of Beats

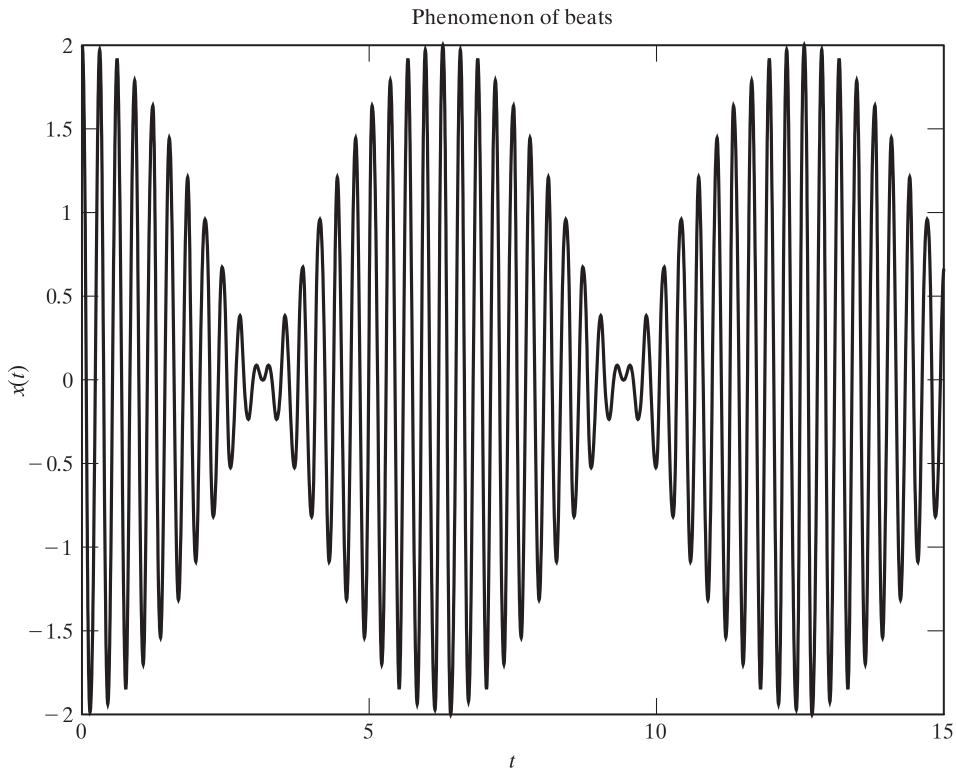
A mass is subjected to two harmonic motions given by $x_1(t) = X \cos \omega t$ and $x_2(t) = X \cos (\omega + \delta) t$ with $X = 1$ cm, $\omega = 20$ rad/sec, and $\delta = 1$ rad/sec. Plot the resulting motion of the mass using MATLAB and identify the beat frequency.

Solution: The resultant motion of the mass, $x(t)$, is given by

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) \\ &= X \cos \omega t + X \cos(\omega + \delta) t \\ &= 2X \cos \frac{\delta t}{2} \cos\left(\omega + \frac{\delta}{2}\right) t \end{aligned} \tag{E.1}$$

The motion can be seen to exhibit the phenomenon of beats with a beat frequency $\omega_b = (\omega + \delta) - (\omega) = \delta = 1$ rad/sec. Equation (E.1) is plotted using MATLAB as shown below.

```
% ex1_22.m
% Plot the Phenomenon of beats
A = 1;
w = 20;
delta = 1;
for i = 1: 1001
     t(i) = 15 * (i-1)/1000;
     x(i) = 2 * A * cos (delta*t(i)/2) * cos ((w + delta/2) *t(i));
end
plot (t,x);
xlabel ('t');
ylabel ('x(t)');
title ('Phenomenon of beats');
```



■

EXAMPLE 1.23 Numerical Fourier Analysis Using MATLAB

Conduct a harmonic analysis of the pressure fluctuations given in Table 1.1 on page 75 and determine the first five harmonics of the Fourier series expansion.

Solution: To find the first five harmonics of the pressure fluctuations (i.e., $a_0, a_1, \ldots, a_5, b_1, \ldots, b_5$), a general-purpose MATLAB program is developed for the harmonic analysis of a function $x(t)$ using Eqs. (1.97) to (1.99). The program, named Program1.m, requires the following input data:

n = number of equidistant points at which the values of $x(t)$ are known

m = number of Fourier coefficients to be computed

time = time period of the function $x(t)$

x = array of dimension n, containing the known values of $x(t)$; $x(i) = x(t_i)$

t = array of dimension n, containing the known values of t; $t(i) = t_i$

The following output is generated by the program:

azero = a_0 of Eq. (1.97)

$i, a(i), b(i); i = 1, 2, \ldots, m$

where $a(i)$ and $b(i)$ denote the computed values of a_i and b_i given by Eqs. (1.98) and (1.99), respectively.

```
>> program1
Fourier series expansion of the function x(t)

Data:

Number of data points in one cycle = 12

Number of Fourier Coefficients required = 5

Time period = 1.200000e-001

Station i      Time at station i: t(i)    x(i) at t(i)
        1            1.000000e-002          2.000000e+004
        2            2.000000e-002          3.400000e+004
        3            3.000000e-002          4.200000e+004
        4            4.000000e-002          4.900000e+004
        5            5.000000e-002          5.300000e+004
        6            6.000000e-002          7.000000e+004
        7            7.000000e-002          6.000000e+004
        8            8.000000e-002          3.600000e+004
        9            9.000000e-002          2.200000e+004
       10            1.000000e-001          1.600000e+004
       11            1.100000e-001          7.000000e+003
       12            1.200000e-001          0.000000e+000

Results of Fourier analysis:

azero=6.816667e+004

values of i             a(i)                   b(i)
          1        -2.699630e+004         8.307582e+003
          2         1.416632e+003         3.608493e+003
          3        -5.833248e+003        -2.333434e+003
          4        -5.834026e+002         2.165061e+003
          5        -2.170284e+003        -6.411708e+002
```

■

1.13 Vibration Literature

The literature on vibrations is large and diverse. Several textbooks are available [1.39], and dozens of technical periodicals regularly publish papers relating to vibrations. This is primarily because vibration affects so many disciplines, from science of materials to machinery analysis to spacecraft structures. Researchers in many fields must be attentive to vibration research.

The most widely circulated journals that publish papers relating to vibrations are *ASME Journal of Vibration and Acoustics; ASME Journal of Applied Mechanics; Journal of Sound and Vibration; AIAA Journal; ASCE Journal of Engineering Mechanics; Earthquake Engineering and Structural Dynamics; Bulletin of the Japan Society of Mechanical Engineers; International Journal of Solids and Structures; International Journal for Numerical Methods in Engineering; Journal of the Acoustical Society of America; Sound and Vibration; Vibrations, Mechanical Systems and Signal Processing; International Journal of Analytical and Experimental Modal Analysis; JSME International Journal Series III—Vibration Control Engineering*; and *Vehicle System Dynamics*. Many of these journals are cited in the chapter references.

In addition, *Shock and Vibration Digest, Applied Mechanics Reviews*, and *Noise and Vibration Worldwide* are monthly abstract journals containing brief discussions of nearly every published vibration paper. Formulas and solutions in vibration engineering can be readily found in references [1.40–1.42].

CHAPTER SUMMARY

In this chapter, we presented the fundamental concepts of vibration along with a brief outline of the history and the importance of the study of vibration. We introduced the concepts of degree of freedom, discrete and continuous systems, and the different classes of vibration. We outlined the basic steps involved in the vibration analysis of a system. We introduced the fundamental type of vibration, namely harmonic motion, along with the associated terminology. We presented harmonic analysis and Fourier series representation of periodic functions as well as numerical determination of Fourier coefficients with examples.

At this point, the reader should be able to achieve the objectives stated at the beginning of the chapter. To help the reader, review questions in the form of questions requiring brief answers, true or false statements, fill in the blanks, multiple choices, and matching of statements are given for self testing with answers available at the Companion Website. Several problems involving different levels of difficulty in applying the basic concepts presented in the chapter are also given. The answers to selected problems can be found at the end of the book.

REFERENCES

1.1 D. C. Miller, *Anecdotal History of the Science of Sound*, Macmillan, New York, 1935.

1.2 N. F. Rieger, "The quest for $\sqrt{k/m}$: Notes on the development of vibration analysis, Part I. genius awakening," *Vibrations*, Vol. 3, No. 3/4, December 1987, pp. 3–10.

1.3 Chinese Academy of Sciences (compiler), *Ancient China's Technology and Science*, Foreign Languages Press, Beijing, 1983.

1.4 R. Taton (ed.), *Ancient and Medieval Science: From the Beginnings to 1450*, A. J. Pomerans (trans.), Basic Books, New York, 1957.

1.5 S. P. Timoshenko, *History of Strength of Materials*, McGraw-Hill, New York, 1953.

1.6 R. B. Lindsay, "The story of acoustics," *Journal of the Acoustical Society of America*, Vol. 39, No. 4, 1966, pp. 629–644.

1.7 J. T. Cannon and S. Dostrovsky, *The Evolution of Dynamics: Vibration Theory from 1687 to 1742*, Springer-Verlag, New York, 1981.

1.8 L. L. Bucciarelli and N. Dworsky, *Sophie Germain: An Essay in the History of the Theory of Elasticity*, D. Reidel Publishing, Dordrecht, Holland, 1980.

1.9 J. W. Strutt (Baron Rayleigh), *The Theory of Sound*, Dover, New York, 1945.

1.10 R. Burton, *Vibration and Impact*, Addison-Wesley, Reading, MA, 1958.

1.11 A. H. Nayfeh, *Perturbation Methods*, Wiley, New York, 1973.

1.12 S. H. Crandall and W. D. Mark, *Random Vibration in Mechanical Systems*, Academic Press, New York, 1963.

1.13 J. D. Robson, *Random Vibration*, Edinburgh University Press, Edinburgh, 1964.

1.14 S. S. Rao, *The Finite Element Method in Engineering* (4th ed.), Elsevier Butterworth Heinemann, Burlington, MA, 2005.

1.15 M. J. Turner, R. W. Clough, H. C. Martin, and L. J. Topp, "Stiffness and deflection analysis of complex structures," *Journal of Aeronautical Sciences*, Vol. 23, 1956, pp. 805–824.

1.16 D. Radaj et al., "Finite element analysis, an automobile engineer's tool," *International Conference on Vehicle Structural Mechanics: Finite Element Application to Design*, Society of Automotive Engineers, Detroit, 1974.

1.17 R. E. D. Bishop, *Vibration* (2nd ed.), Cambridge University Press, Cambridge, 1979.

1.18 M. H. Richardson and K. A. Ramsey, "Integration of dynamic testing into the product design cycle," *Sound and Vibration*, Vol. 15, No. 11, November 1981, pp. 14–27.

1.19 M. J. Griffin and E. M. Whitham, "The discomfort produced by impulsive whole-body vibration," *Journal of the Acoustical Society of America*, Vol. 65, No. 5, 1980, pp. 1277–1284.

1.20 J. E. Ruzicka, "Fundamental concepts of vibration control," *Sound and Vibration*, Vol. 5, No. 7, July 1971, pp. 16–23.

1.21 T. W. Black, "Vibratory finishing goes automatic" (Part 1: Types of machines; Part 2: Steps to automation), *Tool and Manufacturing Engineer*, July 1964, pp. 53–56; and August 1964, pp. 72–76.

1.22 S. Prakash and V. K. Puri, *Foundations for Machines; Analysis and Design*, Wiley, New York, 1988.

1.23 L. Meirovitch, *Fundamentals of Vibrations*, McGraw-Hill, New York, 2001.

1.24 A. Dimarogonas, *Vibration for Engineers*, (2nd ed.), Prentice-Hall, Upper Saddle River, NJ, 1996.

1.25 E. O. Doebelin, *System Modeling and Response*, Wiley, New York, 1980.

1.26 R. W. Fitzgerald, *Mechanics of Materials* (2nd ed.), Addison-Wesley, Reading, Mass., 1982.

1.27 I. Cochin and W. Cadwallender, *Analysis and Design of Dynamic Systems*, (3rd ed.), Addison-Wesley, Reading, MA, 1997.

1.28 F. Y. Chen, *Mechanics and Design of Cam Mechanisms*, Pergamon Press, New York, 1982.

1.29 W. T. Thomson and M. D. Dahleh, *Theory of Vibration with Applications* (5th ed.), Prentice-Hall, Upper Saddle River, NJ, 1998.

1.30 N. O. Myklestad, *Fundamentals of Vibration Analysis*, McGraw-Hill, New York, 1956.

1.31 C. W. Bert, "Material damping: An introductory review of mathematical models, measures, and experimental techniques," *Journal of Sound and Vibration*, Vol. 29, No. 2, 1973, pp. 129–153.

1.32 J. M. Gasiorek and W. G. Carter, *Mechanics of Fluids for Mechanical Engineers*, Hart Publishing, New York, 1968.

1.33 A. Mironer, *Engineering Fluid Mechanics*, McGraw-Hill, New York, 1979.

1.34 F. P. Beer and E. R. Johnston, *Vector Mechanics for Engineers* (6th ed.), McGraw-Hill, New York, 1997.

1.35 A. Higdon and W. B. Stiles, *Engineering Mechanics* (2nd ed.), Prentice-Hall, New York, 1955.

1.36 E. Kreyszig, *Advanced Engineering Mathematics* (9th ed.), Wiley, New York, 2006.

1.37 M. C. Potter and J. Goldberg and E. F. Aboufadel, *Advanced Engineering Mathematics* (3rd ed.), Oxford University Press, New York, 2005.

1.38 S. S. Rao, *Applied Numerical Methods for Engineers and Scientists*, Prentice Hall, Upper Saddle River, NJ, 2002.

1.39 N. F. Rieger, "The literature of vibration engineering," *Shock and Vibration Digest*, Vol. 14, No. 1, January 1982, pp. 5–13.

1.40 R. D. Blevins, *Formulas for Natural Frequency and Mode Shape*, Van Nostrand Reinhold, New York, 1979.

1.41 W. D. Pilkey and P. Y. Chang, *Modern Formulas for Statics and Dynamics*, McGraw-Hill, New York, 1978.

1.42 C. M. Harris (ed.), *Shock and Vibration Handbook* (4th ed.), McGraw-Hill, New York, 1996.

1.43 R. G. Budynas and J. K. Nisbett, *Shigley's Mechanical Engineering Design* (8th ed.), McGraw-Hill, New York, 2008.

1.44 N. P. Chironis (ed.), *Machine Devices and Instrumentation*, McGraw-Hill, New York, 1966.

1.45 D. Morrey and J. E. Mottershead, "Vibratory bowl feeder design using numerical modelling techniques," in *Modern Practice in Stress and Vibration Analysis*, J. E. Mottershead (ed.), Pergamon Press, Oxford, 1989, pp. 211–217.

1.46 K. McNaughton (ed.), *Solids Handling*, McGraw-Hill, New York, 1981.

1.47 M. M. Kamal and J. A. Wolf, Jr. (eds.), *Modern Automotive Structural Analysis*, Van Nostrand Reinhold, New York, 1982.

1.48 D. J. Inman, *Engineering Vibration*, (3rd ed.), Pearson Prentice-Hall, Upper Saddle River, NJ, 2007.

1.49 J. H. Ginsberg, *Mechanical and Structural Vibrations: Theory and Applications*, John Wiley, New York, 2001.

1.50 S. S. Rao, *Vibration of Continuous Systems*, John Wiley, Hoboken, NJ, 2007.

1.51 S. Braun, D. J. Ewins and S. S. Rao (eds.), *Encyclopedia of Vibration*, Vols. 1-3, Academic Press, London, 2002.

1.52 B. R. Munson, D. F. Young, T. H. Okiishi and W. W. Huebsch, *Fundamentals of Fluid Mechanics* (6th ed.), Wiley, Hoboken, NJ, 2009.

1.53 C. W. de Silva (ed.), *Vibration and Shock Handbook*, Taylor & Francis, Boca Raton, FL, 2005.

REVIEW QUESTIONS

1.1 Give brief answers to the following:

1. Give two examples each of the bad and the good effects of vibration.
2. What are the three elementary parts of a vibrating system?
3. Define the number of degrees of freedom of a vibrating system.
4. What is the difference between a discrete and a continuous system? Is it possible to solve any vibration problem as a discrete one?
5. In vibration analysis, can damping always be disregarded?

6. Can a nonlinear vibration problem be identified by looking at its governing differential equation?
7. What is the difference between deterministic and random vibration? Give two practical examples of each.
8. What methods are available for solving the governing equations of a vibration problem?
9. How do you connect several springs to increase the overall stiffness?
10. Define spring stiffness and damping constant.
11. What are the common types of damping?
12. State three different ways of expressing a periodic function in terms of its harmonics.
13. Define these terms: cycle, amplitude, phase angle, linear frequency, period, and natural frequency.
14. How are τ, ω, and f related to each other?
15. How can we obtain the frequency, phase, and amplitude of a harmonic motion from the corresponding rotating vector?
16. How do you add two harmonic motions having different frequencies?
17. What are beats?
18. Define the terms *decibel* and *octave*.
19. Explain Gibbs' phenomenon.
20. What are half-range expansions?

1.2 Indicate whether each of the following statements is true or false:

1. If energy is lost in any way during vibration, the system can be considered to be damped.
2. The superposition principle is valid for both linear and nonlinear systems.
3. The frequency with which an initially disturbed system vibrates on its own is known as natural frequency.
4. Any periodic function can be expanded into a Fourier series.
5. A harmonic motion is a periodic motion.
6. The equivalent mass of several masses at different locations can be found using the equivalence of kinetic energy.
7. The generalized coordinates are not necessarily Cartesian coordinates.
8. Discrete systems are same as lumped parameter systems.
9. Consider the sum of harmonic motions, $x(t) = x_1(t) + x_2(t) = A\cos(\omega t + \alpha)$, with $x_1(t) = 15\cos\omega t$ and $x_2(t) = 20\cos(\omega t + 1)$. The amplitude A is given by 30.8088.
10. Consider the sum of harmonic motions, $x(t) = x_1(t) + x_2(t) = A\cos(\omega t + \alpha)$, with $x_1(t) = 15\cos\omega t$ and $x_2(t) = 20\cos(\omega t + 1)$. The phase angle α is given by 1.57 rad.

1.3 Fill in the blank with the proper word:

1. Systems undergo dangerously large oscillations at _____.
2. Undamped vibration is characterized by no loss of _____.
3. A vibratory system consists of a spring, damper, and _____.
4. If a motion repeats after equal intervals of time, it is called a _____ motion.
5. When acceleration is proportional to the displacement and directed toward the mean position, the motion is called _____ harmonic.
6. The time taken to complete one cycle of motion is called the _____ of vibration.
7. The number of cycles per unit time is called the _____ of vibration.
8. Two harmonic motions having the same frequency are said to be _____.

9. The angular difference between the occurrence of similar points of two harmonic motions is called _____.
10. Continuous or distributed systems can be considered to have _____ number of degrees of freedom.
11. Systems with a finite number of degrees of freedom are called _____ systems.
12. The number of degrees of freedom of a system denotes the minimum number of independent _____ necessary to describe the positions of all parts of the system at any instant of time.
13. If a system vibrates due to initial disturbance only, it is called _____ vibration.
14. If a system vibrates due to an external excitation, it is called _____ vibration.
15. Resonance denotes the coincidence of the frequency of external excitation with a _____ frequency of the system.
16. A function $f(t)$ is called an odd function if _____.
17. The _____ range expansions can be used to represent functions defined only in the interval 0 to τ.
18. _____ analysis deals with the Fourier series representation of periodic functions.
19. The rotational speed of 1000 rpm (revolutions per minute) is equivalent to _____ radians/sec.
20. When the speed of a turbine is 6000 rpm, it takes _____ seconds for the turbine to complete one revolution.

1.4 Select the most appropriate answer from the multiple choices given:

1. The world's first seismograph was invented in
   (a) Japan **(b)** China **(c)** Egypt
2. The first experiments on simple pendulums were conducted by
   (a) Galileo **(b)** Pythagoras **(c)** Aristotle
3. The *Philosophiae Naturalis Principia Mathematica* was published by
   (a) Galileo **(b)** Pythagoras **(c)** Newton
4. Mode shapes of plates, by placing sand on vibrating plates, were first observed by
   (a) Chladni **(b)** D'Alembert **(c)** Galileo
5. The thick beam theory was first presented by
   (a) Mindlin **(b)** Einstein **(c)** Timoshenko
6. The number of degrees of freedom of a simple pendulum is:
   (a) 0 **(b)** 1 **(c)** 2
7. Vibration can be classified in
   (a) one way **(b)** two ways **(c)** several ways
8. Gibbs' phenomenon denotes an anomalous behavior in the Fourier series representation of a
   (a) harmonic function **(b)** periodic function **(c)** random function
9. The graphical representation of the amplitudes and phase angles of the various frequency components of a periodic function is known as a
   (a) spectral diagram **(b)** frequency diagram **(c)** harmonic diagram
10. When a system vibrates in a fluid medium, the damping is
    (a) viscous **(b)** Coulomb **(c)** solid
11. When parts of a vibrating system slide on a dry surface, the damping is
    (a) viscous **(b)** Coulomb **(c)** solid
12. When the stress-strain curve of the material of a vibrating system exhibits a hysteresis loop, the damping is
    (a) viscous **(b)** Coulomb **(c)** solid

13. The equivalent spring constant of two parallel springs with stiffnesses k_1 and k_2 is

(a) $k_1 + k_2$ **(b)** $\dfrac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$ **(c)** $\dfrac{1}{k_1} + \dfrac{1}{k_2}$

14. The equivalent spring constant of two series springs with stiffnesses k_1 and k_2 is

(a) $k_1 + k_2$ **(b)** $\dfrac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$ **(c)** $\dfrac{1}{k_1} + \dfrac{1}{k_2}$

15. The spring constant of a cantilever beam with an end mass m is

(a) $\dfrac{3EI}{l^3}$ **(b)** $\dfrac{l^3}{3EI}$ **(c)** $\dfrac{Wl^3}{3EI}$

16. If $f(-t) = f(t)$, function $f(t)$ is said to be

(a) even **(b)** odd **(c)** continuous

1.5 Match the following:

1. Pythagoras (582–507 B.C.)	**a.** published a book on the theory of sound
2. Euclid (300 B.C.)	**b.** first person to investigate musical sounds on a scientific basis
3. Zhang Heng (132 A.D.)	**c.** wrote a treatise called *Introduction to Harmonics*
4. Galileo (1564–1642)	**d.** founder of modern experimental science
5. Rayleigh (1877)	**e.** invented the world's first seismograph

1.6 Match the following:

1. Imbalance in diesel engines	**a.** can cause failure of turbines and aircraft engines
2. Vibration in machine tools	**b.** cause discomfort in human activityduring metal cutting
3. Blade and disk vibration	**c.** can cause wheels of locomotives to rise off the track
4. Wind-induced vibration	**d.** can cause failure of bridges
5. Transmission of vibration	**e.** can give rise to chatter

1.7 Consider four springs with the spring constants:

$k_1 = 20$ lb/in., $k_2 = 50$ lb/in., $k_3 = 100$ lb/in., $k_4 = 200$ lb/in.

Match the equivalent spring constants:

1. k_1, k_2, k_3, and k_4 are in parallel	**a.** 18.9189 lb/in.
2. k_1, k_2, k_3, and k_4 are in series	**b.** 370.0 lb/in.
3. k_1 and k_2 are in parallel $(k_{eq} = k_{12})$	**c.** 11.7647 lb/in.
4. k_3 and k_4 are in parallel $(k_{eq} = k_{34})$	**d.** 300.0 lb/in.
5. k_1, k_2, and k_3 are in parallel $(k_{eq} = k_{123})$	**e.** 70.0 lb/in.
6. k_{123} is in series with k_4	**f.** 170.0 lb/in.
7. k_2, k_3, and k_4 are in parallel $(k_{eq} = k_{234})$	**g.** 350.0 lb/in.
8. k_1 and k_{234} are in series	**h.** 91.8919 lb/in.

PROBLEMS

Section 1.4 Basic Concepts of Vibration and
Section 1.6 Vibration Analysis Procedure

1.1* A study of the response of a human body subjected to vibration/shock is important in many applications. In a standing posture, the masses of head, upper torso, hips, and legs and the elasticity/damping of neck, spinal column, abdomen, and legs influence the response characteristics. Develop a sequence of three improved approximations for modeling the human body.

1.2* Figure 1.54 shows a human body and a restraint system at the time of an automobile collision [1.47]. Suggest a simple mathematical model by considering the elasticity, mass, and damping of the seat, human body, and restraints for a vibration analysis of the system.

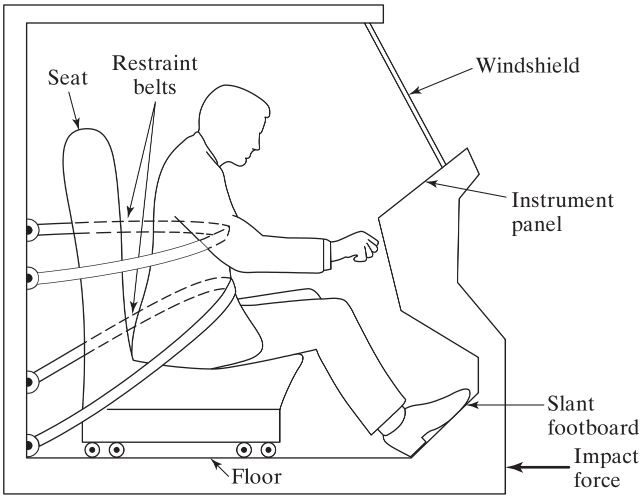


FIGURE 1.62 A human body and a restraint system.

1.3* A reciprocating engine is mounted on a foundation as shown in Fig. 1.63. The unbalanced forces and moments developed in the engine are transmitted to the frame and the foundation. An elastic pad is placed between the engine and the foundation block to reduce the transmission of vibration. Develop two mathematical models of the system using a gradual refinement of the modeling process.

1.4* An automobile moving over a rough road (Fig. 1.64) can be modeled considering (a) weight of the car body, passengers, seats, front wheels, and rear wheels; (b) elasticity of

*The asterisk denotes a design-type problem or a problem with no unique answer.

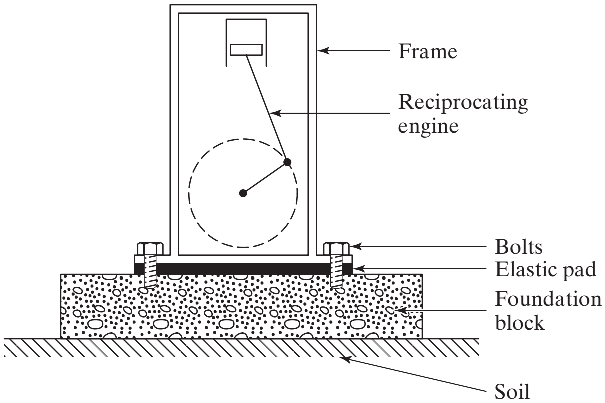


FIGURE 1.63 A reciprocating engine on a foundation.

tires (suspension), main springs, and seats; and (c) damping of the seats, shock absorbers, and tires. Develop three mathematical models of the system using a gradual refinement in the modeling process.

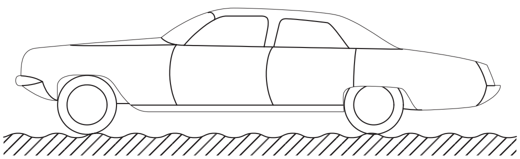


FIGURE 1.64 An automobile moving on a rough road.

1.5* The consequences of a head-on collision of two automobiles can be studied by considering the impact of the automobile on a barrier, as shown in Fig. 1.65. Construct a mathematical model by considering the masses of the automobile body, engine, transmission, and suspension and the elasticity of the bumpers, radiator, sheet metal body, driveline, and engine mounts.

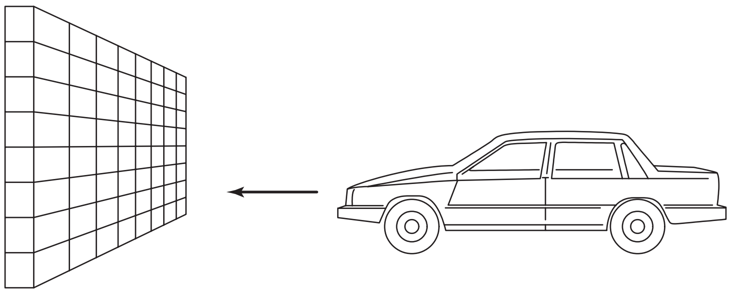


FIGURE 1.65 An automobile colliding with a barrier.

1.6* Develop a mathematical model for the tractor and plow shown in Fig. 1.66 by considering the mass, elasticity, and damping of the tires, shock absorbers, and plows (blades).

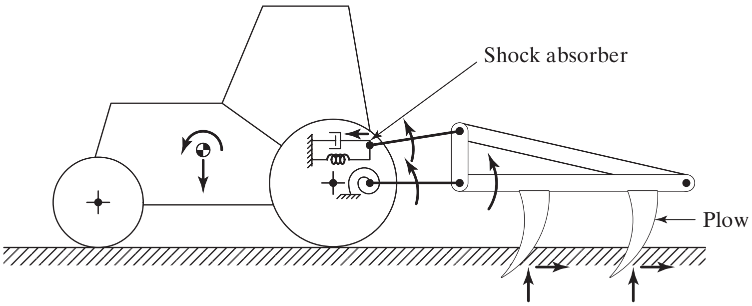


FIGURE 1.66 A tractor and plow.

Section 1.7 Spring Elements

1.7 Determine the equivalent spring constant of the system shown in Fig. 1.67.

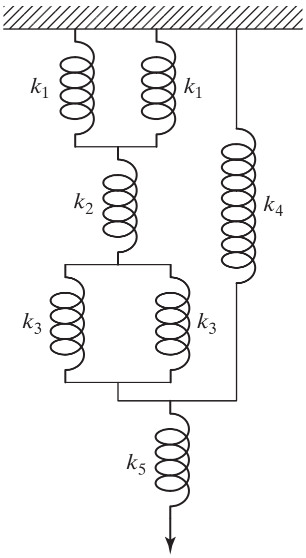


FIGURE 1.67 Springs in series-parallel.

1.8 Consider a system of two springs, with stiffnesses k_1 and k_2, arranged in parallel as shown in Fig. 1.68. The rigid bar to which the two springs are connected remains horizontal when the

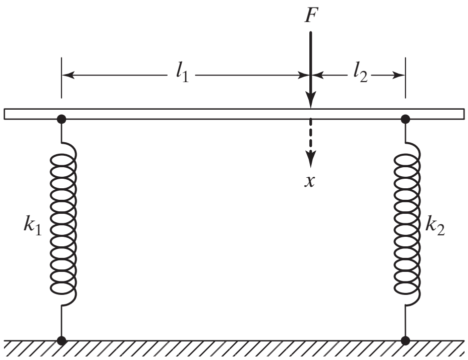


FIGURE 1.68 Parallel springs subjected to load.

force F is zero. Determine the equivalent spring constant of the system (k_e) that relates the force applied (F) to the resulting displacement (x) as

$$F = k_e x$$

Hint: Because the spring constants of the two springs are different and the distances l_1 and l_2 are not the same, the rigid bar will not remain horizontal when the force F is applied.

1.9 In Fig. 1.69, find the equivalent spring constant of the system in the direction of θ.

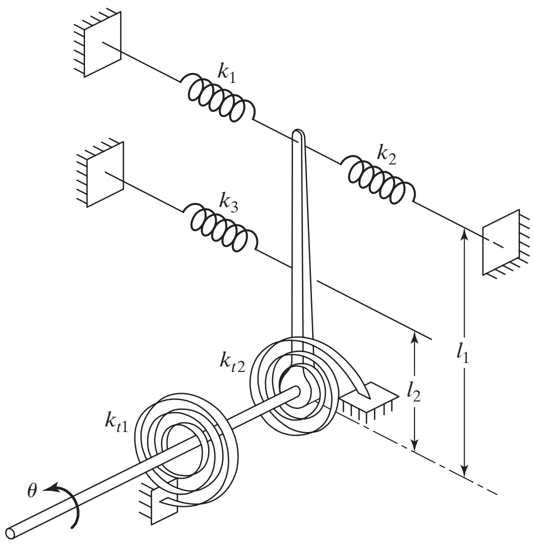


FIGURE 1.69

1.10 Find the equivalent torsional spring constant of the system shown in Fig. 1.70. Assume that k_1, k_2, k_3, and k_4 are torsional and k_5 and k_6 are linear spring constants.

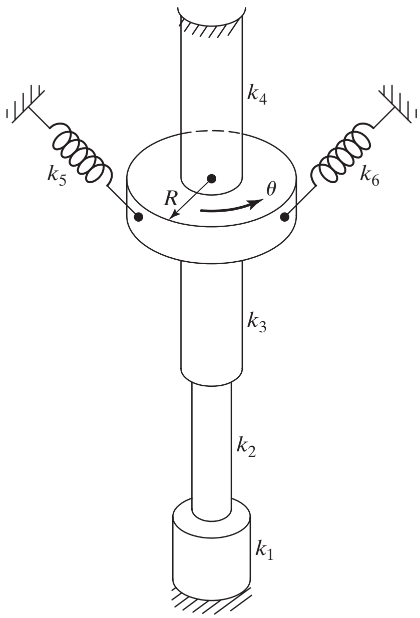


FIGURE 1.70

1.11 A machine of mass $m = 500$ kg is mounted on a simply supported steel beam of length $l = 2$ m having a rectangular cross section (depth = 0.1 m, width = 1.2 m) and Young's modulus $E = 2.06 \times 10^{11}$ N/m^2. To reduce the vertical deflection of the beam, a spring of stiffness k is attached at mid-span, as shown in Fig. 1.71. Determine the value of k needed to reduce the deflection of the beam by

a. 25 percent of its original value.
b. 50 percent of its original value.
c. 75 percent of its original value.

Assume that the mass of the beam is negligible.

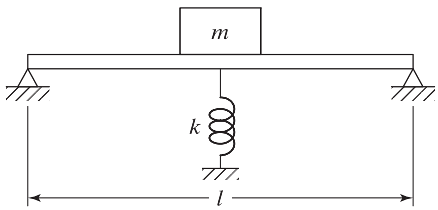


FIGURE 1.71

1.12 A bar of length L and Young's modulus E is subjected to an axial force. Compare the spring constants of bars with cross sections in the form of a solid circle (of diameter d), square (of side d) and hollow circle (of mean diameter d and wall thickness $t = 0.1d$). Determine which of these cross sections leads to an economical design for a specified value of axial stiffness of the bar.

1.13 A cantilever beam of length L and Young's modulus E is subjected to a bending force at its free end. Compare the spring constants of beams with cross sections in the form of a solid circle (of diameter d), square (of side d), and hollow circle (of mean diameter d and wall thickness $t = 0.1d$). Determine which of these cross sections leads to an economical design for a specified value of bending stiffness of the beam.

1.14 An electronic instrument, weighing 200 lb, is supported on a rubber mounting whose force-deflection relationship is given by $F(x) = 800\,x + 40\,x^3$, where the force (F) and the deflection (x) are in pounds and inches, respectively. Determine the following:

a. Equivalent linear spring constant of the mounting at its static equilibrium position.
b. Deflection of the mounting corresponding to the equivalent linear spring constant.

1.15 The force-deflection relation of a steel helical spring used in an engine is found experimentally as $F(x) = 200\,x + 50\,x^2 + 10\,x^3$, where the force ($F$) and deflection ($x$) are measured in pounds and inches, respectively. If the spring undergoes a steady deflection of 0.5 in. during the operation of the engine, determine the equivalent linear spring constant of the spring at its steady deflection.

1.16 Four identical rigid bars—each of length a—are connected to a spring of stiffness k to form a structure for carrying a vertical load P, as shown in Figs. 1.72(a) and (b). Find the equivalent spring constant of the system k_{eq}, for each case, disregarding the masses of the bars and the friction in the joints.

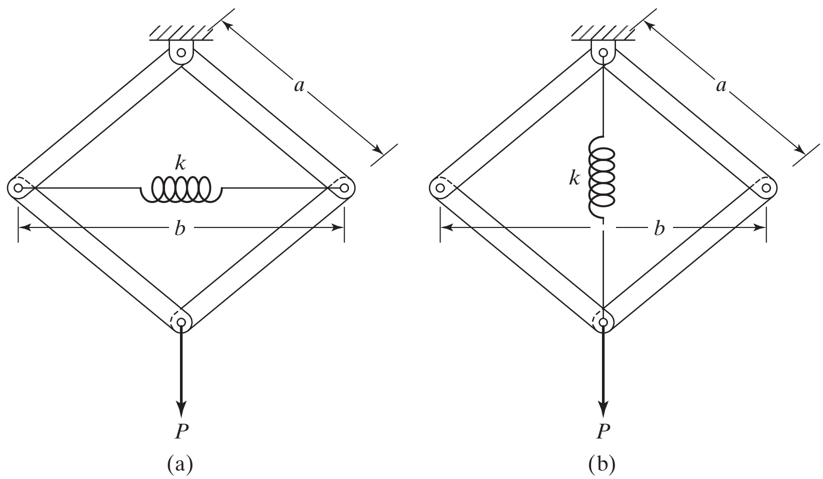


FIGURE 1.72

1.17 The tripod shown in Fig. 1.73 is used for mounting an electronic instrument that finds the distance between two points in space. The legs of the tripod are located symmetrically about the mid-vertical axis, each leg making an angle α with the vertical. If each leg has a length l and axial stiffness k, find the equivalent spring stiffness of the tripod in the vertical direction.

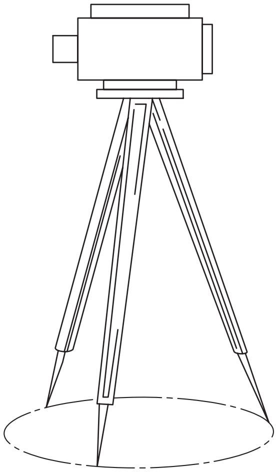


FIGURE 1.73

1.18 The static equilibrium position of a massless rigid bar, hinged at point O and connected with springs k_1 and k_2, is shown in Fig. 1.74. Assuming that the displacement (x) resulting from the application of a force F at point A is small, find the equivalent spring constant of the system, k_e, that relates the applied force F to the displacement x as $F = k_e x$.

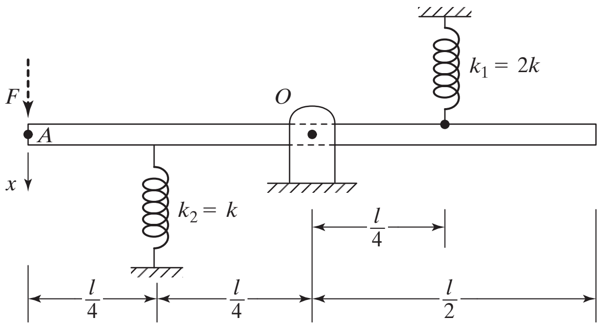


FIGURE 1.74 Rigid bar connected by springs.

1.19 Figure 1.75 shows a system in which the mass m is directly connected to the springs with stiffnesses k_1 and k_2 while the spring with stiffness k_3 or k_4 comes into contact with the mass based on the value of the displacement of the mass. Determine the variation of the spring force exerted on the mass as the displacement of the mass (x) varies.

1.20 Figure 1.76 shows a uniform rigid bar of mass m that is pivoted at point O and connected by springs of stiffnesses k_1 and k_2. Considering a small angular displacement θ of the rigid bar about the point O, determine the equivalent spring constant associated with the restoring moment.

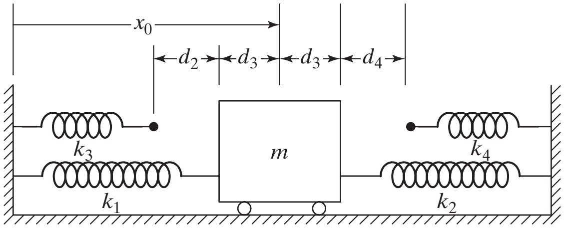


FIGURE 1.75 Mass connected by springs.

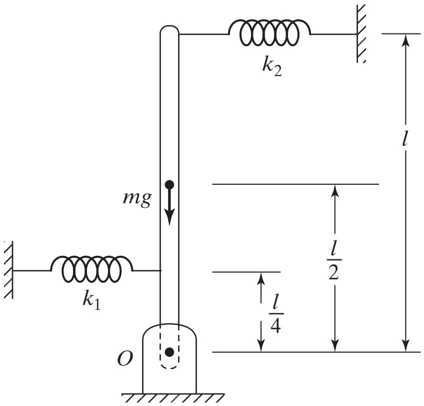


FIGURE 1.76 Rigid bar connected by springs.

1.21 Figure 1.77 shows a U-tube manometer open at both ends and containing a column of liquid mercury of length l and specific weight γ. Considering a small displacement x of the manometer meniscus from its equilibrium position (or datum), determine the equivalent spring constant associated with the restoring force.

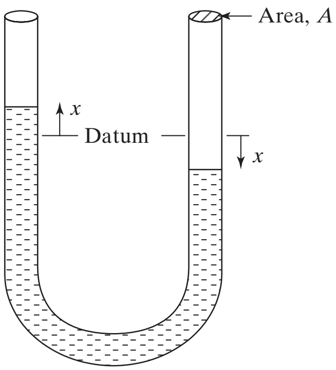


FIGURE 1.77 U-tube manometer.

1.22 An oil drum of diameter d and mass m floats in a bath of sea water of density ρ_w as shown in Fig. 1.78. Considering a small displacement x of the oil drum from its static equilibrium position, determine the equivalent spring constant associated with the restoring force.

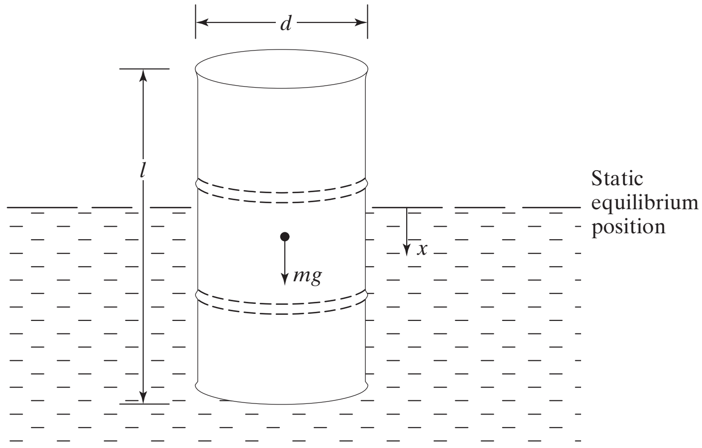


FIGURE 1.78 Oil drum floating in sea water.

1.23 Find the equivalent spring constant and equivalent mass of the system shown in Fig. 1.79 with references to θ. Assume that the bars AOB and CD are rigid with negligible mass.

1.24 Find the length of the equivalent uniform hollow shaft of inner diameter d and thickness t that has the same axial spring constant as that of the solid conical shaft shown in Fig. 1.80.

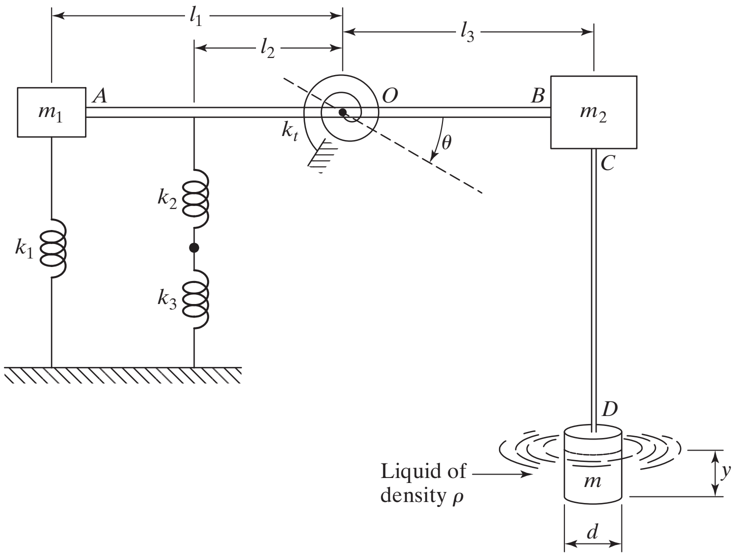


FIGURE 1.79

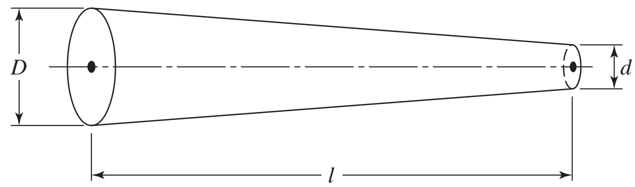


FIGURE 1.80

1.25 Figure 1.81 shows a three-stepped bar fixed at one end and subjected to an axial force F at the other end. The length of step i is l_i and its cross sectional area is A_i, $i = 1, 2, 3$. All the steps are made of the same material with Young's modulus $E_i = E$, $i = 1, 2, 3$.

a. Find the spring constant (or stiffness) k_i of step i in the axial direction $(i = 1, 2, 3)$.
b. Find the equivalent spring constant (or stiffness) of the stepped bar, k_{eq}, in the axial direction so that $F = k_{eq} x$.
c. Indicate whether the steps behave as series or parallel springs.

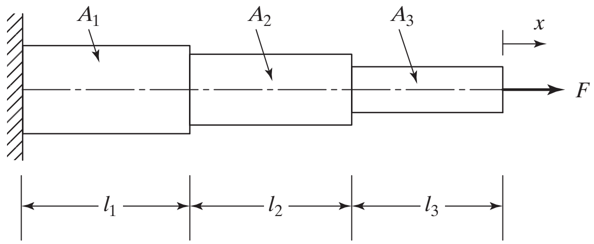


FIGURE 1.81 A stepped bar subjected to axial force

1.26 Find the equivalent spring constant of the system shown in Fig. 1.82.

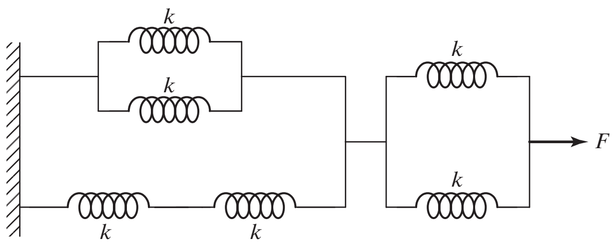


FIGURE 1.82 Springs connected in series-parallel

1.27 Figure 1.83 shows a three-stepped shaft fixed at one end and subjected to a torsional moment T at the other end. The length of step i is l_i and its diameter is D_i, $i = 1, 2, 3$. All the steps are made of the same material with shear modulus $G_i = G$, $i = 1, 2, 3$.

a. Find the torsional spring constant (or stiffness) k_{t_i} of step i $(i = 1, 2, 3)$.

b. Find the equivalent torsional spring constant (or stiffness) of the stepped shaft, $k_{t_{eq}}$, so that $T = k_{t_{eq}}\theta$.

c. Indicate whether the steps behave as series or parallel torsional springs.

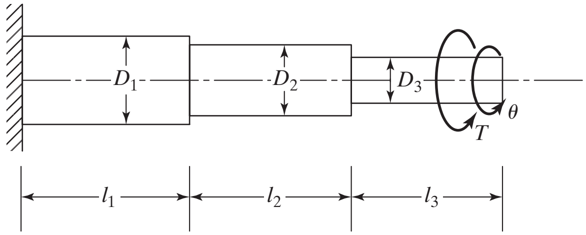


FIGURE 1.83 A stepped shaft subjected to torsional moment.

1.28 The force-deflection characteristic of a spring is described by $F = 500x + 2x^3$, where the force (F) is in Newtons and the deflection (x) is in millimeters. Find (a) the linearized spring constant at $x = 10$ mm and (b) the spring forces at $x = 9$ mm and $x = 11$ mm using the linearized spring constant. Also find the error in the spring forces found in (b).

1.29 Figure 1.84 shows an air spring. This type of spring is generally used for obtaining very low natural frequencies while maintaining zero deflection under static loads. Find the spring constant of this air spring by assuming that the pressure p and volume v change adiabatically when the mass m moves.

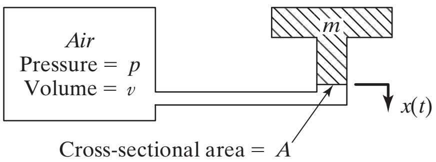


FIGURE 1.84

Hint: pv^γ = constant for an adiabatic process, where γ is the ratio of specific heats. For air, $\gamma = 1.4$.

1.30 Find the equivalent spring constant of the system shown in Fig. 1.85 in the direction of the load P.

1.31 Derive the expression for the equivalent spring constant that relates the applied force F to the resulting displacement x of the system shown in Fig. 1.86. Assume the displacement of the link to be small.

1.32 The spring constant of a helical spring under axial load is given by

$$k = \frac{Gd^4}{8ND^3}$$

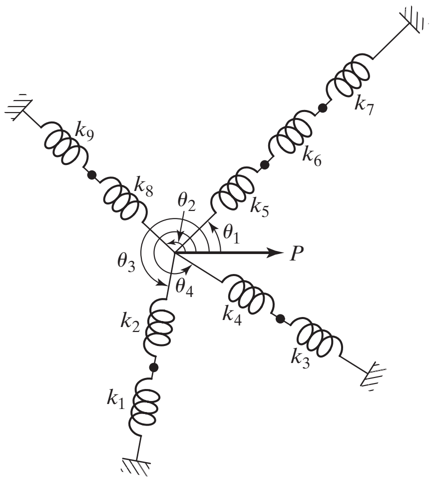


FIGURE 1.85

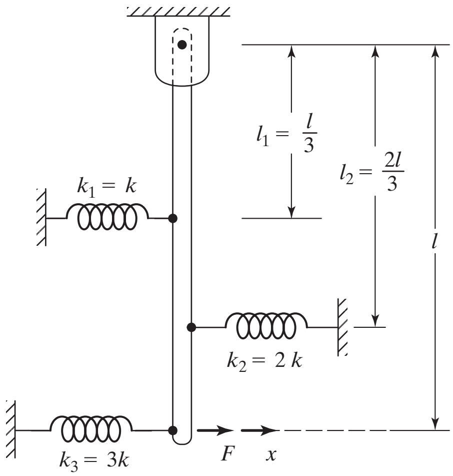


FIGURE 1.86 Rigid bar connected by springs.

where G is the shear modulus, d is the wire diameter, D is the coil diameter, and N is the number of turns. Find the spring constant and the weight of a helical spring made of steel for the following data: $D = 0.2$ m, $d = 0.005$ m, $N = 10$.

1.33 Two helical springs, one made of steel and the other made of aluminum, have identical values of d and D. (a) If the number of turns in the steel spring is 10, determine the number of turns required in the aluminum spring whose weight will be same as that of the steel spring, (b) Find the spring constants of the two springs.

1.34 Figure 1.87 shows three parallel springs, one with stiffness $k_1 = k$ and each of the other two with stiffness $k_2 = k$. The spring with stiffness k_1 has a length l and each of the springs with stiffness k_2 has a length of $l - a$. Find the force-deflection characteristic of the system.

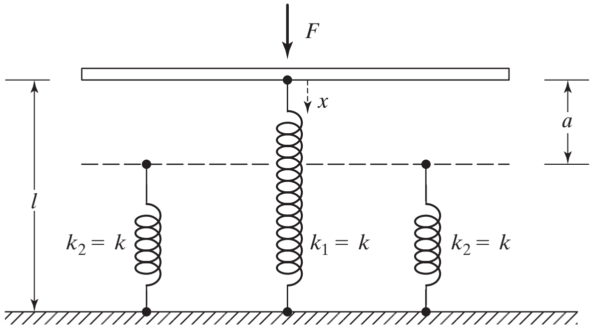


FIGURE 1.87 Nonlinear behavior of linear springs.

1.35* Design an air spring using a cylindrical container and a piston to achieve a spring constant of 75 lb/in. Assume that the maximum air pressure available is 200 psi.

1.36 The force (F)-deflection (x) relationship of a nonlinear spring is given by

$$F = ax + bx^3$$

where a and b are constants. Find the equivalent linear spring constant when the deflection is 0.01 m with $a = 20{,}000$ N/m and $b = 40 \times 10^6$ N/m^3.

1.37 Two nonlinear springs, S_1 and S_2, are connected in two different ways as indicated in Fig. 1.88. The force, F_i, in spring S_i is related to its deflection (x_i) as

$$F_i = a_i x_i + b_i x_i^3, \qquad i = 1, 2$$

where a_i and b_i are constants. If an equivalent linear spring constant, k_{eq}, is defined by $W = k_{eq}x$, where x is the total deflection of the system, find an expression for k_{eq} in each case.

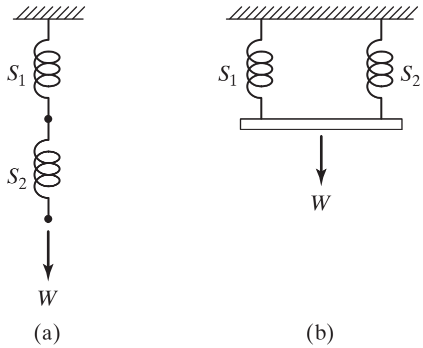


FIGURE 1.88

1.38* Design a steel helical compression spring to satisfy the following requirements:

Spring stiffness $(k) \geq 8000$ N/mm
Fundamental natural frequency of vibration $(f_1) \geq 0.4$ Hz
Spring index $(D/d) \geq 6$
Number of active turns $(N) \geq 10$.

The stiffness and fundamental natural frequency of the spring are given by [1.43]:

$$k = \frac{Gd^4}{8D^3N} \quad \text{and} \quad f_1 = \frac{1}{2}\sqrt{\frac{kg}{W}}$$

where G = shear modulus, d = wire diameter, D = coil diameter, W = weight of the spring, and g = acceleration due to gravity.

1.39 Find the spring constant of the bimetallic bar shown in Fig. 1.89 in axial motion.

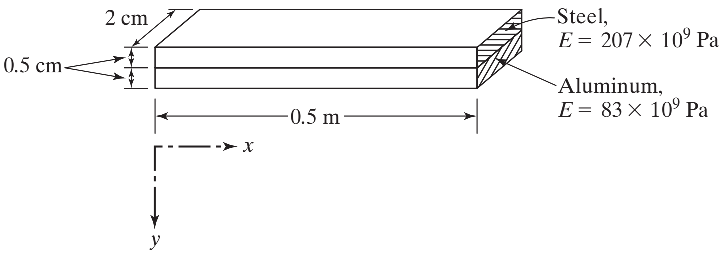


FIGURE 1.89

1.40 Consider a spring of stiffness k stretched by a distance x_0 from its free length. One end of the spring is fixed at point O and the other end is connected to a roller as shown in Fig. 1.90. The

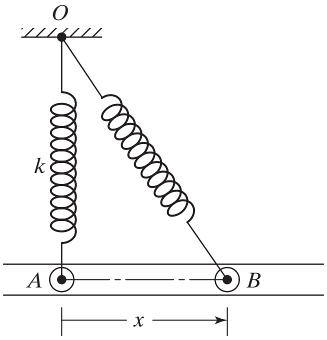


FIGURE 1.90 One end of spring with lateral movement.

roller is constrained to move in the horizontal direction with no friction. Find the force (F)-displacement (x) relationship of the spring when the roller is moved by a horizontal distance x to position B. Discuss the resulting force-displacement relation and identify the stiffness constant $\widetilde{k}$ along the direction of x.

1.41 One end of a helical spring is fixed and the other end is subjected to five different tensile forces. The lengths of the spring measured at various values of the tensile forces are given below:

Tensile force F (N)	0	100	250	330	480	570
Total length of the spring (mm)	150	163	183	194	214	226

Determine the force-deflection relation of the helical spring.

1.42 A tapered solid steel propeller shaft is shown in Fig. 1.91. Determine the torsional spring constant of the shaft.

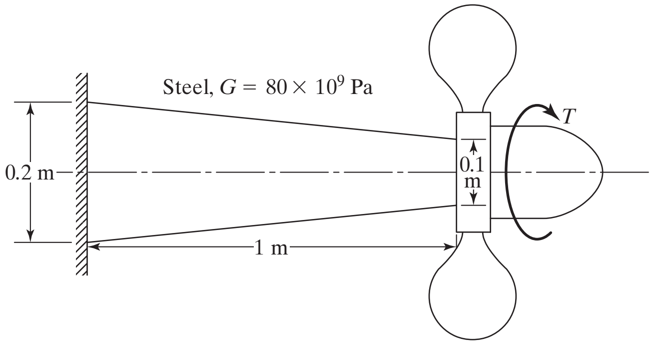


FIGURE 1.91

1.43 A composite propeller shaft, made of steel and aluminum, is shown in Fig. 1.92.

a. Determine the torsional spring constant of the shaft.
b. Determine the torsional spring constant of the composite shaft when the inner diameter of the aluminum tube is 5 cm instead of 10 cm.

1.44 Consider two helical springs with the following characteristics:

Spring 1: material—steel; number of turns—10; mean coil diameter—12 in.; wire diameter—2 in.; free length—15 in.; shear modulus—12×10^6 psi.

Spring 2: material—aluminum; number of turns—10; mean coil diameter—10 in.; wire diameter—1 in.; free length—15 in.; shear modulus—4×10^6 psi.

Determine the equivalent spring constant when (a) spring 2 is placed inside spring 1, and (b) spring 2 is placed on top of spring 1.

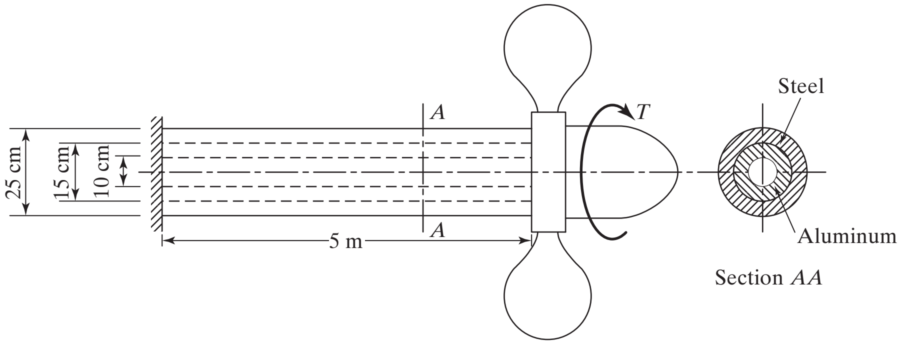


FIGURE 1.92

1.45 Solve Problem 1.44 by assuming the wire diameters of springs 1 and 2 to be 1.0 in. and 0.5 in. instead of 2.0 in. and 1.0 in., respectively.

1.46 The arm *AD* of the excavator shown in Fig. 1.93 can be approximated as a steel tube of outer diameter 10 in., inner diameter 9.5 in., and length 100 in. with a viscous damping coefficient of 0.4. The arm *DE* can be approximated as a steel tube of outer diameter 7 in., inner diameter 6.5 in., and length 75 in. with a viscous damping coefficient of 0.3. Estimate the equivalent spring constant and equivalent damping coefficient of the excavator, assuming that the base *AC* is fixed.

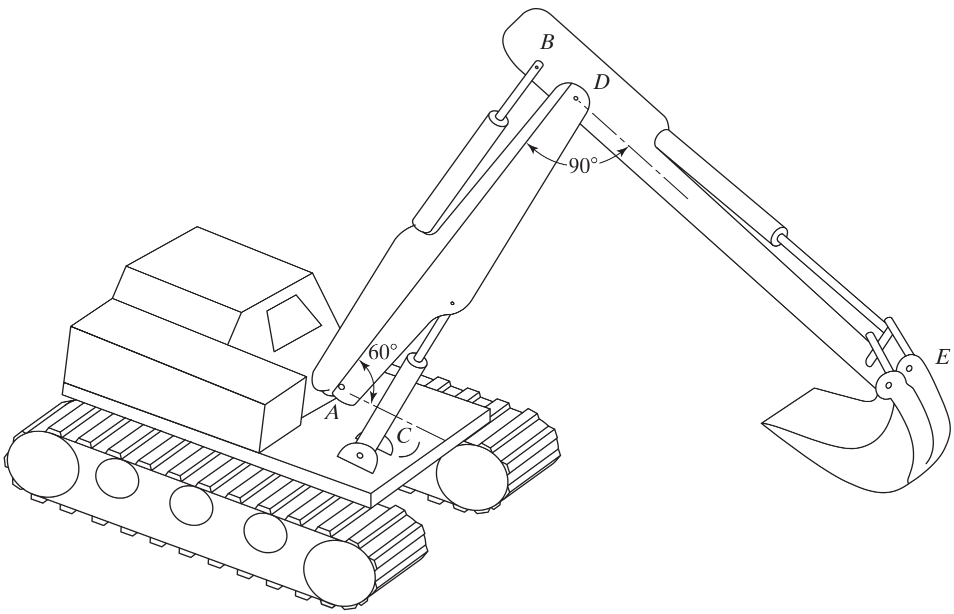


FIGURE 1.93 An excavator.

1.47 A heat exchanger consists of six identical stainless steel tubes connected in parallel as shown in Fig. 1.94. If each tube has an outer diameter 0.30 in., inner diameter 0.29 in., and length 50 in., determine the axial stiffness and the torsional stiffness about the longitudinal axis of the heat exchanger.

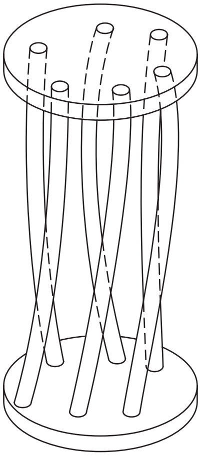


FIGURE 1.94
A heat exchanger.

Section 1.8 Mass or Inertia Elements

1.48 Two sector gears, located at the ends of links 1 and 2, are engaged together and rotate about O_1 and O_2, as shown in Fig. 1.95. If links 1 and 2 are connected to springs k_1 to k_4 and k_{t1} and k_{t2} as shown, find the equivalent torsional spring stiffness and equivalent mass moment of inertia of the system with reference to θ_1. Assume (a) the mass moment of inertia of link 1 (including the sector gear) about O_1 is J_1 and that of link 2 (including the sector gear) about O_2 is J_2, and (b) the angles θ_1 and θ_2 are small.

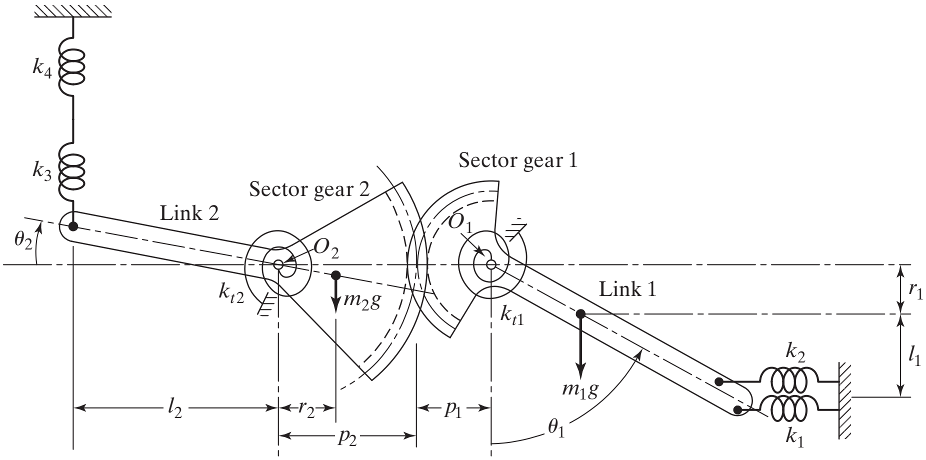


FIGURE 1.95 Two sector gears.

1.49 In Fig. 1.96 find the equivalent mass of the rocker arm assembly with respect to the x coordinate.

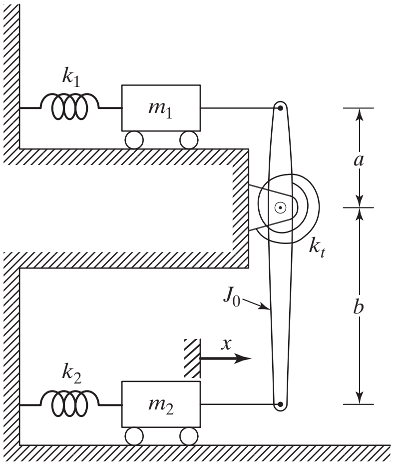


FIGURE 1.96 Rocker arm assembly.

1.50 Find the equivalent mass moment of inertia of the gear train shown in Fig. 1.97 with reference to the driving shaft. In Fig. 1.97, J_i and n_i denote the mass moment of inertia and the number of teeth, respectively, of gear i, $i = 1, 2, \ldots, 2N$.

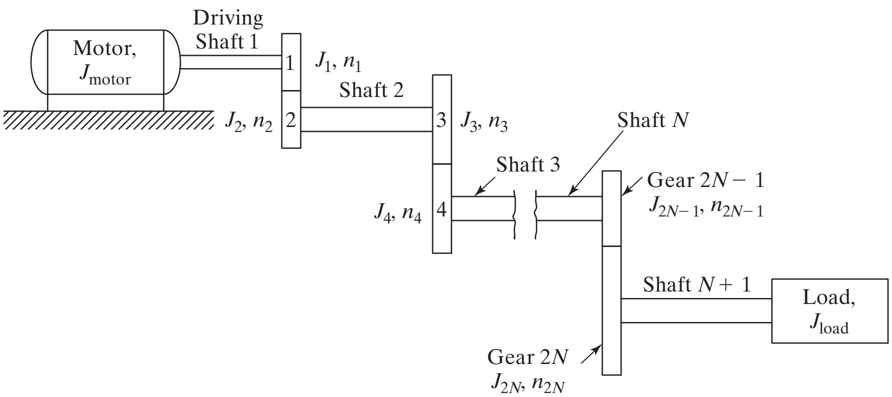


FIGURE 1.97

1.51 Two masses, having mass moments of inertia J_1 and J_2, are placed on rotating rigid shafts that are connected by gears, as shown in Fig. 1.98. If the numbers of teeth on gears 1 and 2 are n_1 and n_2, respectively, find the equivalent mass moment of inertia corresponding to θ_1.

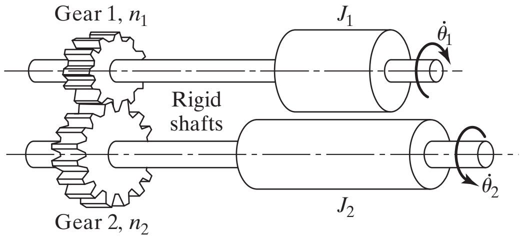


FIGURE 1.98 Rotational masses on geared shafts.

1.52 A simplified model of a petroleum pump is shown in Fig. 1.99, where the rotary motion of the crank is converted to the reciprocating motion of the piston. Find the equivalent mass, m_{eq}, of the system at location A.

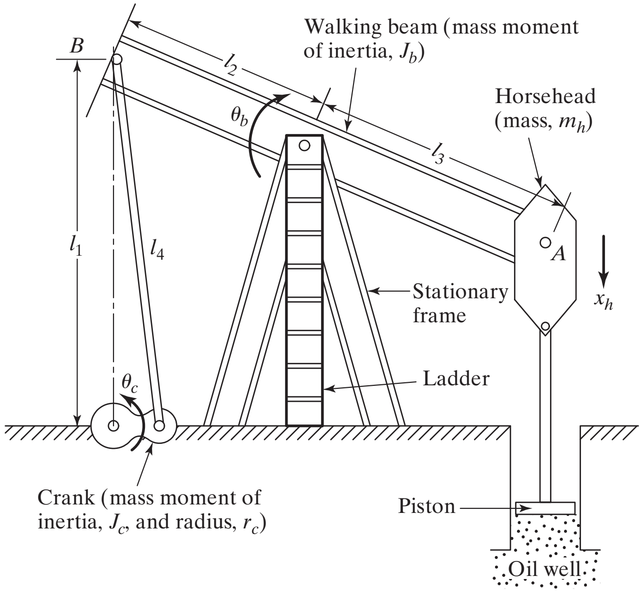


FIGURE 1.99

1.53 Find the equivalent mass of the system shown in Fig. 1.100.

1.54 Figure 1.101 shows an offset slider-crank mechanism with a crank length r, connecting rod length l, and offset δ. If the crank has a mass and mass moment of inertia of m_r and J_r, respectively, at its center of mass A, the connecting rod has a mass and mass moment of inertia of m_c and J_c, respectively, at its center of mass C, and the piston has a mass m_p, determine the equivalent rotational inertia of the system about the center of rotation of the crank, point O.

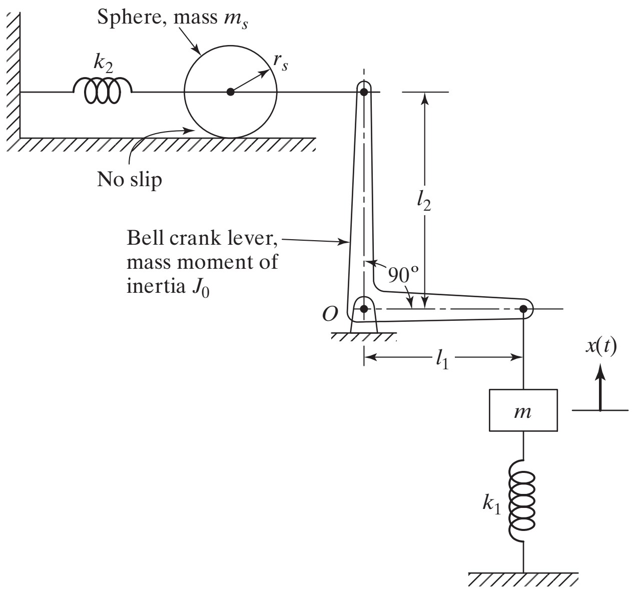


FIGURE 1.100

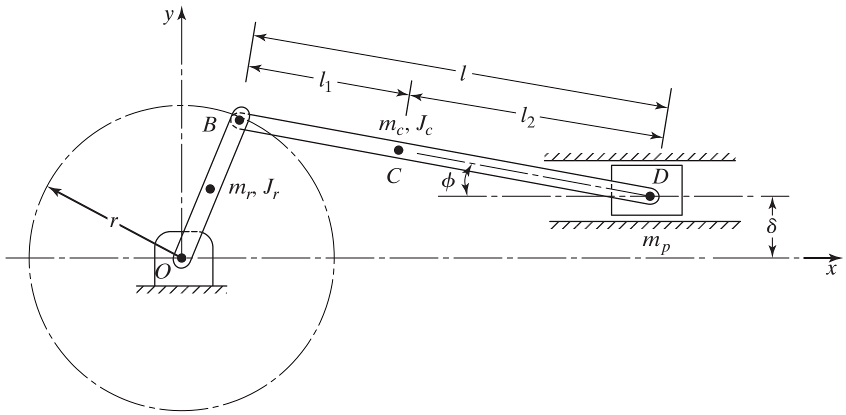


FIGURE 1.101 Slider-crank mechanism.

Section 1.9 Damping Elements

1.55 Find a single equivalent damping constant for the following cases:

a. When three dampers are parallel.

b. When three dampers are in series.

c. When three dampers are connected to a rigid bar (Fig. 1.102) and the equivalent damper is at site c_1.

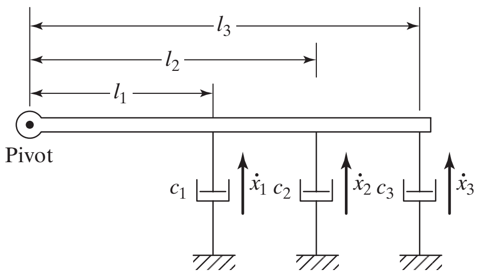


FIGURE 1.102 Dampers connected to a rigid bar.

d. When three torsional dampers are located on geared shafts (Fig. 1.103) and the equivalent damper is at location c_{t_1}.

Hint: The energy dissipated by a viscous damper in a cycle during harmonic motion is given by $\pi c \omega X^2$, where c is the damping constant, ω is the frequency, and X is the amplitude of oscillation.

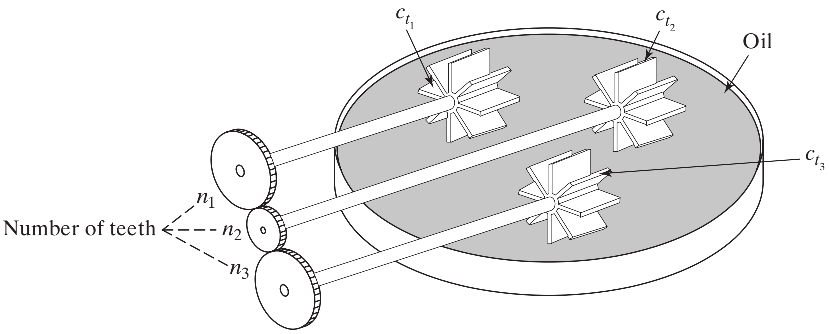


FIGURE 1.103 Dampers located on geared shafts.

1.56 Consider a system of two dampers, with damping constants c_1 and c_2, arranged in parallel as shown in Fig. 1.104. The rigid bar to which the two dampers are connected remains horizontal when the force F is zero. Determine the equivalent damping constant of the system (c_e) that relates the force applied (F) to the resulting velocity (v) as $F = c_e v$.
Hint: Because the damping constants of the two dampers are different and the distances l_1 and l_2 are not the same, the rigid bar will not remain horizontal when the force F is applied.

1.57* Design a piston-cylinder-type viscous damper to achieve a damping constant of 1 lb-sec/in. using a fluid of viscosity 4 μreyn $(1 \text{ reyn} = 1 \text{ lb-sec/in.}^2)$.

1.58* Design a shock absorber (piston-cylinder-type dashpot) to obtain a damping constant of 10^5 lb-sec/in. using SAE 30 oil at 70°F. The diameter of the piston has to be less than 2.5 in.

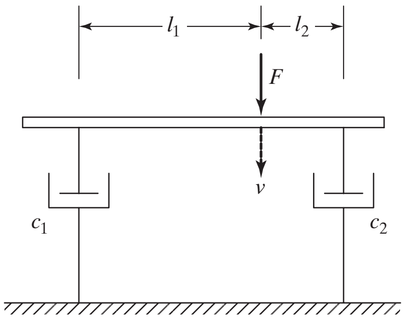


FIGURE 1.104 Parallel dampers subjected to load.

1.59 Develop an expression for the damping constant of the rotational damper shown in Fig. 1.105 in terms of D, d, l, h, ω, and μ, where ω denotes the constant angular velocity of the inner cylinder, and d and h represent the radial and axial clearances between the inner and outer cylinders.

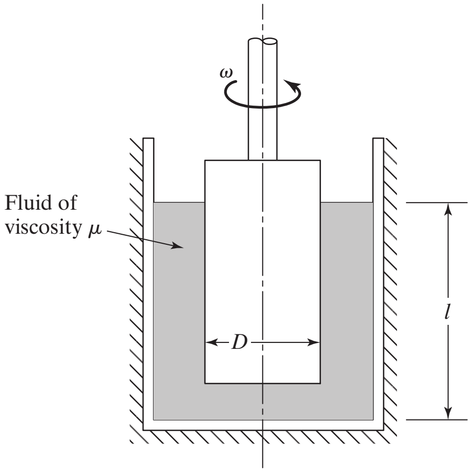


FIGURE 1.105

1.60 Consider two nonlinear dampers with the same force-velocity relationship given by $F = 1000v + 400v^2 + 20v^3$ with F in newtons and v in meters/second. Find the linearized damping constant of the dampers at an operating velocity of 10 m/s.

1.61 If the linearized dampers of Problem 1.60 are connected in parallel, determine the resulting equivalent damping constant.

1.62 If the linearized dampers of Problem 1.60 are connected in series, determine the resulting equivalent damping constant.

1.63 The force-velocity relationship of a nonlinear damper is given by $F = 500v + 100v^2 + 50v^3$, where F is in newtons and v is in meters/second. Find the linearized damping constant of the damper at an operating velocity of 5 m/s. If the resulting linearized damping constant is used at an operating velocity of 10 m/s, determine the error involved.

1.64 The experimental determination of damping force corresponding to several values of the velocity of the damper yielded the following results:

Damping force (newtons)	80	150	250	350	500	600
Velocity of damper (meters/second)	0.025	0.045	0.075	0.110	0.155	0.185

Determine the damping constant of the damper.

1.65 A flat plate with a surface area of 0.25 m^2 moves above a parallel flat surface with a lubricant film of thickness 1.5 mm in between the two parallel surfaces. If the viscosity of the lubricant is 0.5 Pa-s, determine the following:

a. Damping constant.
b. Damping force developed when the plate moves with a velocity of 2 m/s.

1.66 Find the torsional damping constant of a journal bearing for the following data: Viscosity of the lubricant (μ): 0.35 Pa-s, Diameter of the journal or shaft (2 R): 0.05 m, Length of the bearing (l): 0.075 m, Bearing clearance (d): 0.005 m. If the journal rotates at a speed (N) of 3000 rpm, determine the damping torque developed.

1.67 If each of the parameters (μ, R, l, d, and N) of the journal bearing described in Problem 1.66 is subjected to a $\pm 5\%$ variation about the corresponding value given, determine the percentage fluctuation in the values of the torsional damping constant and the damping torque developed.

Note: The variations in the parameters may have several causes, such as measurement error, manufacturing tolerances on dimensions, and fluctuations in the operating temperature of the bearing.

1.68 Consider a piston with an orifice in a cylinder filled with a fluid of viscosity μ as shown in Fig. 1.106. As the piston moves in the cylinder, the fluid flows through the orifice, giving rise to a friction or damping force. Derive an expression for the force needed to move the piston with a velocity v and indicate the type of damping involved.

Hint: The mass flow rate of the fluid (q) passing through an orifice is given by $q = \alpha \sqrt{\Delta p}$, where α is a constant for a given fluid, area of cross section of the cylinder (or area of piston), and area of the orifice [1.52].

1.69 The force (F)-velocity $(\dot{x})$ relationship of a nonlinear damper is given by

$$F = a\dot{x} + b\dot{x}^2$$

where a and b are constants. Find the equivalent linear damping constant when the relative velocity is 5 m/s with $a = 5$ N-s/m and $b = 0.2$ N-s^2/m^2.

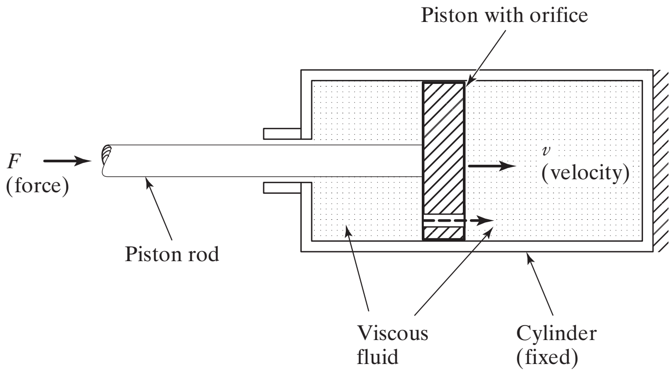


FIGURE 1.106 Piston and cylinder with orifice flow.

1.70 The damping constant (c) due to skin-friction drag of a rectangular plate moving in a fluid of viscosity μ is given by (see Fig. 1.107):

$$c = 100\mu l^2 d$$

Design a plate-type damper (shown in Fig. 1.42) that provides an identical damping constant for the same fluid.

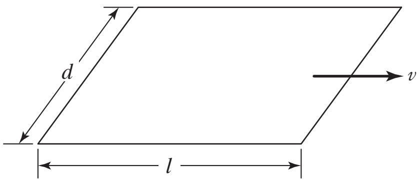


FIGURE 1.107

1.71 The damping constant (c) of the dashpot shown in Fig. 1.108 is given by [1.27]:

$$c = \frac{6\pi\mu l}{h^3}\left[\left(a - \frac{h}{2}\right)^2 - r^2\right]\left[\frac{a^2 - r^2}{a - \frac{h}{2}} - h\right]$$

Determine the damping constant of the dashpot for the following data: $\mu = 0.3445$ Pa-s, $l = 10$ cm, $h = 0.1$ cm, $a = 2$ cm, $r = 0.5$ cm.

1.72 In Problem 1.71, using the given data as reference, find the variation of the damping constant c when

a. r is varied from 0.5 cm to 1.0 cm.
b. h is varied from 0.05 cm to 0.10 cm.
c. a is varied from 2 cm to 4 cm.

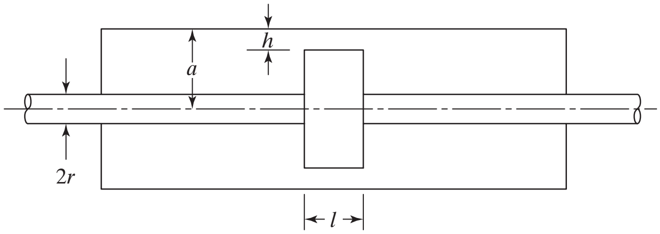


FIGURE 1.108

1.73 A massless bar of length 1 m is pivoted at one end and subjected to a force F at the other end. Two translational dampers, with damping constants $c_1 = 10$ N-s/m and $c_2 = 15$ N-s/m are connected to the bar as shown in Fig. 1.109. Determine the equivalent damping constant, c_{eq}, of the system so that the force F at point A can be expressed as $F = c_{eq}v$, where v is the linear velocity of point A.

1.74 Find an expression for the equivalent translational damping constant of the system shown in Fig. 1.110 so that the force F can be expressed as $F = c_{eq}v$, where v is the velocity of the rigid bar A.

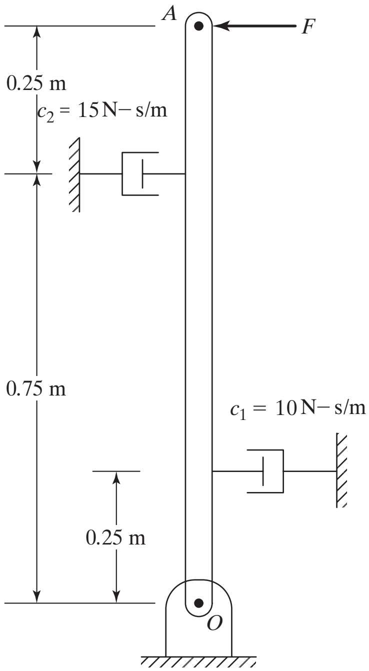


FIGURE 1.109 Rigid bar connected by dampers.

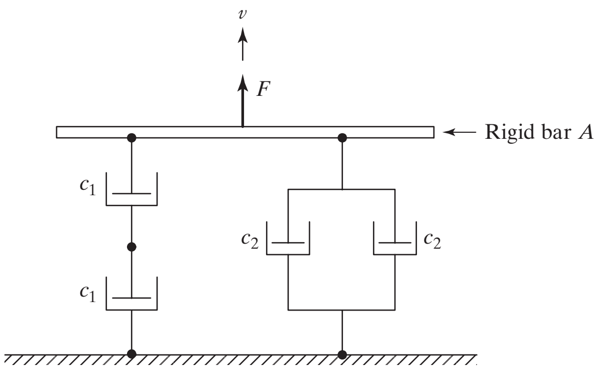


FIGURE 1.110 Dampers connected in series-parallel.

Section 1.10 Harmonic Motion

1.75 Express the complex number $5 + 2i$ in the exponential form $Ae^{i\theta}$.

1.76 Add the two complex numbers $(1 + 2i)$ and $(3 - 4i)$ and express the result in the form $Ae^{i\theta}$.

1.77 Subtract the complex number $(1 + 2i)$ from $(3 - 4i)$ and express the result in the form $Ae^{i\theta}$.

1.78 Find the product of the complex numbers $z_1 = (1 + 2i)$ and $z_2 = (3 - 4i)$ and express the result in the form $Ae^{i\theta}$.

1.79 Find the quotient, z_1/z_2, of the complex numbers $z_1 = (1 + 2i)$ and $z_2 = (3 - 4i)$ and express the result in the form $Ae^{i\theta}$.

1.80 The foundation of a reciprocating engine is subjected to harmonic motions in x and y directions:

$$x(t) = X \cos \omega t$$
$$y(t) = Y \cos(\omega t + \phi)$$

where X and Y are the amplitudes, ω is the angular velocity, and ϕ is the phase difference.

a. Verify that the resultant of the two motions satisfies the equation of the ellipse given by (see Fig. 1.111):

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\frac{xy}{XY}\cos\phi = \sin^2\phi \tag{E.1}$$

b. Discuss the nature of the resultant motion given by Eq. (E.1) for the special cases of $\phi = 0$, $\phi = \frac{\pi}{2}$, and $\phi = \pi$.

Note: The elliptic figure represented by Eq. (E.1) is known as a Lissajous figure and is useful in interpreting certain types of experimental results (motions) displayed by oscilloscopes.

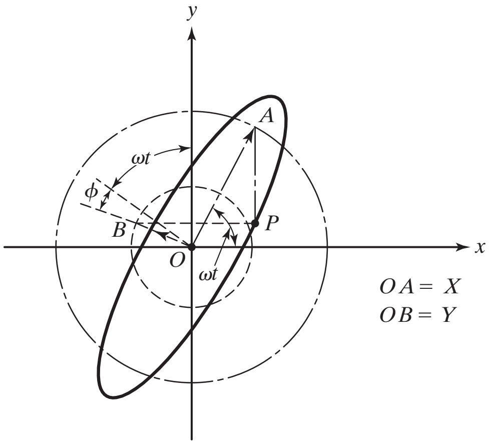


FIGURE 1.111 Lissajous figure.

1.81 The foundation of an air compressor is subjected to harmonic motions (with the same frequency) in two perpendicular directions. The resultant motion, displayed on an oscilloscope, appears as shown in Fig. 1.112. Find the amplitudes of vibration in the two directions and the phase difference between them.

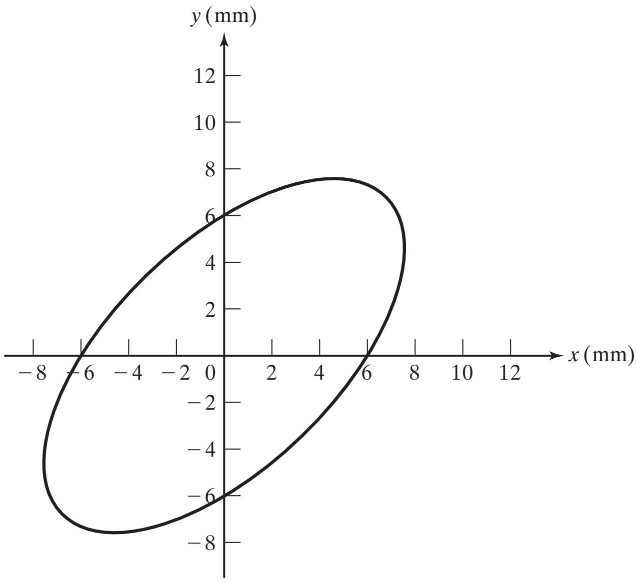


FIGURE 1.112

1.82 A machine is subjected to the motion $x(t) = A\cos(50t + \alpha)$ mm. The initial conditions are given by $x(0) = 3$ mm and $\dot{x}(0) = 1.0$ m/s.

a. Find the constants A and α.
b. Express the motion in the form $x(t) = A_1 \cos \omega t + A_2 \sin \omega t$, and identify the constants A_1 and A_2.

1.83 Show that any linear combination of $\sin \omega t$ and $\cos \omega t$ such that $x(t) = A_1 \cos \omega t + A_2 \sin \omega t$ (A_1, A_2 = constants) represents a simple harmonic motion.

1.84 Find the sum of the two harmonic motions $x_1(t) = 5\cos(3t + 1)$ and $x_2(t) = 10\cos(3t + 2)$. Use:

a. Trigonometric relations
b. Vector addition
c. Complex-number representation

1.85 If one of the components of the harmonic motion $x(t) = 10\sin(\omega t + 60°)$ is $x_1(t) = 5\sin(\omega t + 30°)$, find the other component.

1.86 Consider the two harmonic motions $x_1(t) = \frac{1}{2}\cos\frac{\pi}{2}t$ and $x_2(t) = \sin \pi t$. Is the sum $x_1(t) + x_2(t)$ a periodic motion? If so, what is its period?

1.87 Consider two harmonic motions of different frequencies: $x_1(t) = 2\cos 2t$ and $x_2(t) = \cos 3t$. Is the sum $x_1(t) + x_2(t)$ a harmonic motion? If so, what is its period?

1.88 Consider the two harmonic motions $x_1(t) = \frac{1}{2}\cos\frac{\pi}{2}t$ and $x_2(t) = \cos \pi t$. Is the difference $x(t) = x_1(t) - x_2(t)$ a harmonic motion? If so, what is its period?

1.89 Find the maximum and minimum amplitudes of the combined motion $x(t) = x_1(t) + x_2(t)$ when $x_1(t) = 3\sin 30t$ and $x_2(t) = 3\sin 29t$. Also find the frequency of beats corresponding to $x(t)$.

1.90 A machine is subjected to two harmonic motions, and the resultant motion, as displayed by an oscilloscope, is shown in Fig. 1.113. Find the amplitudes and frequencies of the two motions.

1.91 A harmonic motion has an amplitude of 0.05 m and a frequency of 10 Hz. Find its period, maximum velocity, and maximum acceleration.

1.92 An accelerometer mounted on a building frame indicates that the frame is vibrating harmonically at 15 cps, with a maximum acceleration of $0.5g$. Determine the amplitude and the maximum velocity of the building frame.

1.93 The maximum amplitude and the maximum acceleration of the foundation of a centrifugal pump were found to be $x_{max} = 0.25$ mm and $\ddot{x}_{max} = 0.4g$. Find the operating speed of the pump.

1.94 An exponential function is expressed as $x(t) = Ae^{-\alpha t}$ with the values of $x(t)$ known at $t = 1$ and $t = 2$ as $x(1) = 0.752985$ and $x(2) = 0.226795$. Determine the values of A and α.

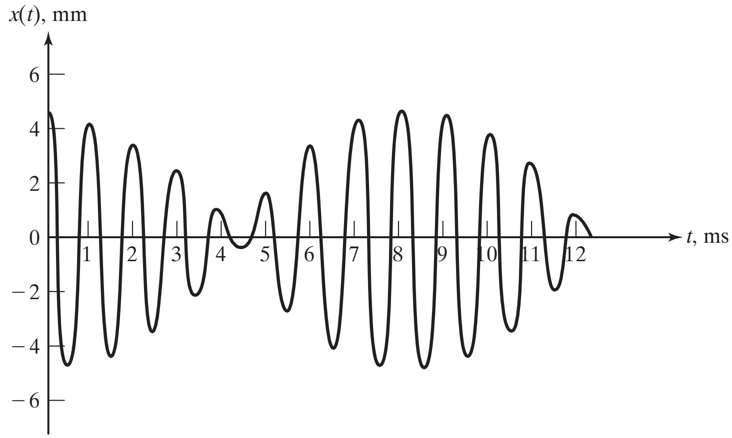


FIGURE 1.113

1.95 When the displacement of a machine is given by $x(t) = 18 \cos 8t$, where x is measured in millimeters and t in seconds, find (a) the period of the machine in sec, and (b) the frequency of oscillation of the machine in rad/sec as well as in Hz.

1.96 If the motion of a machine is described as $8 \sin(5t + 1) = A \sin 5t + B \cos 5t$, determine the values of A and B.

1.97 Express the vibration of a machine given by $x(t) = -3.0 \sin 5t - 2.0 \cos 5t$ in the form $x(t) = A \cos(5t + \phi)$.

1.98 If the displacement of a machine is given by $x(t) = 0.2 \sin(5t + 3)$, where x is in meters and t is in seconds, find the variations of the velocity and acceleration of the machine. Also find the amplitudes of displacement, velocity, and acceleration of the machine.

1.99 If the displacement of a machine is described as $x(t) = 0.15 \sin 4t + 2.0 \cos 4t$, where x is in inches and t is in seconds, find the expressions for the velocity and acceleration of the machine. Also find the amplitudes of displacement, velocity, and acceleration of the machine.

1.100 The displacement of a machine is expressed as $x(t) = 0.05 \sin(6t + \phi)$, where x is in meters and t is in seconds. If the displacement of the machine at $t = 0$ is known to be 0.04 m, determine the value of the phase angle ϕ.

1.101 The displacement of a machine is expressed as $x(t) = A \sin(6t + \phi)$, where x is in meters and t is in seconds. If the displacement and the velocity of the machine at $t = 0$ are known to be 0.05 m and 0.005 m/s, determine the values of A and ϕ.

1.102 A machine is found to vibrate with simple harmonic motion at a frequency of 20 Hz and an amplitude of acceleration of $0.5g$. Determine the displacement and velocity of the machine. Use the value of g as 9.81 m/s^2.

1.103 The amplitudes of displacement and acceleration of an unbalanced turbine rotor are found to be 0.5 mm and $0.5g$, respectively. Find the rotational speed of the rotor using the value of g as 9.81 m/s^2.

1.104 The root mean square (rms) value of a function, $x(t)$, is defined as the square root of the average of the squared value of $x(t)$ over a time period τ:

$$x_{\text{rms}} = \sqrt{\frac{1}{\tau}\int_0^{\tau}[x(t)]^2\,dt}$$

Using this definition, find the rms value of the function

$$x(t) = X\sin\omega t = X\sin\frac{2\pi t}{\tau}$$

1.105 Using the definition given in Problem 1.104, find the rms value of the function shown in Fig. 1.54(a).

Section 1.11 Harmonic Analysis

1.106 Prove that the sine Fourier components (b_n) are zero for even functions—that is, when $x(-t) = x(t)$. Also prove that the cosine Fourier components (a_0 and a_n) are zero for odd functions—that is, when $x(-t) = -x(t)$.

1.107 Find the Fourier series expansions of the functions shown in Figs. 1.58(ii) and (iii). Also, find their Fourier series expansions when the time axis is shifted down by a distance A.

1.108 The impact force created by a forging hammer can be modeled as shown in Fig. 1.114. Determine the Fourier series expansion of the impact force.

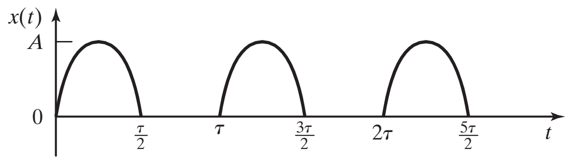


FIGURE 1.114

1.109 Find the Fourier series expansion of the periodic function shown in Fig. 1.115. Also plot the corresponding frequency spectrum.

1.110 Find the Fourier series expansion of the periodic function shown in Fig. 1.116. Also plot the corresponding frequency spectrum.

1.111 Find the Fourier series expansion of the periodic function shown in Fig. 1.117. Also plot the corresponding frequency spectrum.

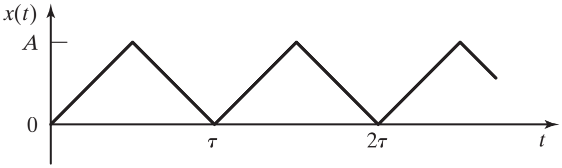


FIGURE 1.115

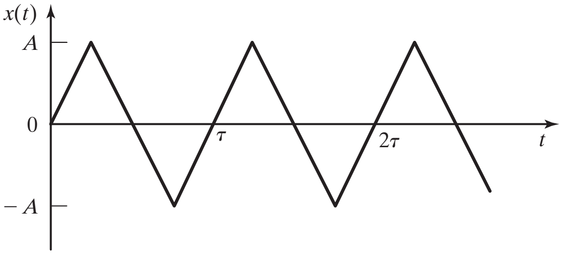


FIGURE 1.116

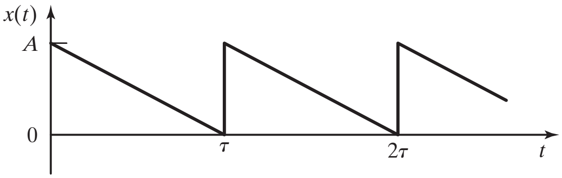


FIGURE 1.117

1.112 The Fourier series of a periodic function, $x(t)$, is an infinite series given by

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \tag{E.1}$$

where

$$a_0 = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} x(t)\, dt \tag{E.2}$$

$$a_n = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} x(t) \cos n\omega t\, dt \tag{E.3}$$

$$b_n = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} x(t) \sin n\omega t\, dt \tag{E.4}$$

ω is the circular frequency and $2\pi/\omega$ is the time period. Instead of including the infinite number of terms in Eq. (E.1), it is often truncated by retaining only k terms as

$$x(t) \approx \widetilde{x}(t) = \frac{\widetilde{a}_0}{2} + \sum_{n=1}^{k} (\widetilde{a}_n \cos n\omega t + \widetilde{b}_n \sin n\omega t) \tag{E.5}$$

so that the error, $e(t)$, becomes

$$e(t) = x(t) - \widetilde{x}(t) \tag{E.6}$$

Find the coefficients $\widetilde{a}_0$, $\widetilde{a}_n$, and $\widetilde{b}_n$ which minimize the square of the error over a time period:

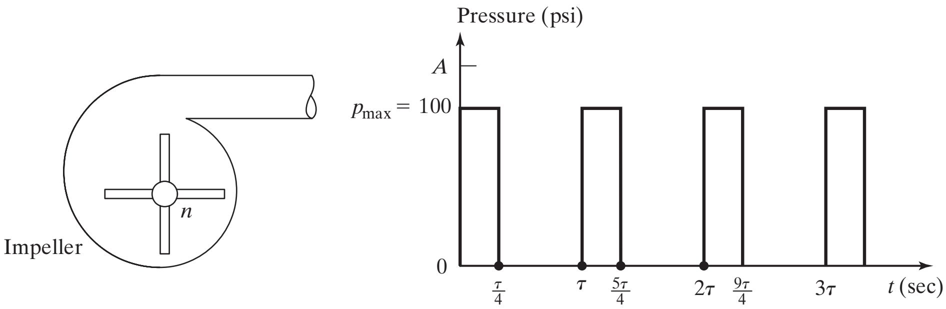
$$\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} e^2(t)\, dt \tag{E.7}$$

Compare the expressions of $\widetilde{a}_0$, $\widetilde{a}_n$, and $\widetilde{b}_n$ with Eqs. (E.2) to (E.4) and state your observation(s).

1.113 Conduct a harmonic analysis, including the first three harmonics, of the function given below:

t_i	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18
x_i	9	13	17	29	43	59	63	57	49
t_i	0.20	0.22	0.24	0.26	0.28	0.30	0.32		
x_i	35	35	41	47	41	13	7		

1.114 In a centrifugal fan (Fig. 1.118(a)), the air at any point is subjected to an impulse each time a blade passes the point, as shown in Fig. 1.118(b). The frequency of these impulses is determined by the speed of rotation of the impeller n and the number of blades, N, in the impeller. For $n = 100$ rpm and $N = 4$, determine the first three harmonics of the pressure fluctuation shown in Fig. 1.118(b).



(a) Centrifugal fan

(b) Ideal pressure fluctuation at a point

FIGURE 1.118

1.115 Solve Problem 1.114 by using the values of n and N as 200 rpm and 6 instead of 100 rpm and 4, respectively.

1.116 The torque (M_t) variation with time, of an internal combustion engine, is given in Table 1.3. Make a harmonic analysis of the torque. Find the amplitudes of the first three harmonics.

TABLE 1.3

t(s)	M_t(N-m)	t(s)	M_t(N-m)	t(s)	M_t(N-m)
0.00050	770	0.00450	1890	0.00850	1050
0.00100	810	0.00500	1750	0.00900	990
0.00150	850	0.00550	1630	0.00950	930
0.00200	910	0.00600	1510	0.01000	890
0.00250	1010	0.00650	1390	0.01050	850
0.00300	1170	0.00700	1290	0.01100	810
0.00350	1370	0.00750	1190	0.01150	770
0.00400	1610	0.00800	1110	0.01200	750

1.117 Make a harmonic analysis of the function shown in Fig. 1.119 including the first three harmonics.

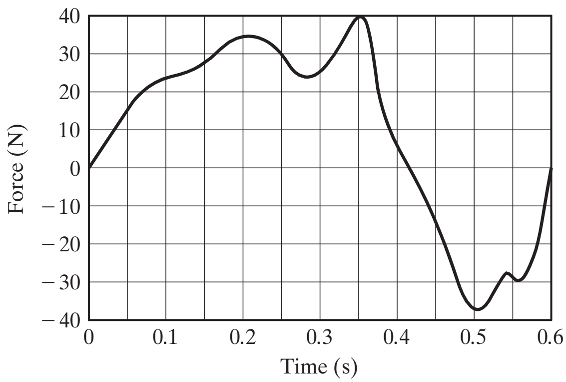


FIGURE 1.119

Section 1.12 Examples Using MATLAB

1.118 Plot the Fourier series expansion of the function $x(t)$ given in Problem 1.113 using MATLAB.

1.119 Use MATLAB to plot the variation of the force with time using the Fourier series expansion determined in Problem 1.117.

1.120 Use MATLAB to plot the variations of the damping constant c with respect to r, h, and a as determined in Problem 1.72.

1.121 Use MATLAB to plot the variation of spring stiffness (k) with deformation (x) given by the relations:

a. $k = 1000x - 100x^2;\ 0 \leq x \leq 4.$
b. $k = 500 + 500x^2;\ 0 \leq x \leq 4.$

1.122 A mass is subjected to two harmonic motions given by $x_1(t) = 3 \sin 30t$ and $x_2(t) = 3 \sin 29t$. Plot the resultant motion of the mass using MATLAB and identify the beat frequency and the beat period.

DESIGN PROJECTS

1.123* A slider-crank mechanism is shown in Fig. 1.120. Derive an expression for the motion of the piston P in terms of the crank length r, the connecting-rod length l, and the constant angular velocity of the crank ω.

a. Discuss the feasibility of using the mechanism for the generation of harmonic motion.
b. Find the value of l/r for which the amplitude of every higher harmonic is smaller than that of the first harmonic by a factor of at least 25.

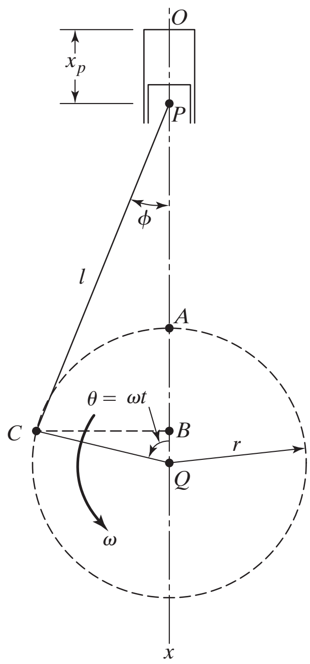


FIGURE 1.120

1.124* The vibration table shown in Fig. 1.121 is used to test certain electronic components for vibration. It consists of two identical mating gears G_1 and G_2 that rotate about the axes O_1 and O_2 attached to the frame F. Two equal masses, m each, are placed symmetrically about the middle vertical axis as shown in Fig. 1.121. During rotation, an unbalanced vertical force of magnitude $P = 2m\omega^2 r \cos\theta$, where $\theta = \omega t$ and ω = angular velocity of gears, will be developed, causing the table to vibrate. Design a vibration table that can develop a force in the range 0–100 N over a frequency range 25–50 Hz.

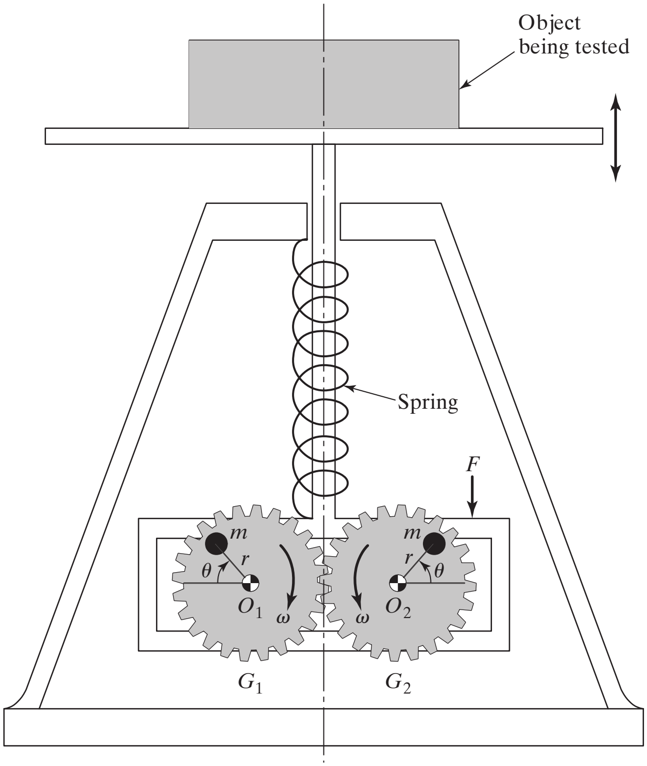


FIGURE 1.121 A vibration table.

1.125* The arrangement shown in Fig. 1.122 is used to regulate the weight of material fed from a hopper to a conveyor [1.44]. The crank imparts a reciprocating motion to the actuating rod through the wedge. The amplitude of motion imparted to the actuating rod can be varied by moving the wedge up or down. Since the conveyor is pivoted about point O, any overload on the conveyor makes the lever OA tilt downward, thereby raising the wedge. This causes a reduction in the amplitude of the actuating rod and hence the feed rate. Design such a weight-regulating system to maintain the weight at 10 ± 0.1 lb per minute.

1.126* Figure 1.123 shows a vibratory compactor. It consists of a plate cam with three profiled lobes and an oscillating roller follower. As the cam rotates, the roller drops after each rise. Correspondingly, the weight attached at the end of the follower also rises and drops. The

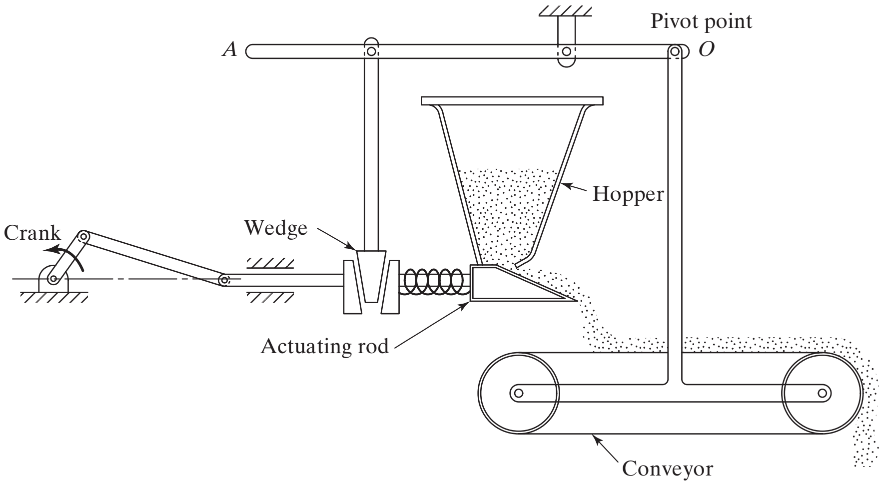


FIGURE 1.122 A vibratory weight-regulating system.

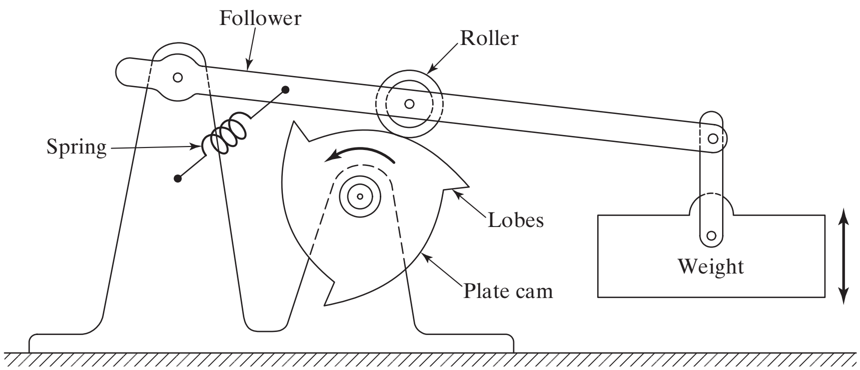


FIGURE 1.123 A vibratory compactor.

contact between the roller and the cam is maintained by the spring. Design a vibration compactor that can apply a force of 200 lb at a frequency of 50 Hz.

1.127* Vibratory bowl feeders are widely used in automated processes where a high volume of identical parts are to be oriented and delivered at a steady rate to a workstation for further tooling [1.45, 1.46]. Basically, a vibratory bowl feeder is separated from the base by a set of inclined elastic members (springs), as shown in Fig. 1.124. An electromagnetic coil mounted between the bowl and the base provides the driving force to the bowl. The vibratory motion of the bowl causes the components to move along the spiral delivery track located inside the bowl with a hopping motion. Special tooling is fixed at suitable positions along the spiral track in order to reject the parts that are defective or out of tolerance or have

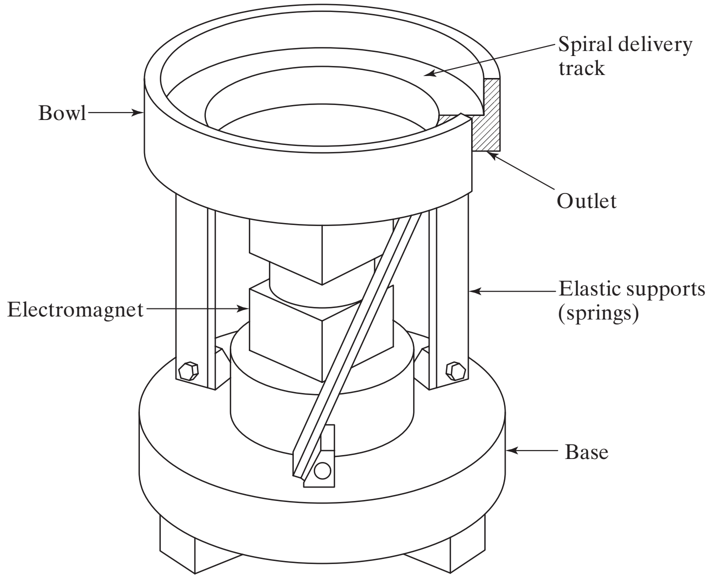


FIGURE 1.124 A vibratory bowl feeder.

incorrect orientation. What factors must be considered in the design of such vibratory bowl feeders?

1.128* The shell-and-tube exchanger shown in Fig. 1.125(a) can be modeled as shown in Fig. 1.125(b) for a simplified vibration analysis. Find the cross-sectional area of the tubes so that the total stiffness of the heat exchanger exceeds a value of 200×10^6 N/m in the axial direction and 20×10^6 N-m/rad in the tangential direction. Assume that the tubes have the same length and cross section and are spaced uniformly.

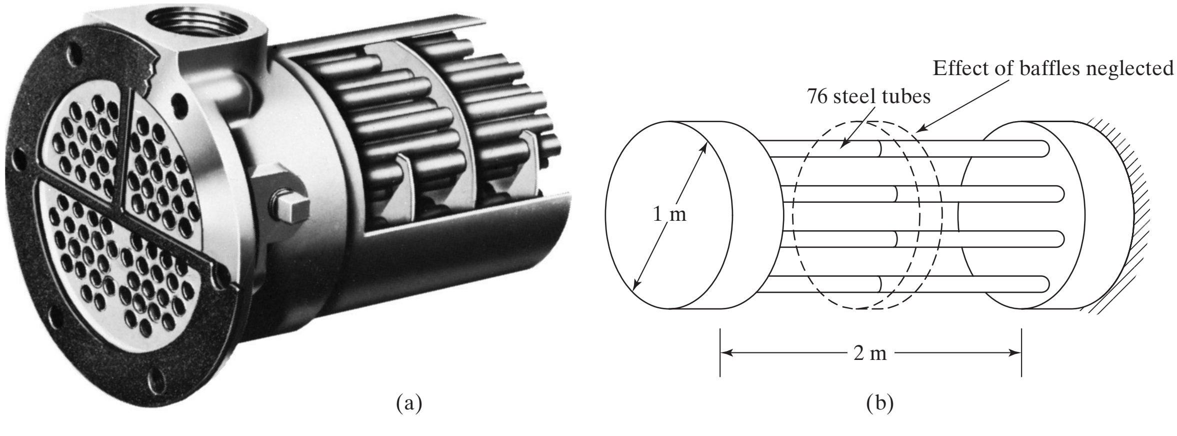


FIGURE 1.125 (Part (a) courtesy of Young Radiator Company.)

Sir Isaac Newton (1642–1727) was an English natural philosopher, a professor of mathematics at Cambridge University, and president of the Royal Society. His *Principia Mathematica* (1687), which deals with the laws and conditions of motion, is considered to be the greatest scientific work ever produced. The definitions of force, mass, and momentum and his three laws of motion crop up continually in dynamics. Quite fittingly, the unit of force named "newton" in SI units happens to be the approximate weight of an average apple, the falling object that inspired him to study the laws of gravity. (Illustration of David Eugene Smith, *History of Mathematics, Vol. I—General Survey of the History of Elementary Mathematics,* Dover Publications, Inc., New York, 1958.)

CHAPTER 2

Free Vibration of Single-Degree-of-Freedom Systems

Chapter Outline

This chapter starts with a consideration of the free vibration of an undamped single-degree-of-freedom (spring-mass) system. Free vibration means that the mass is set into motion due to initial disturbance with no externally applied force other than the spring force, damper force, or gravitational force. To study the free-vibration response of the mass, we need to derive the governing equation, known as the *equation of motion.* The equation of motion of the undamped translational system is derived using four methods. The natural frequency of vibration of the system is defined and the solution of the equation of motion is presented using appropriate initial conditions. The solution is shown to represent harmonic motion. The equation of motion and the solution corresponding to free vibration of an undamped torsional system are presented. The response of first-order systems and the time constant are considered. Rayleigh's method, based on the principle of conservation of energy, is presented with illustrative examples.

Next, the derivation of the equation for the free vibration of a viscously damped single-degree-of-freedom system and its solution are considered. The concepts of critical damping constant, damping ratio, and frequency of damped vibration are introduced. The distinctions between underdamped, critically damped, and overdamped systems are explained. The energy dissipated in viscous damping and the concepts of specific damping and loss coefficient are considered. Viscously damped torsional systems are also considered analogous to viscously damped translational systems with applications. The graphical representation of characteristic roots and the corresponding solutions as well as the concept of parameter variations and root locus plots are considered. The equations of motion and their solutions of single-degree-of-freedom systems with Coulomb and hysteretic damping are presented. The concept of complex stiffness is also presented. The idea of stability and its importance is explained along with an example. The determination of the responses of single-degree-of-freedom systems using MATLAB is illustrated with examples.

Learning Objectives

After completing this chapter, you should be able to do the following:

- Derive the equation of motion of a single-degree-of-freedom system using a suitable technique such as Newton's second law of motion, D'Alembert's principle, the principle of virtual displacements, and the principle of conservation of energy.
- Linearize the nonlinear equation of motion.
- Solve a spring-mass-damper system for different types of free-vibration response depending on the amount of damping.
- Compute the natural frequency, damped frequency, logarithmic decrement, and time constant.
- Determine whether a given system is stable or not.
- Find the responses of systems with Coulomb and hysteretic damping.
- Find the free-vibration response using MATLAB.

2.1 Introduction

A system is said to undergo free vibration when it oscillates only under an initial disturbance with no external forces acting afterward. Some examples are the oscillations of the pendulum of a grandfather clock, the vertical oscillatory motion felt by a bicyclist after hitting a road bump, and the motion of a child on a swing after an initial push.

Figure 2.1(a) shows a spring-mass system that represents the simplest possible vibratory system. It is called a *single-degree-of freedom system*, since one coordinate (x) is sufficient to specify the position of the mass at any time. There is no external force applied to the mass; hence the motion resulting from an initial disturbance will be free vibration.

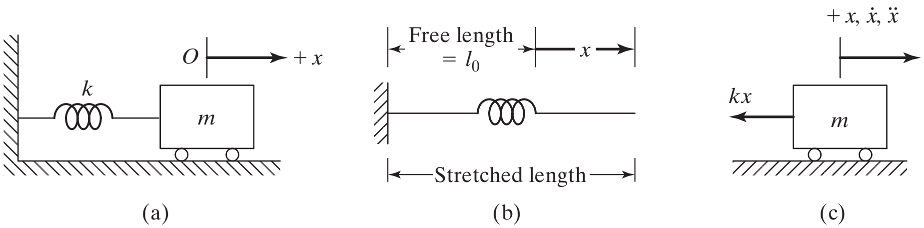


FIGURE 2.1 A spring-mass system in horizontal position.

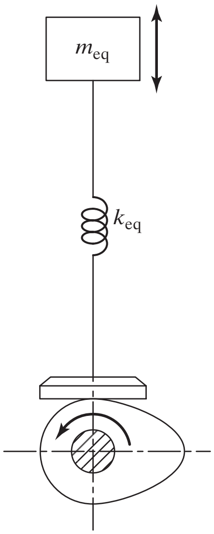


FIGURE 2.2 Equivalent spring-mass system for the cam-follower system of Fig. 1.32.

Since there is no element that causes dissipation of energy during the motion of the mass, the amplitude of motion remains constant with time; it is an *undamped* system. In actual practice, except in a vacuum, the amplitude of free vibration diminishes gradually over time, due to the resistance offered by the surrounding medium (such as air). Such vibrations are said to be *damped*. The study of the free vibration of undamped and damped single-degree-of-freedom systems is fundamental to the understanding of more advanced topics in vibrations.

Several mechanical and structural systems can be idealized as single-degree-of-freedom systems. In many practical systems, the mass is distributed, but for a simple analysis, it can be approximated by a single point mass. Similarly, the elasticity of the system, which may be distributed throughout the system, can also be idealized by a single spring. For the cam-follower system shown in Fig. 1.39, for example, the various masses were replaced by an equivalent mass (m_{eq}) in Example 1.7. The elements of the follower system (pushrod, rocker arm, valve, and valve spring) are all elastic but can be reduced to a single equivalent spring of stiffness k_{eq}. For a simple analysis, the cam-follower system can thus be idealized as a single-degree-of-freedom spring-mass system, as shown in Fig. 2.2.

Similarly, the structure shown in Fig. 2.3 can be considered a cantilever beam that is fixed at the ground. For the study of transverse vibration, the top mass can be considered a

FIGURE 2.3 The space needle (structure).

point mass and the supporting structure (beam) can be approximated as a spring to obtain the single-degree-of-freedom model shown in Fig. 2.4. The building frame shown in Fig. 2.5(a) can also be idealized as a spring-mass system, as shown in Fig. 2.5(b). In this case, since the spring constant k is merely the ratio of force to deflection, it can be determined from the geometric and material properties of the columns. The mass of the idealized system is the same as that of the floor if we assume the mass of the columns to be negligible.

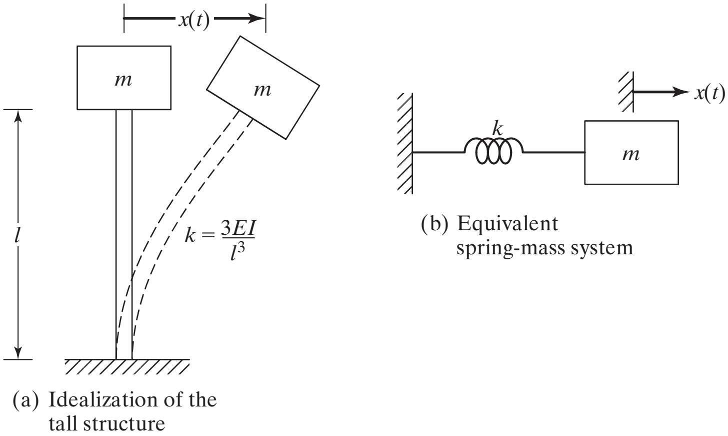


FIGURE 2.4 Modeling of tall structure as spring-mass system.

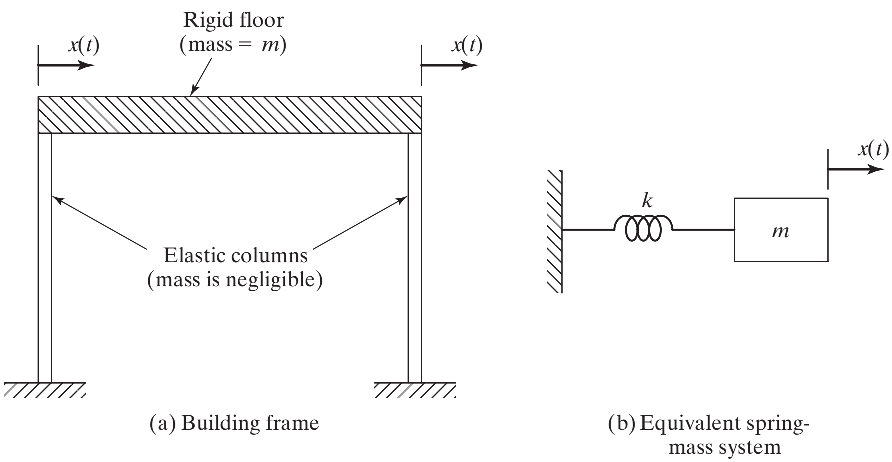


FIGURE 2.5 Idealization of a building frame.

2.2 Free Vibration of an Undamped Translational System

2.2.1 Equation of Motion Using Newton's Second Law of Motion

Using Newton's second law of motion, in this section we will consider the derivation of the equation of motion. The procedure we will use can be summarized as follows:

1. Select a suitable coordinate to describe the position of the mass or rigid body in the system. Use a linear coordinate to describe the linear motion of a point mass or the centroid of a rigid body, and an angular coordinate to describe the angular motion of a rigid body.
2. Determine the static equilibrium configuration of the system and measure the displacement of the mass or rigid body from its static equilibrium position.
3. Draw the free-body diagram of the mass or rigid body when a positive displacement and velocity are given to it. Indicate all the active and reactive forces acting on the mass or rigid body.
4. Apply Newton's second law of motion to the mass or rigid body shown by the free-body diagram. Newton's second law of motion can be stated as follows:

The rate of change of momentum of a mass is equal to the force acting on it.

Thus, if mass m is displaced a distance $\vec{x}(t)$ when acted upon by a resultant force $\vec{F}(t)$ in the same direction, Newton's second law of motion gives

$$\vec{F}(t) = \frac{d}{dt}\left(m\frac{d\vec{x}(t)}{dt}\right)$$

If mass m is constant, this equation reduces to

$$\vec{F}(t) = m\frac{d^2\vec{x}(t)}{dt^2} = m\ddot{\vec{x}} \tag{2.1}$$

where

$$\ddot{\vec{x}} = \frac{d^2\vec{x}(t)}{dt^2}$$

is the acceleration of the mass. Equation (2.1) can be stated in words as

$$\text{Resultant force on the mass} = \text{mass} \times \text{acceleration}$$

For a rigid body undergoing rotational motion, Newton's law gives

$$\vec{M}(t) = J\ddot{\vec{\theta}} \tag{2.2}$$

where $\vec{M}$ is the resultant moment acting on the body and $\vec{\theta}$ and $\ddot{\vec{\theta}} = d^2\theta(t)/dt^2$ are the resulting angular displacement and angular acceleration, respectively. Equation (2.1) or (2.2) represents the equation of motion of the vibrating system.

The procedure is now applied to the undamped single-degree-of-freedom system shown in Fig. 2.1(a). Here the mass is supported on frictionless rollers and can have translatory motion in the horizontal direction. When the mass is displaced a distance $+x$ from its static equilibrium position, the force in the spring is kx, and the free-body diagram of the mass can be represented as shown in Fig. 2.1(c). The application of Eq. (2.1) to mass m yields the equation of motion

$$F(t) = -kx = m\ddot{x}$$

or

$$m\ddot{x} + kx = 0 \tag{2.3}$$

2.2.2 Equation of Motion Using Other Methods

As stated in Section 1.6, the equations of motion of a vibrating system can be derived using several methods. The applications of D'Alembert's principle, the principle of virtual displacements, and the principle of conservation of energy are considered in this section.

D'Alembert's Principle. The equations of motion, Eqs. (2.1) and (2.2), can be rewritten as

$$\vec{F}(t) - m\ddot{\vec{x}} = 0 \tag{2.4a}$$

$$\vec{M}(t) - J\ddot{\vec{\theta}} = 0 \tag{2.4b}$$

These equations can be considered equilibrium equations provided that $-m\ddot{\vec{x}}$ and $-J\ddot{\vec{\theta}}$ are treated as a force and a moment. This fictitious force (or moment) is known as the inertia force (or inertia moment) and the artificial state of equilibrium implied by Eq. (2.4a) or (2.4b) is known as dynamic equilibrium. This principle, implied in Eq. (2.4a) or (2.4b), is called D'Alembert's principle. Applying it to the system shown in Fig. 2.1(c) yields the equation of motion:

$$-kx - m\ddot{x} = 0 \quad \text{or} \quad m\ddot{x} + kx = 0 \tag{2.3}$$

Principle of Virtual Displacements. The principle of virtual displacements states that "if a system that is in equilibrium under the action of a set of forces is subjected to a virtual displacement, then the total virtual work done by the forces will be zero." Here the virtual displacement is defined as an imaginary infinitesimal displacement given instantaneously. It must be a physically possible displacement that is compatible with the constraints of the system. The virtual work is defined as the work done by all the forces, including the inertia forces for a dynamic problem, due to a virtual displacement.

Consider a spring-mass system in a displaced position as shown in Fig. 2.6(a), where x denotes the displacement of the mass. Figure 2.6(b) shows the free-body diagram of the mass with the reactive and inertia forces indicated. When the mass is given a virtual displacement δx, as shown in Fig. 2.6(b), the virtual work done by each force can be computed as follows:

$$\text{Virtual work done by the spring force} = \delta W_s = -(kx)\,\delta x$$

$$\text{Virtual work done by the inertia force} = \delta W_i = -(m\ddot{x})\,\delta x$$

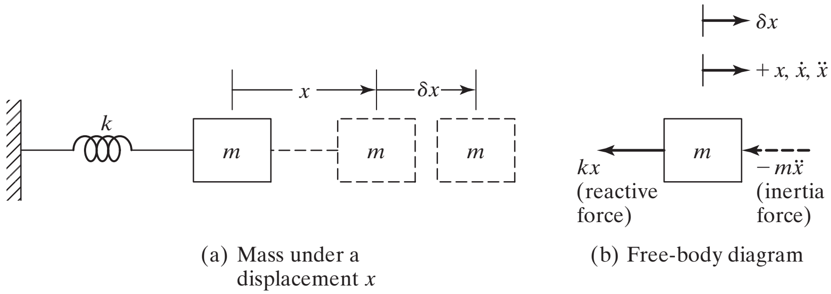


FIGURE 2.6 Mass under virtual displacement.

When the total virtual work done by all the forces is set equal to zero, we obtain

$$-m\ddot{x}\delta x - kx\,\delta x = 0 \tag{2.5}$$

Since the virtual displacement can have an arbitrary value, $\delta x \neq 0$, Eq. (2.5) gives the equation of motion of the spring-mass system as

$$m\ddot{x} + kx = 0 \tag{2.3}$$

Principle of Conservation of Energy. A system is said to be conservative if no energy is lost due to friction or energy-dissipating nonelastic members. If no work is done on a conservative system by external forces (other than gravity or other potential forces), then the total energy of the system remains constant. Since the energy of a vibrating system is partly potential and partly kinetic, the sum of these two energies remains constant. The kinetic energy T is stored in the mass by virtue of its velocity, and the potential energy U is stored in the spring by virtue of its elastic deformation. Thus the principle of conservation of energy can be expressed as:

$$T + U = \text{constant}$$

or

$$\frac{d}{dt}(T + U) = 0 \tag{2.6}$$

The kinetic and potential energies are given by

$$T = \tfrac{1}{2}m\dot{x}^2 \tag{2.7}$$

and

$$U = \tfrac{1}{2}kx^2 \tag{2.8}$$

Substitution of Eqs. (2.7) and (2.8) into Eq. (2.6) yields the desired equation

$$m\ddot{x} + kx = 0 \tag{2.3}$$

2.2.3 Equation of Motion of a Spring-Mass System in Vertical Position

Consider the configuration of the spring-mass system shown in Fig. 2.7(a). The mass hangs at the lower end of a spring, which in turn is attached to a rigid support at its upper end. At rest, the mass will hang in a position called the *static equilibrium position,* in which the upward spring force exactly balances the downward gravitational force on the mass. In this position the length of the spring is $l_0 + \delta_{st}$, where δ_{st} is the static deflection—the elongation due to the weight W of the mass m. From Fig. 2.7(a), we find that, for static equilibrium,

$$W = mg = k\delta_{st} \tag{2.9}$$

where g is the acceleration due to gravity. Let the mass be deflected a distance $+x$ from its static equilibrium position; then the spring force is $-k(x + \delta_{st})$, as shown in Fig. 2.7(c). The application of Newton's second law of motion to mass m gives

$$m\ddot{x} = -k(x + \delta_{st}) + W$$

and since $k\delta_{st} = W$, we obtain

$$m\ddot{x} + kx = 0 \tag{2.10}$$

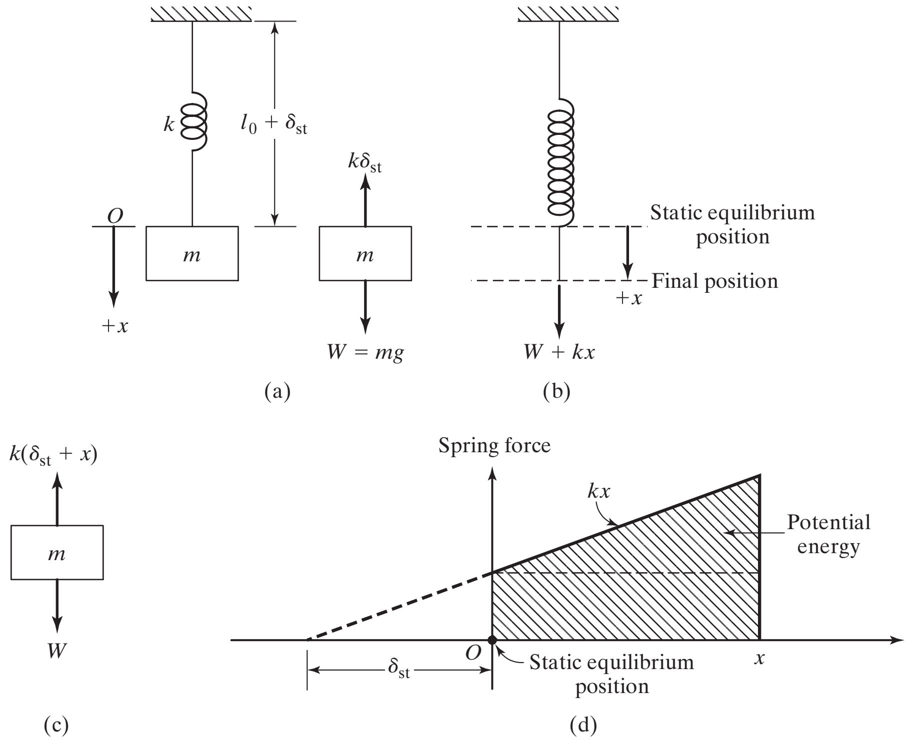


FIGURE 2.7 A spring-mass system in vertical position.

Notice that Eqs. (2.3) and (2.10) are identical. This indicates that when a mass moves in a vertical direction, we can ignore its weight, provided we measure x from its static equilibrium position.

Note: Equation (2.10), the equation of motion of the system shown in Fig. 2.7, can also be derived using D'Alembert's principle, the principle of virtual displacements, or the principle of conservation of energy. If we use the latter, for example, we note that the expression for the kinetic energy, T, remains the same as Eq. (2.7). However, the expression for the potential energy, U, is to be derived by considering the weight of the mass. For this we note that the spring force at static equilibrium position $(x = 0)$ is mg. When the spring deflects by an amount x, its potential energy is given by (see Fig. 2.7(d)):

$$mgx + \frac{1}{2}kx^2$$

Furthermore, the potential energy of the system due to the change in elevation of the mass (note that $+x$ is downward) is $-mgx$. Thus the net potential energy of the system about the static equilibrium position is given by

$$\begin{aligned} U &= \text{potential energy of the spring} \\ &\quad + \text{change in potential energy due to change in elevation of the mass } m \\ &= mgx + \frac{1}{2}kx^2 - mgx = \frac{1}{2}kx^2 \end{aligned}$$

Since the expressions of T and U remain unchanged, the application of the principle of conservation of energy gives the same equation of motion, Eq. (2.3).

2.2.4 Solution

The solution of Eq. (2.3) can be found by assuming

$$x(t) = Ce^{st} \tag{2.11}$$

where C and s are constants to be determined. Substitution of Eq. (2.11) into Eq. (2.3) gives

$$C(ms^2 + k) = 0$$

Since C cannot be zero, we have

$$ms^2 + k = 0 \tag{2.12}$$

and hence

$$s = \pm\left(-\frac{k}{m}\right)^{1/2} = \pm i\omega_n \tag{2.13}$$

where $i = (-1)^{1/2}$ and

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \tag{2.14}$$

Equation (2.12) is called the *auxiliary* or the *characteristic* equation corresponding to the differential Eq. (2.3). The two values of s given by Eq. (2.13) are the roots of the characteristic equation, also known as the *eigenvalues* or the *characteristic values* of the problem. Since both values of s satisfy Eq. (2.12), the general solution of Eq. (2.3) can be expressed as

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} \tag{2.15}$$

where C_1 and C_2 are constants. By using the identities

$$e^{\pm i\alpha t} = \cos \alpha t \pm i \sin \alpha t$$

Eq. (2.15) can be rewritten as

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{2.16}$$

where A_1 and A_2 are new constants. The constants C_1 and C_2 or A_1 and A_2 can be determined from the initial conditions of the system. Two conditions are to be specified to evaluate these constants uniquely. Note that the number of conditions to be specified is the same as the order of the governing differential equation. In the present case, if the values of displacement $x(t)$ and velocity $\dot{x}(t) = (dx/dt)(t)$ are specified as x_0 and $\dot{x}_0$ at $t = 0$, we have, from Eq. (2.16),

$$\begin{aligned} x(t = 0) &= A_1 = x_0 \\ \dot{x}(t = 0) &= \omega_n A_2 = \dot{x}_0 \end{aligned} \tag{2.17}$$

Hence $A_1 = x_0$ and $A_2 = \dot{x}_0/\omega_n$. Thus the solution of Eq. (2.3) subject to the initial conditions of Eq. (2.17) is given by

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \tag{2.18}$$

2.2.5 Harmonic Motion

Equations (2.15), (2.16), and (2.18) are harmonic functions of time. The motion is symmetric about the equilibrium position of the mass m. The velocity is a maximum and the acceleration is zero each time the mass passes through this position. At the extreme displacements, the velocity is zero and the acceleration is a maximum. Since this represents simple harmonic motion (see Section 1.10), the spring-mass system itself is called a *harmonic oscillator*. The quantity ω_n given by Eq. (2.14), represents the system's natural frequency of vibration.

Equation (2.16) can be expressed in a different form by introducing the notation

$$\begin{aligned} A_1 &= A \cos \phi \\ A_2 &= A \sin \phi \end{aligned} \tag{2.19}$$

where A and ϕ are the new constants, which can be expressed in terms of A_1 and A_2 as

$$A = (A_1^2 + A_2^2)^{1/2} = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2} = \text{amplitude}$$

$$\phi = \tan^{-1}\left(\frac{A_2}{A_1}\right) = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right) = \text{phase angle} \tag{2.20}$$

Introducing Eq. (2.19) into Eq. (2.16), the solution can be written as

$$x(t) = A\cos(\omega_n t - \phi) \tag{2.21}$$

By using the relations

$$A_1 = A_0 \sin\phi_0$$
$$A_2 = A_0 \cos\phi_0 \tag{2.22}$$

Eq. (2.16) can also be expressed as

$$x(t) = A_0 \sin(\omega_n t + \phi_0) \tag{2.23}$$

where

$$A_0 = A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2} \tag{2.24}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0\omega_n}{\dot{x}_0}\right) \tag{2.25}$$

The nature of harmonic oscillation can be represented graphically as in Fig. 2.8(a). If $\vec{A}$ denotes a vector of magnitude A, which makes an angle $\omega_n t - \phi$ with respect to the vertical (x) axis, then the solution, Eq. (2.21), can be seen to be the projection of the vector $\vec{A}$ on the x-axis. The constants A_1 and A_2 of Eq. (2.16), given by Eq. (2.19), are merely the rectangular components of $\vec{A}$ along two orthogonal axes making angles ϕ and $-(\frac{\pi}{2} - \phi)$ with respect to the vector $\vec{A}$. Since the angle $\omega_n t - \phi$ is a linear function of time, it increases linearly with time; the entire diagram thus rotates counterclockwise at an angular velocity ω_n. As the diagram (Fig. 2.8a) rotates, the projection of $\vec{A}$ onto the x-axis varies harmonically so that the motion repeats itself every time the vector $\vec{A}$ sweeps an angle of 2π. The projection of $\vec{A}$, namely $x(t)$, is shown plotted in Fig. 2.8(b) as a function of $\omega_n t$, and as a function of t in Fig. 2.8(c). The phase angle ϕ can also be interpreted as the angle between the origin and the first peak.

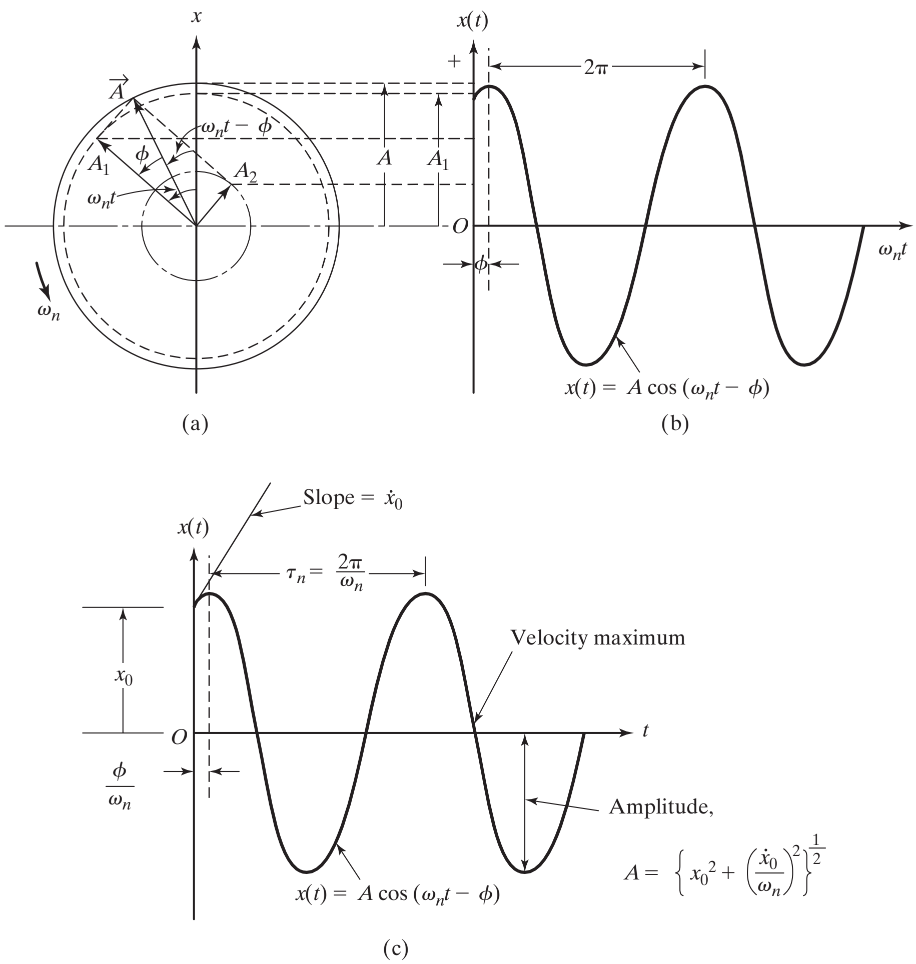


FIGURE 2.8 Graphical representation of the motion of a harmonic oscillator.

Note the following aspects of the spring-mass system:

1. If the spring-mass system is in a vertical position, as shown in Fig. 2.7(a), the circular natural frequency can be expressed as

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \tag{2.26}$$

The spring constant k can be expressed in terms of the mass m from Eq. (2.9) as

$$k = \frac{W}{\delta_{\text{st}}} = \frac{mg}{\delta_{\text{st}}} \tag{2.27}$$

Substitution of Eq. (2.27) into Eq. (2.14) yields

$$\omega_n = \left(\frac{g}{\delta_{\text{st}}}\right)^{1/2} \tag{2.28}$$

Hence the natural frequency in cycles per second and the natural period are given by

$$f_n = \frac{1}{2\pi}\left(\frac{g}{\delta_{\text{st}}}\right)^{1/2} \tag{2.29}$$

$$\tau_n = \frac{1}{f_n} = 2\pi\left(\frac{\delta_{\text{st}}}{g}\right)^{1/2} \tag{2.30}$$

Thus, when the mass vibrates in a vertical direction, we can compute the natural frequency and the period of vibration by simply measuring the static deflection δ_{st}. We don't need to know the spring stiffness k and the mass m.

2. From Eq. (2.21), the velocity $\dot{x}(t)$ and the acceleration $\ddot{x}(t)$ of the mass m at time t can be obtained as

$$\dot{x}(t) = \frac{dx}{dt}(t) = -\omega_n A \sin(\omega_n t - \phi) = \omega_n A \cos\left(\omega_n t - \phi + \frac{\pi}{2}\right)$$

$$\ddot{x}(t) = \frac{d^2x}{dt^2}(t) = -\omega_n^2 A \cos(\omega_n t - \phi) = \omega_n^2 A \cos(\omega_n t - \phi + \pi) \tag{2.31}$$

Equation (2.31) shows that the velocity leads the displacement by $\pi/2$ and the acceleration leads the displacement by π.

3. If the initial displacement (x_0) is zero, Eq. (2.21) becomes

$$x(t) = \frac{\dot{x}_0}{\omega_n}\cos\left(\omega_n t - \frac{\pi}{2}\right) = \frac{\dot{x}_0}{\omega_n}\sin\omega_n t \tag{2.32}$$

If the initial velocity $(\dot{x}_0)$ is zero, however, the solution becomes

$$x(t) = x_0 \cos\omega_n t \tag{2.33}$$

4. The response of a single-degree-of-freedom system can be represented in the displacement (x)-velocity -plane, known as the state space or phase plane. For this we consider the displacement given by Eq. (2.21) and the corresponding velocity:

$$x(t) = A\cos(\omega_n t - \phi)$$

or

$$\cos(\omega_n t - \phi) = \frac{x}{A}$$

$$\dot{x}(t) = -A\omega_n \sin(\omega_n t - \phi) \tag{2.34}$$

or

$$\sin(\omega_n t - \phi) = -\frac{\dot{x}}{A\omega_n} = -\frac{y}{A} \tag{2.35}$$

where $y = \dot{x}/\omega_n$. By squaring and adding Eqs. (2.34) and (2.35), we obtain

$$\cos^2(\omega_n t - \phi) + \sin^2(\omega_n t - \phi) = 1$$

or

$$\frac{x^2}{A^2} + \frac{y^2}{A^2} = 1 \tag{2.36}$$

The graph of Eq. (2.36) in the (x, y)-plane is a circle, as shown in Fig. 2.9(a), and it constitutes the phase-plane or state-space representation of the undamped system. The radius of the circle, A, is determined by the initial conditions of motion. Note that the graph of Eq. (2.36) in the $(x, \dot{x})$-plane will be an ellipse, as shown in Fig. 2.9(b).

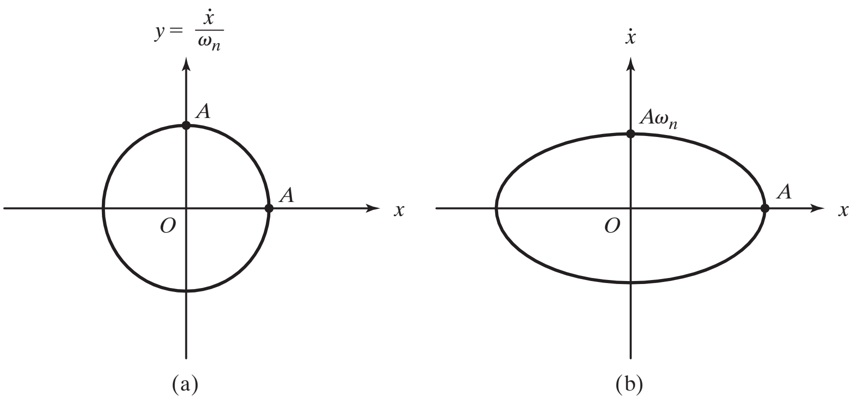


FIGURE 2.9 Phase-plane representation of an undamped system.

EXAMPLE 2.1 Harmonic Response of a Water Tank

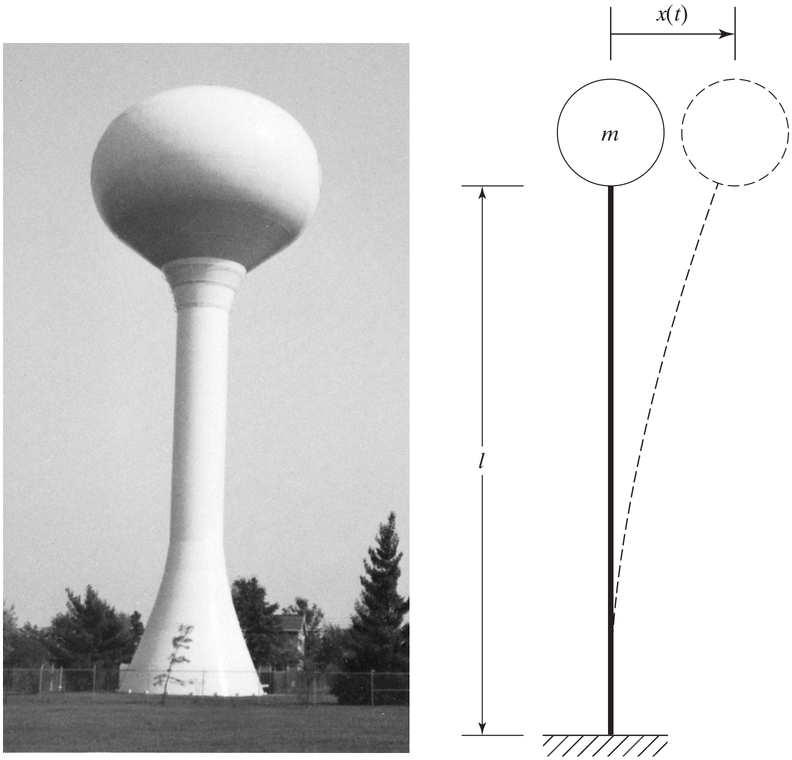
The column of the water tank shown in Fig. 2.10(a) is 300 ft high and is made of reinforced concrete with a tubular cross section of inner diameter 8 ft and outer diameter 10 ft. The tank weighs 6×10^5 lb when filled with water. By neglecting the mass of the column and assuming the Young's modulus of reinforced concrete as 4×10^6 psi, determine the following:

a. the natural frequency and the natural time period of transverse vibration of the water tank.
b. the vibration response of the water tank due to an initial transverse displacement of 10 in.
c. the maximum values of the velocity and acceleration experienced by the water tank.

Solution: Assuming that the water tank is a point mass, the column has a uniform cross section, and the mass of the column is negligible, the system can be modeled as a cantilever beam with a concentrated load (weight) at the free end as shown in Fig. 2.10(b).

a. The transverse deflection of the beam, δ, due to a load P is given by $\frac{Pl^3}{3EI}$, where l is the length, E is the Young's modulus, and I is the area moment of inertia of the beam's cross section. The stiffness of the beam (column of the tank) is given by

$$k = \frac{P}{\delta} = \frac{3EI}{l^3}$$



(a) (b)

FIGURE 2.10 Elevated tank. (Photo courtesy of West Lafayette Water Company.)

In the present case, $l = 3600$ in., $E = 4 \times 10^6$ psi,

$$I = \frac{\pi}{64}(d_0^4 - d_i^4) = \frac{\pi}{64}(120^4 - 96^4) = 600.9554 \times 10^4 \text{ in.}^4$$

and hence

$$k = \frac{3(4 \times 10^6)(600.9554 \times 10^4)}{3600^3} = 1545.6672 \text{ lb/in.}$$

The natural frequency of the water tank in the transverse direction is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1545.6672 \times 386.4}{6 \times 10^5}} = 0.9977 \text{ rad/sec}$$

The natural time period of transverse vibration of the tank is given by

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{0.9977} = 6.2977 \text{ sec}$$

b. Using the initial displacement of $x_0 = 10$ in. and the initial velocity of the water tank $(\dot{x}_0)$ as zero, the harmonic response of the water tank can be expressed, using Eq. (2.23), as

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where the amplitude of transverse displacement (A_0) is given by

$$A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2} = x_0 = 10 \text{ in.}$$

and the phase angle (ϕ_0) by

$$\phi_0 = \tan^{-1}\left(\frac{x_0\omega_n}{0}\right) = \frac{\pi}{2}$$

Thus

$$x(t) = 10 \sin\left(0.9977t + \frac{\pi}{2}\right) = 10 \cos 0.9977t \text{ in.} \tag{E.1}$$

c. The velocity of the water tank can be found by differentiating Eq. (E.1) as

$$\dot{x}(t) = 10(0.9977) \cos\left(0.9977t + \frac{\pi}{2}\right) \tag{E.2}$$

and hence

$$\dot{x}_{\max} = A_0\omega_n = 10(0.9977) = 9.977 \text{ in./sec}$$

The acceleration of the water tank can be determined by differentiating Eq. (E.2) as

$$\ddot{x}(t) = -10(0.9977)^2 \sin\left(0.9977t + \frac{\pi}{2}\right) \tag{E.3}$$

and hence the maximum value of acceleration is given by

$$\ddot{x}_{\max} = A_0(\omega_n)^2 = 10(0.9977)^2 = 9.9540 \text{ in./sec}^2$$

■

EXAMPLE 2.2 Free-Vibration Response Due to Impact

A cantilever beam carries a mass M at the free end as shown in Fig. 2.11(a). A mass m falls from a height h onto the mass M and adheres to it without rebounding. Determine the resulting transverse vibration of the beam.

Solution: When the mass m falls through a height h, it will strike the mass M with a velocity of $v_m = \sqrt{2gh}$, where g is the acceleration due to gravity. Since the mass m adheres to M without rebounding, the velocity of the combined mass $(M + m)$ immediately after the impact $(\dot{x}_0)$ can be found using the principle of conservation of momentum:

$$mv_m = (M + m)\dot{x}_0$$

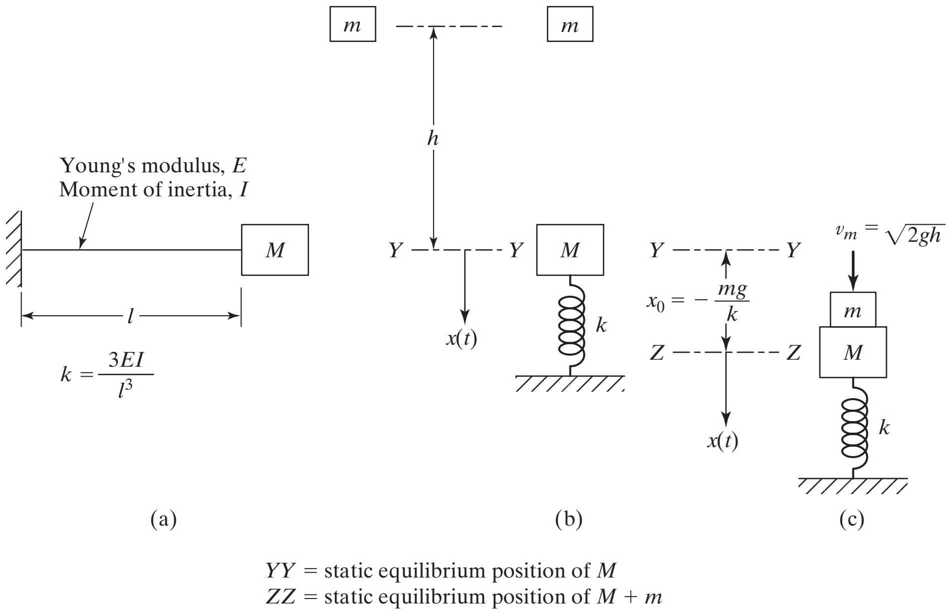


FIGURE 2.11 Response due to impact.

or

$$\dot{x}_0 = \left(\frac{m}{M + m}\right)v_m = \left(\frac{m}{M + m}\right)\sqrt{2gh} \tag{E.1}$$

The static equilibrium position of the beam with the new mass $(M + m)$ is located at a distance of $\frac{mg}{k}$ below the static equilibrium position of the original mass (M) as shown in Fig. 2.11(c). Here k denotes the stiffness of the cantilever beam, given by

$$k = \frac{3EI}{l^3}$$

Since free vibration of the beam with the new mass $(M + m)$ occurs about its own static equilibrium position, the initial conditions of the problem can be stated as

$$x_0 = -\frac{mg}{k}, \qquad \dot{x}_0 = \left(\frac{m}{M + m}\right)\sqrt{2gh} \tag{E.2}$$

Thus the resulting free transverse vibration of the beam can be expressed as (see Eq. (2.21)):

$$x(t) = A\cos(\omega_n t - \phi)$$

where

$$A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right]^{1/2}$$

$$\phi = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right)$$

$$\omega_n = \sqrt{\frac{k}{M + m}} = \sqrt{\frac{3EI}{l^3(M + m)}}$$

with x_0 and $\dot{x}_0$ given by Eq. (E.2).

■

EXAMPLE 2.3 Young's Modulus from Natural Frequency Measurement

A simply supported beam of square cross section 5 mm × 5 mm and length 1 m, carrying a mass of 2.3 kg at the middle, is found to have a natural frequency of transverse vibration of 30 rad/s. Determine the Young's modulus of elasticity of the beam.

Solution: By neglecting the self weight of the beam, the natural frequency of transverse vibration of the beam can be expressed as

$$\omega_n = \sqrt{\frac{k}{m}} \tag{E.1}$$

where

$$k = \frac{192EI}{l^3} \tag{E.2}$$

where E is the Young's modulus, l is the length, and I is the area moment of inertia of the beam:

$$I = \frac{1}{12}(5 \times 10^{-3})(5 \times 10^{-3})^3 = 0.5208 \times 10^{-10} \text{ m}^4$$

Since $m = 2.3$ kg, $l = 1.0$ m, and $\omega_n = 30.0$ rad/s, Eqs. (E.1) and (E.2) yield

$$k = \frac{192EI}{l^3} = m\omega_n^2$$

or

$$E = \frac{m\omega_n^2 l^3}{192I} = \frac{2.3(30.0)^2(1.0)^3}{192(0.5208 \times 10^{-10})} = 207.0132 \times 10^9 \text{ N/m}^2$$

This indicates that the material of the beam is probably carbon steel. ■

EXAMPLE 2.4 Natural Frequency of Cockpit of a Firetruck

The cockpit of a firetruck is located at the end of a telescoping boom, as shown in Fig. 2.12(a). The cockpit, along with the fireman, weighs 2000 N. Find the cockpit's natural frequency of vibration in the vertical direction.

Data: Young's modulus of the material: $E = 2.1 \times 10^{11}$ N/m^2; lengths: $l_1 = l_2 = l_3 = 3$ m; cross-sectional areas: $A_1 = 20$ cm^2, $A_2 = 10$ cm^2, $A_3 = 5$ cm^2.

Solution: To determine the system's natural frequency of vibration, we find the equivalent stiffness of the boom in the vertical direction and use a single-degree-of-freedom idealization. For this we assume that the mass of the telescoping boom is negligible and the telescoping boom can deform only in the axial direction (with no bending). Since the force induced at any cross section O_1O_2 is equal to the axial load applied at the end of the boom, as shown in Fig. 2.12(b), the axial stiffness of the boom (k_b) is given by

$$\frac{1}{k_b} = \frac{1}{k_{b_1}} + \frac{1}{k_{b_2}} + \frac{1}{k_{b_3}} \tag{E.1}$$

where k_{b_i} denotes the axial stiffness of the ith segment of the boom:

$$k_{b_i} = \frac{A_i E_i}{l_i}; \qquad i = 1, 2, 3 \tag{E.2}$$

From the known data ($l_1 = l_2 = l_3 = 3$ m, $A_1 = 20$ cm^2, $A_2 = 10$ cm^2; $A_3 = 5$ cm^2, $E_1 = E_2 = E_3 = 2.1 \times 10^{11}$ N/m^2),

$$k_{b_1} = \frac{(20 \times 10^{-4})(2.1 \times 10^{11})}{3} = 14 \times 10^7 \text{ N/m}$$

$$k_{b_2} = \frac{(10 \times 10^{-4})(2.1 \times 10^{11})}{3} = 7 \times 10^7 \text{ N/m}$$

$$k_{b_3} = \frac{(5 \times 10^{-4})(2.1 \times 10^{11})}{3} = 3.5 \times 10^7 \text{ N/m}$$

Thus Eq. (E.1) gives

$$\frac{1}{k_b} = \frac{1}{14 \times 10^7} + \frac{1}{7 \times 10^7} + \frac{1}{3.5 \times 10^7} = \frac{1}{2 \times 10^7}$$

or

$$k_b = 2 \times 10^7 \text{ N/m}$$

The stiffness of the telescoping boom in the vertical direction, k, can be determined as

$$k = k_b \cos^2 45° = 10^7 \text{ N/m}$$

The natural frequency of vibration of the cockpit in the vertical direction is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{(10^7)(9.81)}{2000}} = 221.4723 \text{ rad/s}$$

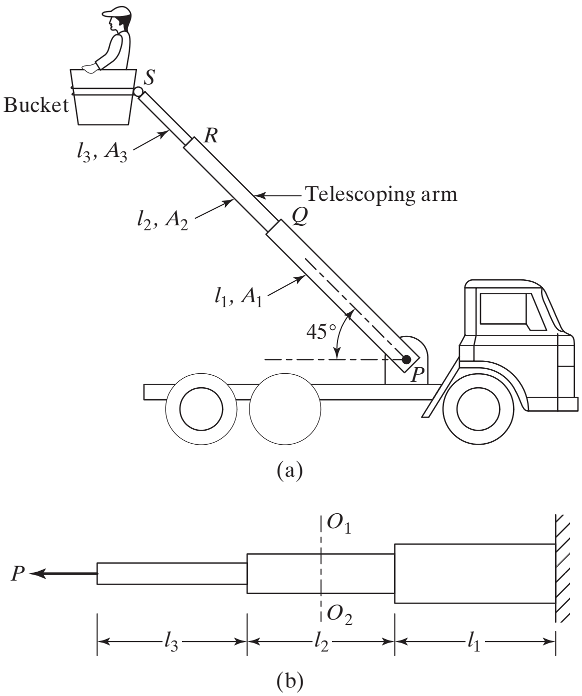


FIGURE 2.12 Telescoping boom of a fire truck.

■

EXAMPLE 2.5 Natural Frequency of Pulley System

Determine the natural frequency of the system shown in Fig. 2.13(a). Assume the pulleys to be frictionless and of negligible mass.

Solution: To determine the natural frequency, we find the equivalent stiffness of the system and solve it as a single-degree-of-freedom problem. Since the pulleys are frictionless and massless, the tension in the rope is constant and is equal to the weight W of the mass m. From the static equilibrium of the pulleys and the mass (see Fig. 2.13(b)), it can be seen that the upward force acting on pulley 1 is $2W$ and the downward force acting on pulley 2 is $2W$. The center of pulley 1 (point A) moves up by a distance $2W/k_1$, and the center of pulley 2 (point B) moves down by $2W/k_2$. Thus the total movement of the mass m (point O) is

$$2\left(\frac{2W}{k_1} + \frac{2W}{k_2}\right)$$

as the rope on either side of the pulley is free to move the mass downward. If k_{eq} denotes the equivalent spring constant of the system,

$$\frac{\text{Weight of the mass}}{\text{Equivalent spring constant}} = \text{Net displacement of the mass}$$

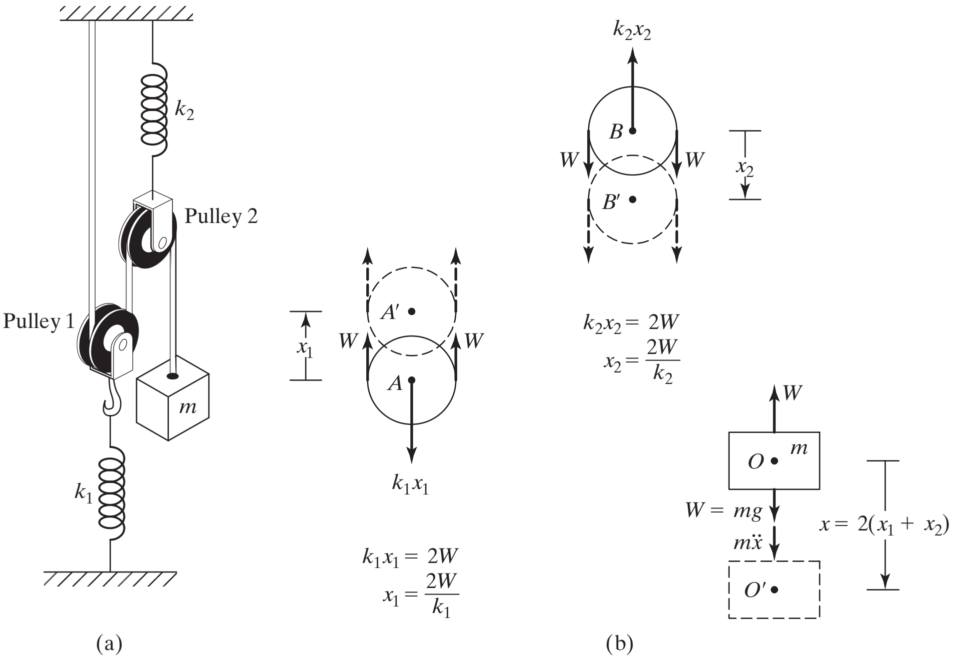


FIGURE 2.13 Pulley system.

$$\frac{W}{k_{\text{eq}}} = 4W\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = \frac{4W(k_1 + k_2)}{k_1 k_2}$$

$$k_{\text{eq}} = \frac{k_1 k_2}{4(k_1 + k_2)} \tag{E.1}$$

By displacing mass m from the static equilibrium position by x, the equation of motion of the mass can be written as

$$m\ddot{x} + k_{\text{eq}}x = 0 \tag{E.2}$$

and hence the natural frequency is given by

$$\omega_n = \left(\frac{k_{\text{eq}}}{m}\right)^{1/2} = \left[\frac{k_1 k_2}{4m(k_1 + k_2)}\right]^{1/2} \text{rad/sec} \tag{E.3}$$

or

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{4\pi}\left[\frac{k_1 k_2}{m(k_1 + k_2)}\right]^{1/2} \text{cycles/sec} \tag{E.4}$$

■

2.3 Free Vibration of an Undamped Torsional System

If a rigid body oscillates about a specific reference axis, the resulting motion is called *torsional vibration.* In this case, the displacement of the body is measured in terms of an angular coordinate. In a torsional vibration problem, the restoring moment may be due to the torsion of an elastic member or to the unbalanced moment of a force or couple.

Figure 2.14 shows a disc, which has a polar mass moment of inertia J_0, mounted at one end of a solid circular shaft, the other end of which is fixed. Let the angular rotation of the disc about the axis of the shaft be θ; θ also represents the shaft's angle of twist. From the theory of torsion of circular shafts [2.1], we have the relation

$$M_t = \frac{GI_0}{l}\theta \tag{2.37}$$

where M_t is the torque that produces the twist θ, G is the shear modulus, l is the length of the shaft, I_o is the polar moment of inertia of the cross section of the shaft, given by

$$I_0 = \frac{\pi d^4}{32} \tag{2.38}$$

and d is the diameter of the shaft. If the disc is displaced by θ from its equilibrium position, the shaft provides a restoring torque of magnitude M_t. Thus the shaft acts as a torsional spring with a torsional spring constant

$$k_t = \frac{M_t}{\theta} = \frac{GI_0}{l} = \frac{\pi G d^4}{32l} \tag{2.39}$$

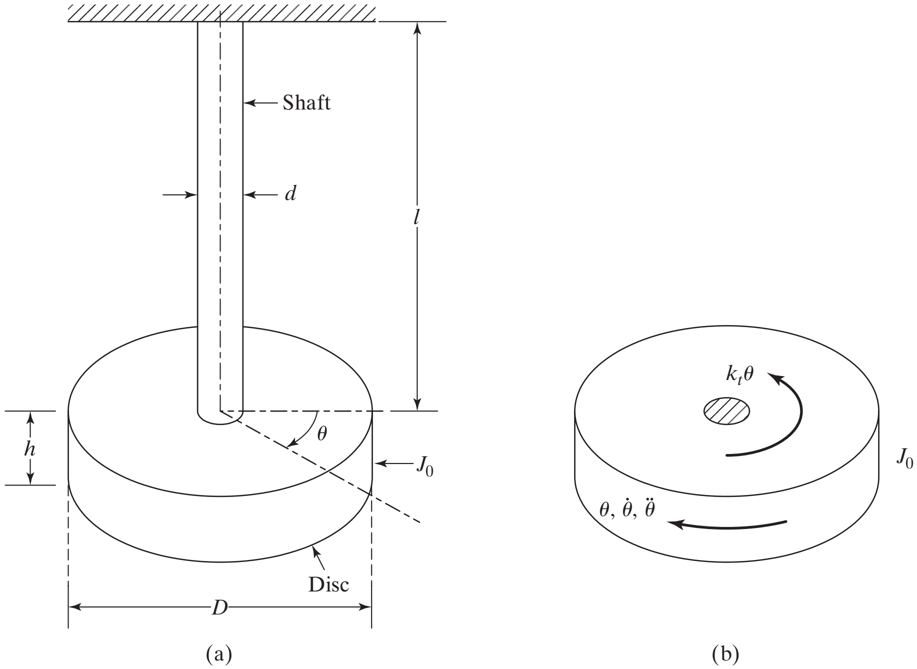


FIGURE 2.14 Torsional vibration of a disc.

2.3.1 Equation of Motion

The equation of the angular motion of the disc about its axis can be derived by using Newton's second law or any of the methods discussed in Section 2.2.2. By considering the free-body diagram of the disc (Fig. 2.14b), we can derive the equation of motion by applying Newton's second law of motion:

$$J_0\ddot{\theta} + k_t\theta = 0 \tag{2.40}$$

which can be seen to be identical to Eq. (2.3) if the polar mass moment of inertia J_0, the angular displacement θ, and the torsional spring constant k_t are replaced by the mass m, the displacement x, and the linear spring constant k, respectively. Thus the natural circular frequency of the torsional system is

$$\omega_n = \left(\frac{k_t}{J_0}\right)^{1/2} \tag{2.41}$$

and the period and frequency of vibration in cycles per second are

$$\tau_n = 2\pi\left(\frac{J_0}{k_t}\right)^{1/2} \tag{2.42}$$

$$f_n = \frac{1}{2\pi}\left(\frac{k_t}{J_0}\right)^{1/2} \tag{2.43}$$

Note the following aspects of this system:

1. If the cross section of the shaft supporting the disc is not circular, an appropriate torsional spring constant is to be used [2.4, 2.5].
2. The polar mass moment of inertia of a disc is given by

$$J_0 = \frac{\rho h \pi D^4}{32} = \frac{W D^2}{8g}$$

where ρ is the mass density, h is the thickness, D is the diameter, and W is the weight of the disc.
3. The torsional spring-inertia system shown in Fig. 2.14 is referred to as a *torsional pendulum*. One of the most important applications of a torsional pendulum is in a mechanical clock, where a ratchet and pawl convert the regular oscillation of a small torsional pendulum into the movements of the hands.

2.3.2 Solution

The general solution of Eq. (2.40) can be obtained, as in the case of Eq. (2.3):

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{2.44}$$

where ω_n is given by Eq. (2.41) and A_1 and A_2 can be determined from the initial conditions. If

$$\theta(t = 0) = \theta_0 \quad \text{and} \quad \dot{\theta}(t = 0) = \frac{d\theta}{dt}(t = 0) = \dot{\theta}_0 \tag{2.45}$$

the constants A_1 and A_2 can be found:

$$\begin{aligned} A_1 &= \theta_0 \\ A_2 &= \dot{\theta}_0/\omega_n \end{aligned} \tag{2.46}$$

Equation (2.44) can also be seen to represent a simple harmonic motion.

EXAMPLE 2.6 Natural Frequency of Compound Pendulum

Any rigid body pivoted at a point other than its center of mass will oscillate about the pivot point under its own gravitational force. Such a system is known as a compound pendulum (Fig. 2.15). Find the natural frequency of such a system.

Solution: Let O be the point of suspension and G be the center of mass of the compound pendulum, as shown in Fig. 2.15. Let the rigid body oscillate in the xy-plane so that the coordinate θ can be used to describe its motion. Let d denote the distance between O and G, and J_0 the mass moment of inertia of the body about the z-axis (perpendicular to both x and y). For a displacement θ, the restoring torque (due to the weight of the body W) is $(Wd \sin \theta)$ and the equation of motion is

$$J_0 \ddot{\theta} + Wd \sin \theta = 0 \tag{E.1}$$

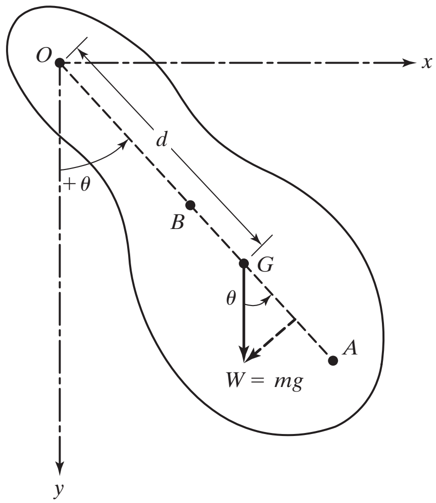


FIGURE 2.15 Compound pendulum.

Note that Eq. (E.1) is a second-order nonlinear ordinary differential equation. Although it is possible to find an exact solution of this equation (see Section 13.3), exact solutions cannot be found for most nonlinear differential equations. An approximate solution of Eq. (E.1) can be found by one of two methods. A numerical procedure can be used to integrate Eq. (E.1). Alternatively, Eq. (E.1) can be approximated by a linear equation whose exact solution can be determined readily. To use the latter approach, we assume small angular displacements so that θ is small and $\sin\theta \approx \theta$. Hence Eq. (E.1) can be approximated by the linear equation:

$$J_0\ddot{\theta} + Wd\theta = 0 \tag{E.2}$$

This gives the natural frequency of the compound pendulum:

$$\omega_n = \left(\frac{Wd}{J_0}\right)^{1/2} = \left(\frac{mgd}{J_0}\right)^{1/2} \tag{E.3}$$

Comparing Eq. (E.3) with the natural frequency of a simple pendulum, $\omega_n = (g/l)^{1/2}$ (see Problem 2.61), we can find the length of the equivalent simple pendulum:

$$l = \frac{J_0}{md} \tag{E.4}$$

If J_0 is replaced by mk_0^2, where k_0 is the radius of gyration of the body about O, Eqs. (E.3) and (E.4) become

$$\omega_n = \left(\frac{gd}{k_0^2}\right)^{1/2} \tag{E.5}$$

$$l = \left(\frac{k_0^2}{d}\right) \tag{E.6}$$

If k_G denotes the radius of gyration of the body about G, we have

$$k_0^2 = k_G^2 + d^2 \tag{E.7}$$

and Eq. (E.6) becomes

$$l = \left(\frac{k_G^2}{d} + d\right) \tag{E.8}$$

If the line OG is extended to point A such that

$$GA = \frac{k_G^2}{d} \tag{E.9}$$

Eq. (E.8) becomes

$$l = GA + d = OA \tag{E.10}$$

Hence, from Eq. (E.5), ω_n is given by

$$\omega_n = \left\{\frac{g}{(k_0^2/d)}\right\}^{1/2} = \left(\frac{g}{l}\right)^{1/2} = \left(\frac{g}{OA}\right)^{1/2} \tag{E.11}$$

This equation shows that, no matter whether the body is pivoted from O or A, its natural frequency is the same. The point A is called the *center of percussion.*

■

Center of Percussion. The concepts of compound pendulum and center of percussion can be used in many practical applications:

1. A hammer can be shaped to have the center of percussion at the hammer head while the center of rotation is at the handle. In this case, the impact force at the hammer head will not cause any normal reaction at the handle (Fig. 2.16(a)).
2. In a baseball bat, if on one hand the ball is made to strike at the center of percussion while the center of rotation is at the hands, no reaction perpendicular to the bat will be experienced by the batter (Fig. 2.16(b)). On the other hand, if the ball strikes the bat near the free end or near the hands, the batter will experience pain in the hands as a result of the reaction perpendicular to the bat.
3. In Izod (impact) testing of materials, the specimen is suitably notched and held in a vise fixed to the base of the machine (see Fig. 2.16(c)). A pendulum is released from a standard height, and the free end of the specimen is struck by the pendulum as it passes through its lowest position. The deformation and bending of the pendulum can be reduced if the center of percussion is located near the striking edge. In this case, the pivot will be free of any impulsive reaction.
4. In an automobile (shown in Fig. 2.16(d)), if the front wheels strike a bump, the passengers will not feel any reaction if the center of percussion of the vehicle is located near the rear axle. Similarly, if the rear wheels strike a bump at point A, no reaction

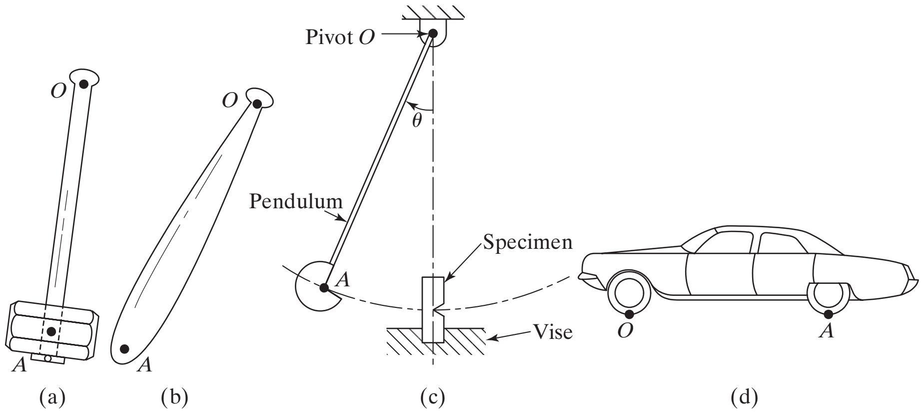


FIGURE 2.16 Applications of center of percussion.

will be felt at the front axle (point O) if the center of percussion is located near the front axle. It is desirable, therefore, to have the center of oscillation of the vehicle at one axle and the center of percussion at the other axle [2.2].

2.4 Response of First-Order Systems and Time Constant

Consider a turbine rotor mounted in bearings as shown in Fig. 2.17(a). The viscous fluid (lubricant) in the bearings offers viscous damping torque during the rotation of the turbine rotor. Assuming the mass moment of inertia of the rotor about the axis of rotation as J and the rotational damping constant of the bearings as c_t, the application of Newton's second law of motion yields the equation of motion of the rotor as

$$J\dot{\omega} + c_t\omega = 0 \tag{2.47}$$

where ω is the angular velocity of the rotor, $\dot{\omega} = \dfrac{d\omega}{dt}$ is the time rate of change of the angular velocity, and the external torque applied to the system is assumed to be zero. We assume the initial angular velocity, $\omega(t = 0) = \omega_0$, as the input and the angular velocity of the rotor as the output of the system. Note that the angular velocity, instead of the angular displacement, is considered as the output in order to obtain the equation of motion as a first order differential equation.

The solution of the equation of motion of the rotor, Eq. (2.47), can be found by assuming the trial solution as

$$\omega(t) = Ae^{st} \tag{2.48}$$

where A and s are unknown constants. By using the initial condition, $\omega(t = 0) = \omega_0$, Eq. (2.48) can be written as

$$\omega(t) = \omega_0 e^{st} \tag{2.49}$$

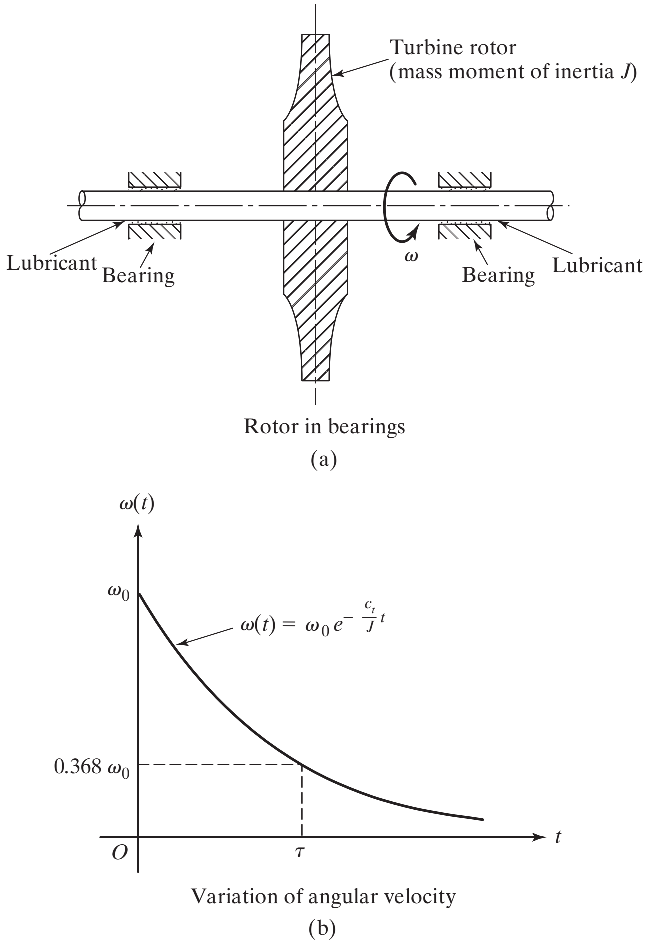


FIGURE 2.17

By substituting Eq. (2.49) into Eq. (2.47), we obtain

$$\omega_0 e^{st}(Js + c_t) = 0 \tag{2.50}$$

Since $\omega_0 = 0$ leads to "no motion" of the rotor, we assume $\omega_0 \neq 0$ and Eq. (2.50) can be satisfied only if

$$Js + c_t = 0 \tag{2.51}$$

Equation (2.51) is known as the characteristic equation which yields $s = -\dfrac{c_t}{J}$. Thus the solution, Eq. (2.49), becomes

$$\omega(t) = \omega_0 e^{-\frac{c_t}{J}t} \tag{2.52}$$

The variation of the angular velocity, given by Eq. (2.52), with time is shown in Fig. 2.17(b). The curve starts at ω_0, decays and approaches zero as t increases without limit. In dealing with exponentially decaying responses, such as the one given by Eq. (2.52), it is convenient to describe the response in terms of a quantity known as the *time constant* (τ). The time constant is defined as the value of time which makes the exponent in Eq. (2.52) equal to -1. Because the exponent of Eq. (2.52) is known to be $-\frac{c_t}{J}t$, the time constant will be equal to

$$\tau = \frac{J}{c_t} \tag{2.53}$$

so that Eq. (2.52) gives, for $t = \tau$,

$$\omega(t) = \omega_0 e^{-\frac{c_t}{J}\tau} = \omega_0 e^{-1} = 0.368\omega_0 \tag{2.54}$$

Thus the response reduces to 0.368 times its initial value at a time equal to the time constant of the system.

2.5 Rayleigh's Energy Method

For a single-degree-of-freedom system, the equation of motion was derived using the energy method in Section 2.2.2. In this section, we shall use the energy method to find the natural frequencies of single-degree-of-freedom systems. The principle of conservation of energy, in the context of an undamped vibrating system, can be restated as

$$T_1 + U_1 = T_2 + U_2 \tag{2.55}$$

where the subscripts 1 and 2 denote two different instants of time. Specifically, we use the subscript 1 to denote the time when the mass is passing through its static equilibrium position and choose $U_1 = 0$ as reference for the potential energy. If we let the subscript 2 indicate the time corresponding to the maximum displacement of the mass, we have $T_2 = 0$. Thus Eq. (2.55) becomes

$$T_1 + 0 = 0 + U_2 \tag{2.56}$$

If the system is undergoing harmonic motion, then T_1 and U_2 denote the maximum values of T and U, respectively, and Eq. (2.56) becomes

$$T_{\max} = U_{\max} \tag{2.57}$$

The application of Eq. (2.57), which is also known as *Rayleigh's energy method,* gives the natural frequency of the system directly, as illustrated in the following examples.

EXAMPLE 2.7 Manometer for Diesel Engine

The exhaust from a single-cylinder four-stroke diesel engine is to be connected to a silencer, and the pressure therein is to be measured with a simple U-tube manometer (see Fig. 2.18). Calculate the minimum length of the manometer tube so that the natural frequency of oscillation of the mercury column will be 3.5 times slower than the frequency of the pressure fluctuations in the silencer at an engine speed of 600 rpm. The frequency of pressure fluctuation in the silencer is equal to

$$\frac{\text{Number of cylinders} \times \text{Speed of the engine}}{2}$$

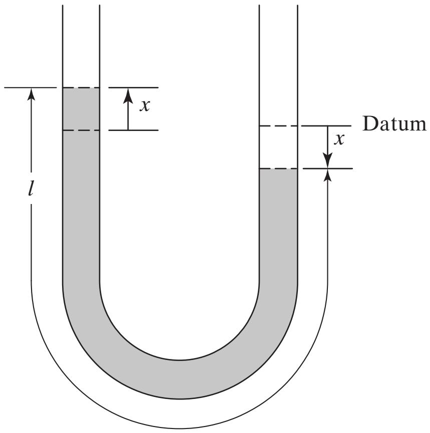


FIGURE 2.18 U-tube manometer.

Solution

1. *Natural frequency of oscillation of the liquid column:* Let the datum in Fig. 2.18 be taken as the equilibrium position of the liquid. If the displacement of the liquid column from the equilibrium position is denoted by x, the change in potential energy is given by

$$\begin{aligned} U &= \text{potential energy of raised liquid column} + \text{potential energy of depressed liquid column} \\ &= (\text{weight of mercury raised} \times \text{displacement of the C.G. of the segment}) + (\text{weight of} \\ &\quad \text{mercury depressed} \times \text{displacement of the C.G. of the segment}) \\ &= (Ax\gamma)\frac{x}{2} + (Ax\gamma)\frac{x}{2} = A\gamma x^2 \end{aligned} \tag{E.1}$$

where A is the cross-sectional area of the mercury column and γ is the specific weight of mercury. The change in kinetic energy is given by

$$\begin{aligned} T &= \frac{1}{2}(\text{mass of mercury})(\text{velocity})^2 \\ &= \frac{1}{2}\frac{Al\gamma}{g}\dot{x}^2 \end{aligned} \tag{E.2}$$

where l is the length of the mercury column. By assuming harmonic motion, we can write

$$x(t) = X\cos\omega_n t \tag{E.3}$$

where X is the maximum displacement and ω_n is the natural frequency. By substituting Eq. (E.3) into Eqs. (E.1) and (E.2), we obtain

$$U = U_{\max}\cos^2\omega_n t \tag{E.4}$$

$$T = T_{\max}\sin^2\omega_n t \tag{E.5}$$

where

$$U_{\max} = A\gamma X^2 \tag{E.6}$$

and

$$T_{\max} = \frac{1}{2}\frac{A\gamma l\omega_n^2}{g}X^2 \tag{E.7}$$

By equating $U_{\max}$ to $T_{\max}$, we obtain the natural frequency:

$$\omega_n = \left(\frac{2g}{l}\right)^{1/2} \tag{E.8}$$

2. *Length of the mercury column:* The frequency of pressure fluctuations in the silencer

$$\begin{aligned} &= \frac{1 \times 600}{2} \\ &= 300 \text{ rpm} \\ &= \frac{300 \times 2\pi}{60} = 10\pi \text{ rad/sec} \end{aligned} \tag{E.9}$$

Thus the frequency of oscillations of the liquid column in the manometer is $10\pi/3.5 = 9.0$ rad/sec. By using Eq. (E.8), we obtain

$$\left(\frac{2g}{l}\right)^{1/2} = 9.0 \tag{E.10}$$

or

$$l = \frac{2.0 \times 9.81}{(9.0)^2} = 0.243 \text{ m} \tag{E.11}$$

■

EXAMPLE 2.8 Effect of Mass on ω_n of a Spring

Determine the effect of the mass of the spring on the natural frequency of the spring-mass system shown in Fig. 2.19.

Solution: To find the effect of the mass of the spring on the natural frequency of the spring-mass system, we add the kinetic energy of the system to that of the attached mass and use the energy method to determine the natural frequency. Let l be the total length of the spring. If x denotes the displacement of the lower end of the spring (or mass m), the displacement at distance y from the support is given by $y(x/l)$. Similarly, if $\dot{x}$ denotes the velocity of the mass m, the velocity of a spring element located at distance y from the support is given by $y(\dot{x}/l)$. The kinetic energy of the spring element of length dy is

$$dT_s = \frac{1}{2}\left(\frac{m_s}{l}dy\right)\left(\frac{y\dot{x}}{l}\right)^2 \tag{E.1}$$

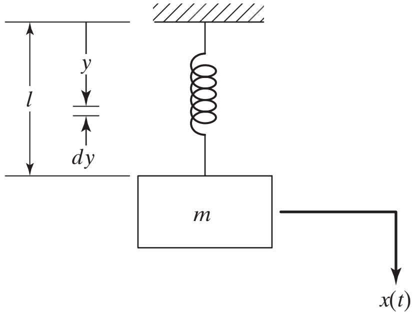


FIGURE 2.19 Equivalent mass of a spring.

where m_s is the mass of the spring. The total kinetic energy of the system can be expressed as

$$\begin{aligned} T &= \text{kinetic energy of mass } (T_m) + \text{kinetic energy of spring } (T_s) \\ &= \frac{1}{2} m \dot{x}^2 + \int_{y=0}^{l} \frac{1}{2}\left(\frac{m_s}{l} dy\right)\left(\frac{y^2 \dot{x}^2}{l^2}\right) \\ &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2 \end{aligned} \tag{E.2}$$

The total potential energy of the system is given by

$$U = \tfrac{1}{2} k x^2 \tag{E.3}$$

By assuming a harmonic motion

$$x(t) = X \cos \omega_n t \tag{E.4}$$

where X is the maximum displacement of the mass and ω_n is the natural frequency, the maximum kinetic and potential energies can be expressed as

$$T_{\max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right) X^2 \omega_n^2 \tag{E.5}$$

$$U_{\max} = \frac{1}{2} k X^2 \tag{E.6}$$

By equating $T_{\max}$ and $U_{\max}$, we obtain the expression for the natural frequency:

$$\omega_n = \left(\frac{k}{m + \frac{m_s}{3}}\right)^{1/2} \tag{E.7}$$

Thus the effect of the mass of the spring can be accounted for by adding one-third of its mass to the main mass [2.3].

■

EXAMPLE 2.9 Effect of Mass of Column on Natural Frequency of Water Tank

Find the natural frequency of transverse vibration of the water tank considered in Example 2.1 and Fig. 2.10 by including the mass of the column.

Solution: To include the mass of the column, we find the equivalent mass of the column at the free end using the equivalence of kinetic energy and use a single-degree-of-freedom model to find the natural frequency of vibration. The column of the tank is considered as a cantilever beam fixed at one end (ground) and carrying a mass M (water tank) at the other end. The static deflection of a cantilever beam under a concentrated end load is given by (see Fig. 2.20):

$$y(x) = \frac{Px^2}{6EI}(3l - x) = \frac{y_{\max}x^2}{2l^3}(3l - x)$$
$$= \frac{y_{\max}}{2l^3}(3x^2l - x^3) \tag{E.1}$$

The maximum kinetic energy of the beam itself ($T_{\max}$) is given by

$$T_{\max} = \frac{1}{2}\int_0^l \frac{m}{l}\{\dot{y}(x)\}^2\,dx \tag{E.2}$$

where m is the total mass and (m/l) is the mass per unit length of the beam. Equation (E.1) can be used to express the velocity variation, $\dot{y}(x)$, as

$$\dot{y}(x) = \frac{\dot{y}_{\max}}{2l^3}(3x^2l - x^3) \tag{E.3}$$

and hence Eq. (E.2) becomes

$$T_{\max} = \frac{m}{2l}\left(\frac{\dot{y}_{\max}}{2l^3}\right)^2\int_0^l (3x^2l - x^3)^2\,dx$$
$$= \frac{1}{2}\,\frac{m}{l}\,\frac{\dot{y}^2_{\max}}{4l^6}\left(\frac{33}{35}l^7\right) = \frac{1}{2}\left(\frac{33}{140}m\right)\dot{y}^2_{\max} \tag{E.4}$$

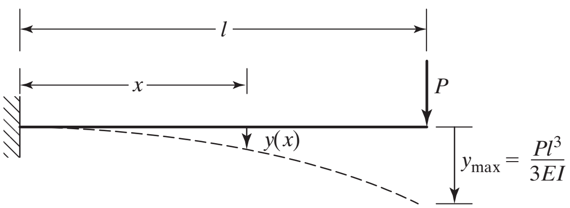


FIGURE 2.20 Equivalent mass of the column.

If m_{eq} denotes the equivalent mass of the cantilever (water tank) at the free end, its maximum kinetic energy can be expressed as

$$T_{max} = \frac{1}{2} m_{eq} \dot{y}_{max}^2 \tag{E.5}$$

By equating Eqs. (E.4) and (E.5), we obtain

$$m_{eq} = \frac{33}{140} m \tag{E.6}$$

Thus the total effective mass acting at the end of the cantilever beam is given by

$$M_{eff} = M + m_{eq} \tag{E.7}$$

where M is the mass of the water tank. The natural frequency of transverse vibration of the water tank is given by

$$\omega_n = \sqrt{\frac{k}{M_{eff}}} = \sqrt{\frac{k}{M + \frac{33}{140} m}} \tag{E.8}$$

■

2.6 Free Vibration with Viscous Damping

2.6.1 Equation of Motion

As stated in Section 1.9, the viscous damping force F is proportional to the velocity $\dot{x}$ or v and can be expressed as

$$F = -c\dot{x} \tag{2.58}$$

where c is the damping constant or coefficient of viscous damping and the negative sign indicates that the damping force is opposite to the direction of velocity. A single-degree-of-freedom system with a viscous damper is shown in Fig. 2.21. If x is measured from the equilibrium position of the mass m, the application of Newton's law yields the equation of motion:

$$m\ddot{x} = -c\dot{x} - kx$$

or

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.59}$$

2.6.2 Solution

To solve Eq. (2.59), we assume a solution in the form

$$x(t) = Ce^{st} \tag{2.60}$$

where C and s are undetermined constants. Inserting this function into Eq. (2.59) leads to the characteristic equation

$$ms^2 + cs + k = 0 \tag{2.61}$$

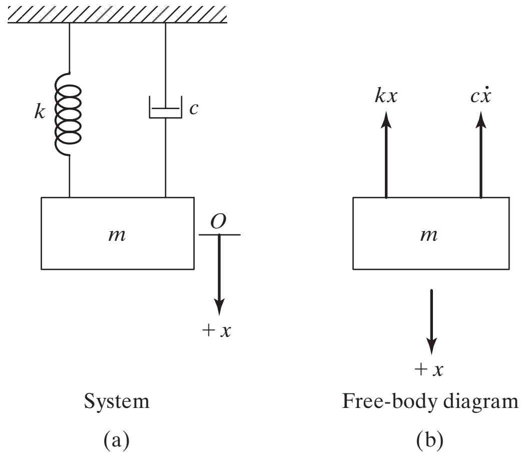


FIGURE 2.21 Single-degree-of-freedom system with viscous damper.

the roots of which are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{2.62}$$

These roots give two solutions to Eq. (2.59):

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t} \tag{2.63}$$

Thus the general solution of Eq. (2.59) is given by a combination of the two solutions $x_1(t)$ and $x_2(t)$:

$$\begin{aligned} x(t) &= C_1 e^{s_1 t} + C_2 e^{s_2 t} \\ &= C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} \end{aligned} \tag{2.64}$$

where C_1 and C_2 are arbitrary constants to be determined from the initial conditions of the system.

Critical Damping Constant and the Damping Ratio. The critical damping c_c is defined as the value of the damping constant c for which the radical in Eq. (2.62) becomes zero:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

or

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n \tag{2.65}$$

For any damped system, the damping ratio ζ is defined as the ratio of the damping constant to the critical damping constant:

$$\zeta = c/c_c \tag{2.66}$$

Using Eqs. (2.66) and (2.65), we can write

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta\omega_n \tag{2.67}$$

and hence

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \tag{2.68}$$

Thus the solution, Eq. (2.64), can be written as

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{2.69}$$

The nature of the roots s_1 and s_2 and hence the behavior of the solution, Eq. (2.69), depends upon the magnitude of damping. It can be seen that the case $\zeta = 0$ leads to the undamped vibrations discussed in Section 2.2. Hence we assume that $\zeta \neq 0$ and consider the following three cases.

Case 1. *Underdamped system* ($\zeta < 1$ or $c < c_c$ or $c/2m < \sqrt{k/m}$). For this condition, $(\zeta^2 - 1)$ is negative and the roots s_1 and s_2 can be expressed as

$$s_1 = \left(-\zeta + i\sqrt{1 - \zeta^2}\right)\omega_n$$
$$s_2 = \left(-\zeta - i\sqrt{1 - \zeta^2}\right)\omega_n$$

and the solution, Eq. (2.69), can be written in different forms:

$$\begin{aligned}
x(t) &= C_1 e^{\left(-\zeta+i\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-\zeta-i\sqrt{1-\zeta^2}\right)\omega_n t} \\
&= e^{-\zeta\omega_n t}\left\{C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t}\right\} \\
&= e^{-\zeta\omega_n t}\left\{(C_1 + C_2)\cos\sqrt{1-\zeta^2}\omega_n t + i(C_1 - C_2)\sin\sqrt{1-\zeta^2}\omega_n t\right\} \\
&= e^{-\zeta\omega_n t}\left\{C_1'\cos\sqrt{1-\zeta^2}\omega_n t + C_2'\sin\sqrt{1-\zeta^2}\omega_n t\right\}
\end{aligned}$$

$$= X_0 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_0\right)$$

$$= X e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi\right) \tag{2.70}$$

where (C_1', C_2'), (X, ϕ), and (X_0, ϕ_0) are arbitrary constants to be determined from the initial conditions.

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, C_1' and C_2' can be found:

$$C_1' = x_0 \quad \text{and} \quad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \tag{2.71}$$

and hence the solution becomes

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\} \tag{2.72}$$

The constants (X, ϕ) and (X_0, ϕ_0) can be expressed as

$$X = X_0 = \sqrt{(C_1')^2 + (C_2')^2} = \frac{\sqrt{x_0^2\omega_n^2 + \dot{x}_0^2 + 2x_0\dot{x}_0\zeta\omega_n}}{\sqrt{1-\zeta^2}\,\omega_n} \tag{2.73}$$

$$\phi_0 = \tan^{-1}\left(\frac{C_1'}{C_2'}\right) = \tan^{-1}\left(\frac{x_0\omega_n\sqrt{1-\zeta^2}}{\dot{x}_0 + \zeta\omega_n x_0}\right) \tag{2.74}$$

$$\phi = \tan^{-1}\left(\frac{C_2'}{C_1'}\right) = \tan^{-1}\left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{x_0\omega_n\sqrt{1-\zeta^2}}\right) \tag{2.75}$$

The motion described by Eq. (2.72) is a damped harmonic motion of angular frequency $\sqrt{1-\zeta^2}\,\omega_n$, but because of the factor $e^{-\zeta\omega_n t}$, the amplitude decreases exponentially with time, as shown in Fig. 2.22. The quantity

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \tag{2.76}$$

is called the *frequency of damped vibration.* It can be seen that the frequency of damped vibration ω_d is always less than the undamped natural frequency ω_n. The decrease in the frequency of damped vibration with increasing amount of damping, given by Eq. (2.76), is shown graphically in Fig. 2.23. The underdamped case is very important in the study of mechanical vibrations, as it is the only case that leads to an oscillatory motion [2.10].

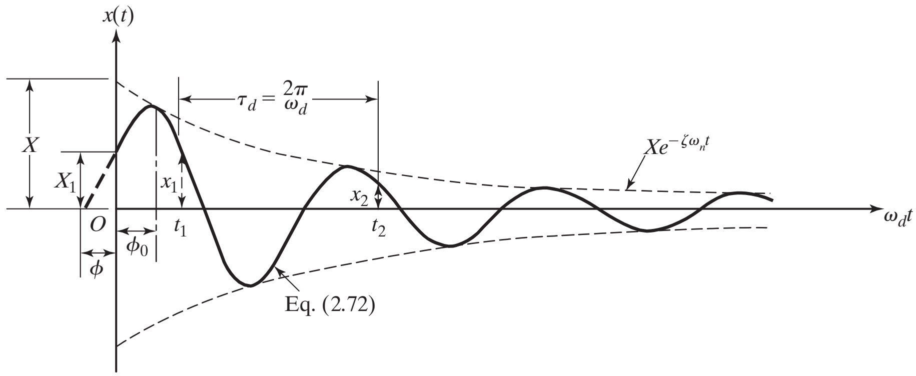


FIGURE 2.22 Underdamped solution.

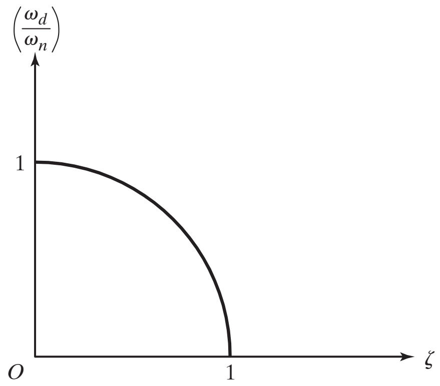


FIGURE 2.23 Variation of ω_d with damping.

Case 2. *Critically damped system* ($\zeta = 1$ or $c = c_c$ or $c/2m = \sqrt{k/m}$). In this case the two roots s_1 and s_2 in Eq. (2.68) are equal:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n \tag{2.77}$$

Because of the repeated roots, the solution of Eq. (2.59) is given by [2.6][1]

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{2.78}$$

[1]Equation (2.78) can also be obtained by making ζ approach unity in the limit in Eq. (2.72). As $\zeta \to 1$, $\omega_n \to 0$; hence $\cos \omega_d t \to 1$ and $\sin \omega_d t \to \omega_d t$. Thus Eq. (2.72) yields

$$x(t) = e^{-\omega_n t}(C_1' + C_2'\omega_d t) = (C_1 + C_2 t)e^{-\omega_n t}$$

where $C_1 = C_1'$ and $C_2 = C_2'\omega_d$ are new constants.

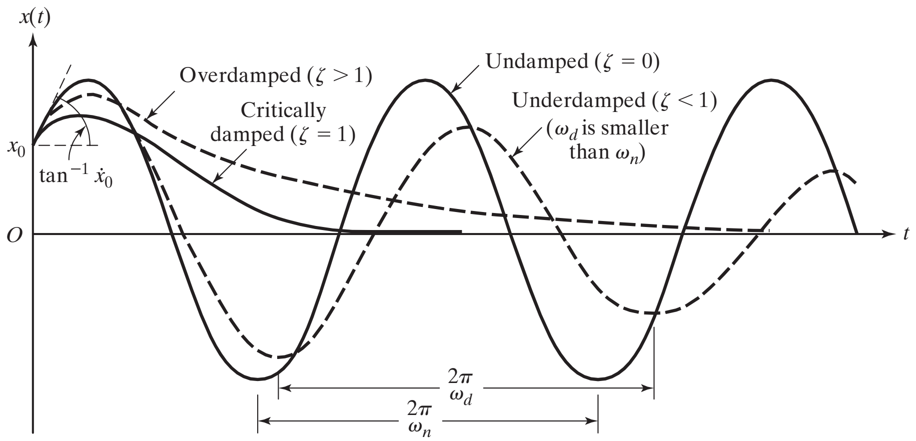


FIGURE 2.24 Comparison of motions with different types of damping.

The application of the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$ for this case gives

$$C_1 = x_0$$
$$C_2 = \dot{x}_0 + \omega_n x_0 \tag{2.79}$$

and the solution becomes

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t} \tag{2.80}$$

It can be seen that the motion represented by Eq. (2.80) is *aperiodic* (i.e., nonperiodic). Since $e^{-\omega_n t} \rightarrow 0$ as $t \rightarrow \infty$, the motion will eventually diminish to zero, as indicated in Fig. 2.24.

Case 3. *Overdamped system* ($\zeta > 1$ or $c > c_c$ or $c/2m > \sqrt{k/m}$). As $\sqrt{\zeta^2 - 1} > 0$, Eq. (2.68) shows that the roots s_1 and s_2 are real and distinct and are given by

$$s_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega_n < 0$$
$$s_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega_n < 0$$

with $s_2 \ll s_1$. In this case, the solution, Eq. (2.69), can be expressed as

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \tag{2.81}$$

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, the constants C_1 and C_2 can be obtained:

$$C_1 = \frac{x_0\omega_n(\zeta + \sqrt{\zeta^2 - 1}) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0\omega_n(\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{2.82}$$

Equation (2.81) shows that the motion is aperiodic regardless of the initial conditions imposed on the system. Since roots s_1 and s_2 are both negative, the motion diminishes exponentially with time, as shown in Fig. 2.24.

Note the following aspects of these systems:

1. The graphical representation of different types of the characteristics roots s_1 and s_2, and the corresponding responses (solutions) of the system are presented in Section 2.7. The representation of the roots s_1 and s_2 with varying values of the system parameters c, k and m in the complex plane (known as the root locus plots) is considered in Section 2.8.
2. A critically damped system will have the smallest damping required for aperiodic motion; hence the mass returns to the position of rest in the shortest possible time without overshooting. The property of critical damping is used in many practical applications. For example, large guns have dashpots with critical damping value, so that they return to their original position after recoil in the minimum time without vibrating. If the damping provided were more than the critical value, some delay would be caused before the next firing.
3. The free damped response of a single-degree-of-freedom system can be represented in phase-plane or state space as indicated in Fig. 2.25.

2.6.3 Logarithmic Decrement

The logarithmic decrement represents the rate at which the amplitude of a free-damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes. Let t_1 and t_2 denote the times corresponding to two consecutive amplitudes

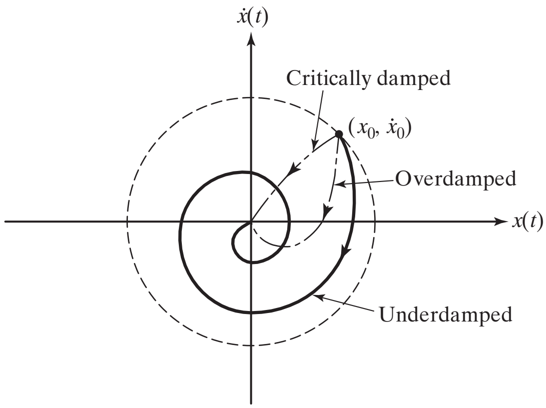


FIGURE 2.25 Phase plane of a damped system.

(displacements), measured one cycle apart for an underdamped system, as in Fig. 2.22. Using Eq. (2.70), we can form the ratio

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1} \cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2} \cos(\omega_d t_2 - \phi_0)} \tag{2.83}$$

But $t_2 = t_1 + \tau_d$, where $\tau_d = 2\pi/\omega_d$ is the period of damped vibration. Hence $\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$, and Eq. (2.83) can be written as

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d} \tag{2.84}$$

The logarithmic decrement δ can be obtained from Eq. (2.84):

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m} \tag{2.85}$$

For small damping, Eq. (2.85) can be approximated:

$$\delta \simeq 2\pi\zeta \qquad \text{if} \qquad \zeta \ll 1 \tag{2.86}$$

Figure 2.26 shows the variation of the logarithmic decrement δ with ζ as given by Eqs. (2.85) and (2.86). It can be noticed that for values up to $\zeta = 0.3$, the two curves are difficult to distinguish.

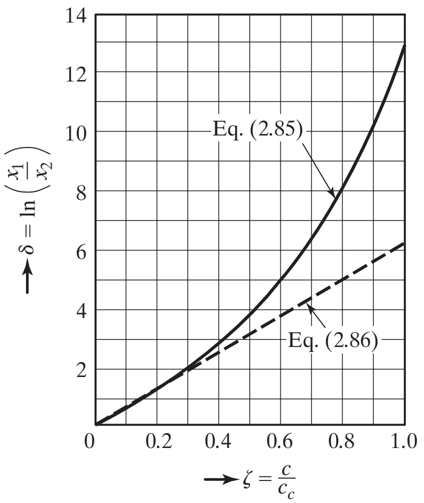


FIGURE 2.26 Variation of logarithmic decrement with damping.

The logarithmic decrement is dimensionless and is actually another form of the dimensionless damping ratio ζ. Once δ is known, ζ can be found by solving Eq. (2.85):

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \tag{2.87}$$

If we use Eq. (2.86) instead of Eq. (2.85), we have

$$\zeta \simeq \frac{\delta}{2\pi} \tag{2.88}$$

If the damping in the given system is not known, we can determine it experimentally by measuring any two consecutive displacements x_1 and x_2. By taking the natural logarithm of the ratio of x_1 and x_2, we obtain δ. By using Eq. (2.87), we can compute the damping ratio ζ. In fact, the damping ratio ζ can also be found by measuring two displacements separated by any number of complete cycles. If x_1 and x_{m+1} denote the amplitudes corresponding to times t_1 and $t_{m+1} = t_1 + m\tau_d$, where m is an integer, we obtain

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4}\cdots\frac{x_m}{x_{m+1}} \tag{2.89}$$

Since any two successive displacements separated by one cycle satisfy the equation

$$\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d} \tag{2.90}$$

Eq. (2.89) becomes

$$\frac{x_1}{x_{m+1}} = \left(e^{\zeta\omega_n\tau_d}\right)^m = e^{m\zeta\omega_n\tau_d} \tag{2.91}$$

Equations (2.91) and (2.85) yield

$$\delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right) \tag{2.92}$$

which can be substituted into Eq. (2.87) or Eq. (2.88) to obtain the viscous damping ratio ζ.

2.6.4 Energy Dissipated in Viscous Damping

In a viscously damped system, the rate of change of energy with time (dW/dt) is given by

$$\frac{dW}{dt} = \text{force} \times \text{velocity} = Fv = -cv^2 = -c\left(\frac{dx}{dt}\right)^2 \tag{2.93}$$

using Eq. (2.58). The negative sign in Eq. (2.93) denotes that energy dissipates with time. Assume a simple harmonic motion as $x(t) = X \sin \omega_d t$, where X is the amplitude of motion and the energy dissipated in a complete cycle is given by[2]

[2]In the case of a damped system, simple harmonic motion $x(t) = X \cos \omega_d t$ is possible only when the steady-state response is considered under a harmonic force of frequency ω_d (see Section 3.4). The loss of energy due to the damper is supplied by the excitation under steady-state forced vibration [2.7].

$$\Delta W = \int_{t=0}^{(2\pi/\omega_d)} c\left(\frac{dx}{dt}\right)^2 dt = \int_0^{2\pi} cX^2\omega_d \cos^2\omega_d t \cdot d(\omega_d t)$$

$$= \pi c \omega_d X^2 \tag{2.94}$$

This shows that the energy dissipated is proportional to the square of the amplitude of motion. Note that it is not a constant for given values of damping and amplitude, since ΔW is also a function of the frequency ω_d.

Equation (2.94) is valid even when there is a spring of stiffness k parallel to the viscous damper. To see this, consider the system shown in Fig. 2.27. The total force resisting motion can be expressed as

$$F = -kx - cv = -kx - c\dot{x} \tag{2.95}$$

If we assume simple harmonic motion

$$x(t) = X \sin \omega_d t \tag{2.96}$$

as before, Eq. (2.95) becomes

$$F = -kX \sin \omega_d t - c\omega_d X \cos \omega_d t \tag{2.97}$$

The energy dissipated in a complete cycle will be

$$\Delta W = \int_{t=0}^{2\pi/\omega_d} Fv\, dt$$

$$= \int_0^{2\pi/\omega_d} kX^2\omega_d \sin \omega_d t \cdot \cos \omega_d t \cdot d(\omega_d t)$$

$$+ \int_0^{2\pi/\omega_d} c\omega_d X^2 \cos^2 \omega_d t \cdot d(\omega_d t) = \pi c \omega_d X^2 \tag{2.98}$$

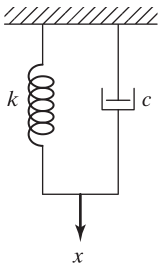


FIGURE 2.27
Spring and damper in parallel.

which can be seen to be identical with Eq. (2.94). This result is to be expected, since the spring force will not do any net work over a complete cycle or any integral number of cycles.

We can also compute the fraction of the total energy of the vibrating system that is dissipated in each cycle of motion $(\Delta W/W)$, as follows. The total energy of the system W can be expressed either as the maximum potential energy $(\frac{1}{2}kX^2)$ or as the maximum kinetic energy $(\frac{1}{2}mv_{\max}^2 = \frac{1}{2}mX^2\omega_d^2)$, the two being approximately equal for small values of damping. Thus

$$\frac{\Delta W}{W} = \frac{\pi c \omega_d X^2}{\frac{1}{2}m\omega_d^2 X^2} = 2\left(\frac{2\pi}{\omega_d}\right)\left(\frac{c}{2m}\right) = 2\delta \simeq 4\pi\zeta = \text{constant} \tag{2.99}$$

using Eqs. (2.85) and (2.88). The quantity $\Delta W/W$ is called the *specific damping capacity* and is useful in comparing the damping capacity of engineering materials. Another quantity known as the *loss coefficient* is also used for comparing the damping capacity of engineering materials. The loss coefficient is defined as the ratio of the energy dissipated per radian and the total strain energy:

$$\text{loss coefficient} = \frac{(\Delta W/2\pi)}{W} = \frac{\Delta W}{2\pi W} \tag{2.100}$$

2.6.5 Torsional Systems with Viscous Damping

The methods presented in Sections 2.6.1 through 2.6.4 for linear vibrations with viscous damping can be extended directly to viscously damped torsional (angular) vibrations. For this, consider a single-degree-of-freedom torsional system with a viscous damper, as shown in Fig. 2.28(a). The viscous damping torque is given by (Fig. 2.28(b)):

$$T = -c_t\dot{\theta} \tag{2.101}$$

where c_t is the torsional viscous damping constant, $\dot{\theta} = d\theta/dt$ is the angular velocity of the disc, and the negative sign denotes that the damping torque is opposite the direction of angular velocity. The equation of motion can be derived as

$$J_0\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0 \tag{2.102}$$

where J_0 = mass moment of inertia of the disc, k_t = spring constant of the system (restoring torque per unit angular displacement), and θ = angular displacement of the disc. The solution of Eq. (2.102) can be found exactly as in the case of linear vibrations. For example, in the underdamped case, the frequency of damped vibration is given by

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \tag{2.103}$$

where

$$\omega_n = \sqrt{\frac{k_t}{J_0}} \tag{2.104}$$

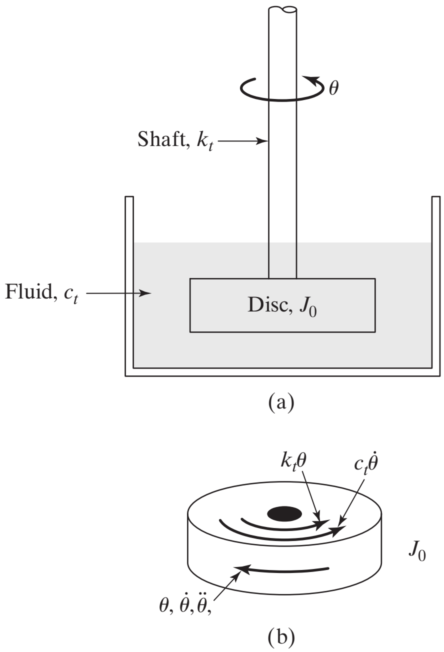


FIGURE 2.28 Torsional viscous damper.

and

$$\zeta = \frac{c_t}{c_{tc}} = \frac{c_t}{2J_0\omega_n} = \frac{c_t}{2\sqrt{k_t J_0}} \tag{2.105}$$

where c_{tc} is the critical torsional damping constant.

EXAMPLE 2.10 Response of Anvil of a Forging Hammer

The anvil of a forging hammer weighs 5,000 N and is mounted on a foundation that has a stiffness of 5×10^6 N/m and a viscous damping constant of 10,000 N-s/m. During a particular forging operation, the tup (i.e., the falling weight or the hammer), weighing 1,000 N, is made to fall from a height of 2 m onto the anvil (Fig. 2.29(a)). If the anvil is at rest before impact by the tup, determine the response of the anvil after the impact. Assume that the coefficient of restitution between the anvil and the tup is 0.4.

Solution: First we use the principle of conservation of momentum and the definition of the coefficient of restitution to find the initial velocity of the anvil. Let the velocities of the tup just before and just after impact with the anvil be v_{t1} and v_{t2}, respectively. Similarly, let v_{a1} and v_{a2} be the velocities of the anvil just before and just after the impact, respectively (Fig. 2.29(b)). Note that the

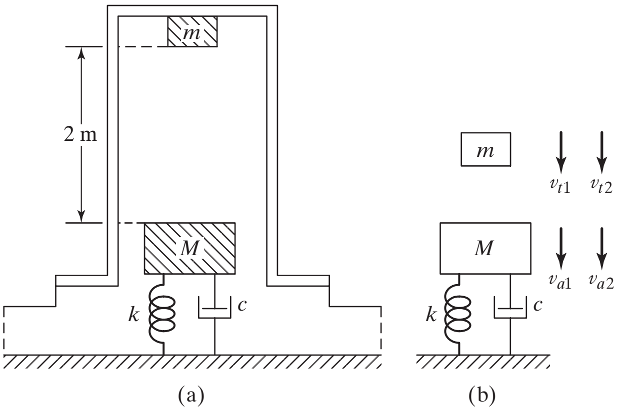


FIGURE 2.29 Forging hammer.

displacement of the anvil is measured from its static equilibrium position and all velocities are assumed to be positive when acting downward. The principle of conservation of momentum gives

$$M(v_{a2} - v_{a1}) = m(v_{t1} - v_{t2}) \tag{E.1}$$

where $v_{a1} = 0$ (anvil is at rest before the impact) and v_{t1} can be determined by equating its kinetic energy just before impact to its potential energy before dropping from a height of $h = 2$ m:

$$\frac{1}{2}mv_{t1}^2 = mgh \tag{E.2}$$

or

$$v_{t1} = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 2} = 6.26099 \text{ m/s}$$

Thus Eq. (E.1) becomes

$$\frac{5000}{9.81}(v_{a2} - 0) = \frac{1000}{9.81}(6.26099 - v_{t2})$$

that is,

$$510.204082\,v_{a2} = 638.87653 - 102.040813\,v_{t2} \tag{E.3}$$

The definition of the coefficient of restitution (r) yields:

$$r = -\left(\frac{v_{a2} - v_{t2}}{v_{a1} - v_{t1}}\right) \tag{E.4}$$

—that is,

$$0.4 = -\left(\frac{v_{a2} - v_{t2}}{0 - 6.26099}\right)$$

—that is,

$$v_{a2} = v_{t2} + 2.504396 \tag{E.5}$$

The solution of Eqs. (E.3) and (E.5) gives

$$v_{a2} = 1.460898 \text{ m/s}; \qquad v_{t2} = -1.043498 \text{ m/s}$$

Thus the initial conditions of the anvil are given by

$$x_0 = 0; \qquad \dot{x}_0 = 1.460898 \text{ m/s}$$

The damping coefficient is equal to

$$\zeta = \frac{c}{2\sqrt{kM}} = \frac{1000}{2\sqrt{(5 \times 10^6)\left(\frac{5000}{9.81}\right)}} = 0.0989949$$

The undamped and damped natural frequencies of the anvil are given by

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{5 \times 10^6}{\left(\frac{5000}{9.81}\right)}} = 98.994949 \text{ rad/s}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 98.994949\sqrt{1 - 0.0989949^2} = 98.024799 \text{ rad/s}$$

The displacement response of the anvil is given by Eq. (2.72):

$$x(t) = e^{-\zeta\,\omega_n\,t}\left\{\frac{\dot{x}_0}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-9.799995\,t}\,\{0.01490335 \sin 98.024799\, t\}\text{ m}$$

■

EXAMPLE 2.11 Shock Absorber for a Motorcycle

An underdamped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig. 2.30(a)). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig. 2.30(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 s and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.

Approach: We use the equation for the logarithmic decrement in terms of the damping ratio, equation for the damped period of vibration, time corresponding to maximum displacement for an underdamped system, and envelope passing through the maximum points of an underdamped system.

Solution: Since $x_{1.5} = x_1/4$, $x_2 = x_{1.5}/4 = x_1/16$. Hence the logarithmic decrement becomes

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln(16) = 2.7726 = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}} \tag{E.1}$$

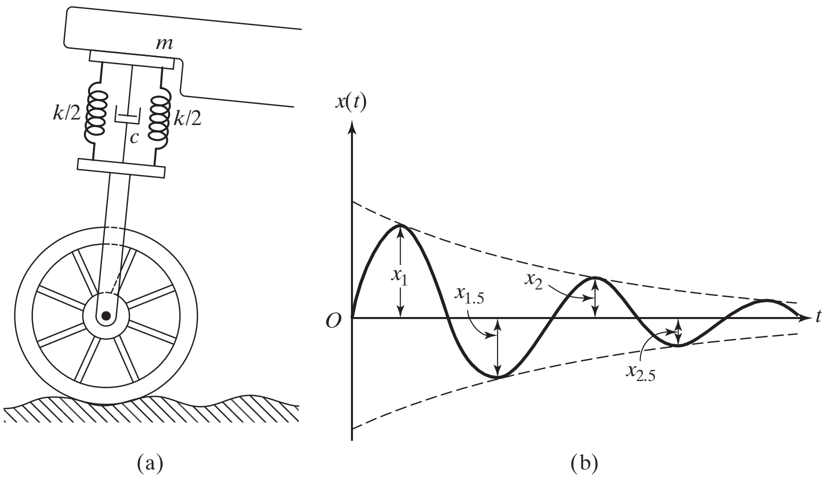


FIGURE 2.30 Shock absorber of a motorcycle.

from which the value of ζ can be found as $\zeta = 0.4037$. The damped period of vibration is given to be 2 s. Hence

$$2 = \tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1 - \zeta^2}}$$

$$\omega_n = \frac{2\pi}{2\sqrt{1 - (0.4037)^2}} = 3.4338 \text{ rad/s}$$

The critical damping constant can be obtained:

$$c_c = 2m\omega_n = 2(200)(3.4338) = 1373.54 \text{ N-s/m}$$

Thus the damping constant is given by

$$c = \zeta c_c = (0.4037)(1373.54) = 554.4981 \text{ N-s/m}$$

and the stiffness by

$$k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652 \text{ N/m}$$

The displacement of the mass will attain its maximum value at time t_1, given by

$$\sin \omega_d t_1 = \sqrt{1 - \zeta^2}$$

(See Problem 2.99.) This gives

$$\sin \omega_d t_1 = \sin \pi t_1 = \sqrt{1 - (0.4037)^2} = 0.9149$$

or

$$t_1 = \frac{\sin^{-1}(0.9149)}{\pi} = 0.3678 \text{ sec}$$

The envelope passing through the maximum points (see Problem 2.99) is given by

$$x = \sqrt{1-\zeta^2}Xe^{-\zeta\omega_n t} \tag{E.2}$$

Since $x = 250$ mm, Eq. (E.2) gives at t_1

$$0.25 = \sqrt{1-(0.4037)^2}\, Xe^{-(0.4037)(3.4338)(0.3678)}$$

or

$$X = 0.4550 \text{ m}$$

The velocity of the mass can be obtained by differentiating the displacement

$$x(t) = Xe^{-\zeta\omega_n t}\sin\omega_d t$$

as

$$\dot{x}(t) = Xe^{-\zeta\omega_n t}(-\zeta\omega_n\sin\omega_d t + \omega_d\cos\omega_d t) \tag{E.3}$$

When $t = 0$, Eq. (E.3) gives

$$\dot{x}(t=0) = \dot{x}_0 = X\omega_d = X\omega_n\sqrt{1-\zeta^2} = (0.4550)(3.4338)\sqrt{1-(0.4037)^2}$$
$$= 1.4294 \text{ m/s}$$

■

EXAMPLE 2.12 Analysis of Cannon

The schematic diagram of a large cannon is shown in Fig. 2.31 [2.8]. When the gun is fired, high-pressure gases accelerate the projectile inside the barrel to a very high velocity. The reaction force pushes the gun barrel in the direction opposite that of the projectile. Since it is desirable to bring the gun barrel to rest in the shortest time without oscillation, it is made to translate backward against a critically damped spring-damper system called the *recoil mechanism*. In a particular case, the gun barrel and the recoil mechanism have a mass of 500 kg with a recoil spring of stiffness 10,000 N/m. The gun recoils 0.4 m upon firing. Find (1) the critical damping coefficient of the damper, (2) the initial recoil velocity of the gun, and (3) the time taken by the gun to return to a position 0.1 m from its initial position.

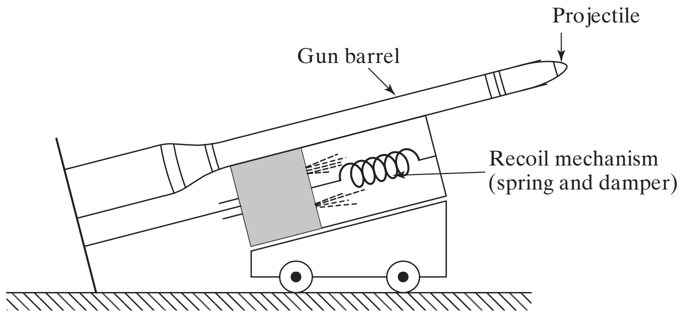


FIGURE 2.31 Recoil of cannon.

Solution

1. The undamped natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000}{500}} = 4.4721 \text{ rad/s}$$

and the critical damping coefficient (Eq. 2.65) of the damper is

$$c_c = 2m\omega_n = 2(500)(4.4721) = 4472.1 \text{ N-s/m}$$

2. The response of a critically damped system is given by Eq. (2.78):

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{E.1}$$

where $C_1 = x_0$ and $C_2 = \dot{x}_0 + \omega_n x_0$. The time t_1 at which $x(t)$ reaches a maximum value can be obtained by setting $\dot{x}(t) = 0$. The differentiation of Eq. (E.1) gives

$$\dot{x}(t) = C_2 e^{-\omega_n t} - \omega_n(C_1 + C_2 t)e^{-\omega_n t}$$

Hence $\dot{x}(t) = 0$ yields

$$t_1 = \left(\frac{1}{\omega_n} - \frac{C_1}{C_2}\right) \tag{E.2}$$

In this case, $x_0 = C_1 = 0$; hence Eq. (E.2) leads to $t_1 = 1/\omega_n$. Since the maximum value of $x(t)$ or the recoil distance is given to be $x_{\max} = 0.4$ m, we have

$$x_{\max} = x(t = t_1) = C_2 t_1 e^{-\omega_n t_1} = \frac{\dot{x}_0}{\omega_n} e^{-1} = \frac{\dot{x}_0}{e\omega_n}$$

or

$$\dot{x}_0 = x_{\max}\omega_n e = (0.4)(4.4721)(2.7183) = 4.8626 \text{ m/s}$$

3. If t_2 denotes the time taken by the gun to return to a position 0.1 m from its initial position, we have

$$0.1 = C_2 t_2 e^{-\omega_n t_2} = 4.8626 t_2 e^{-4.4721 t_2} \tag{E.3}$$

The solution of Eq. (E.3) gives $t_2 = 0.8258$ s.

■

2.7 Graphical Representation of Characteristic Roots and Corresponding Solutions[†]

2.7.1 Roots of the Characteristic Equation

The free vibration of a single-degree-of-freedom spring-mass-viscous-damper system shown in Fig. 2.21 is governed by Eq. (2.59):

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.106}$$

whose characteristic equation can be expressed as (Eq. (2.61)):

$$ms^2 + cs + k = 0 \tag{2.107}$$

or

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{2.108}$$

[†]If necessary, sections 2.7 and 2.8 can be skipped without losing continuity.

The roots of this characteristic equation, called the *characteristic roots* or, simply, *roots*, help us in understanding the behavior of the system. The roots of Eq. (2.107) or (2.108) are given by (see Eqs. (2.62) and (2.68)):

$$s_1, s_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2\,m} \tag{2.109}$$

or

$$s_1, s_2 = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} \tag{2.110}$$

2.7.2 Graphical Representation of Roots and Corresponding Solutions

The roots given by Eq. (2.110) can be plotted in a complex plane, also known as the s-plane, by denoting the real part along the horizontal axis and the imaginary part along the vertical axis. Noting that the response of the system is given by

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \tag{2.111}$$

where C_1 and C_2 are constants, the following observations can be made by examining Eqs. (2.110) and (2.111):

1. Because the exponent of a larger real negative number (such as e^{-2t}) decays faster than the exponent of a smaller real negative number (such as e^{-t}), the roots lying farther to the left in the s-plane indicate that the corresponding responses decay faster than those associated with roots closer to the imaginary axis.
2. If the roots have positive real values of s—that is, the roots lie in the right half of the s-plane—the corresponding response grows exponentially and hence will be unstable.
3. If the roots lie on the imaginary axis (with zero real value), the corresponding response will be naturally stable.
4. If the roots have a zero imaginary part, the corresponding response will not oscillate.
5. The response of the system will exhibit an oscillatory behavior only when the roots have nonzero imaginary parts.
6. The farther the roots lie to the left of the s-plane, the faster the corresponding response decreases.
7. The larger the imaginary part of the roots, the higher the frequency of oscillation of the corresponding response of the system.

Figure 2.32 shows some representative locations of the characteristic roots in the s-plane and the corresponding responses [2.15]. The characteristics that describe the behavior of the response of a system include oscillatory nature, frequency of oscillation, and response time. These characteristics are inherent to the system (depend on the values of m, c, and k) and are determined by the characteristic roots of the system but not by the initial conditions. The initial conditions determine only the amplitudes and phase angles.

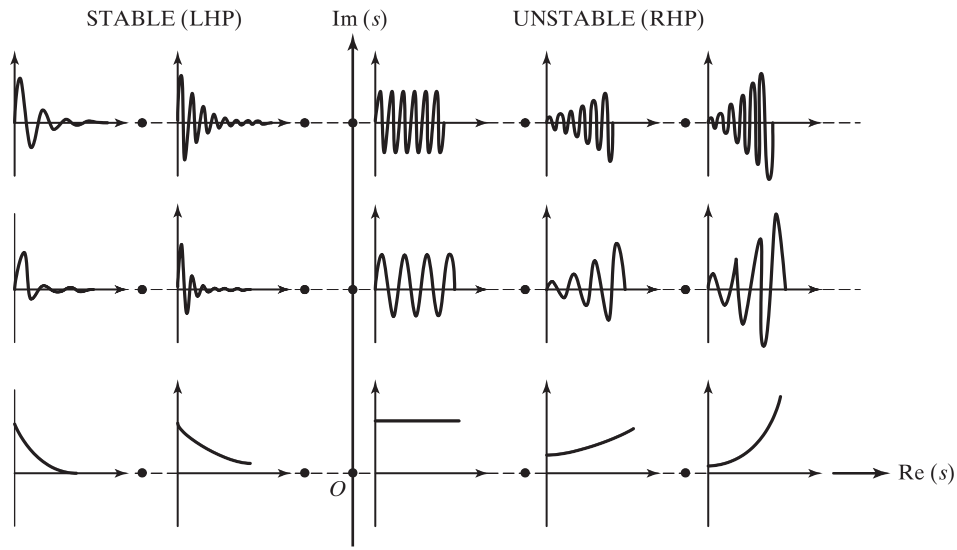


FIGURE 2.32 Locations of characteristic roots (•) and the corresponding responses of the system.

2.8 Parameter Variations and Root Locus Representations

2.8.1 Interpretations of ω_n, ω_d, ζ, and τ in the s-plane

Although the roots s_1 and s_2 appear as complex conjugates, we consider only the roots in the upper half of the s-plane. The root s_1 is plotted as point A with the real value as $\zeta\omega_n$ and the complex value as $\omega_n \sqrt{1 - \zeta^2}$, so that the length of OA is ω_n (Fig. 2.33). Thus the roots lying on the circle of radius ω_n correspond to the same natural frequency (ω_n) of the system (PAQ denotes a quarter of the circle). Thus different concentric circles represent systems with different natural frequencies as shown in Fig. 2.34. The horizontal line passing through point A corresponds to the damped natural frequency, $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. Thus, lines parallel to the real axis denote systems having different damped natural frequencies, as shown in Fig. 2.35.

It can be seen, from Fig. 2.33, that the angle made by the line OA with the imaginary axis is given by

$$\sin\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta \tag{2.112}$$

or

$$\theta = \sin^{-1}\zeta \tag{2.113}$$

Thus, radial lines passing through the origin correspond to different damping ratios, as shown in Fig. 2.36. Therefore, when $\zeta = 0$, we have no damping $(\theta = 0)$, and the damped natural frequency will reduce to the undamped natural frequency. Similarly, when $\zeta = 1$,

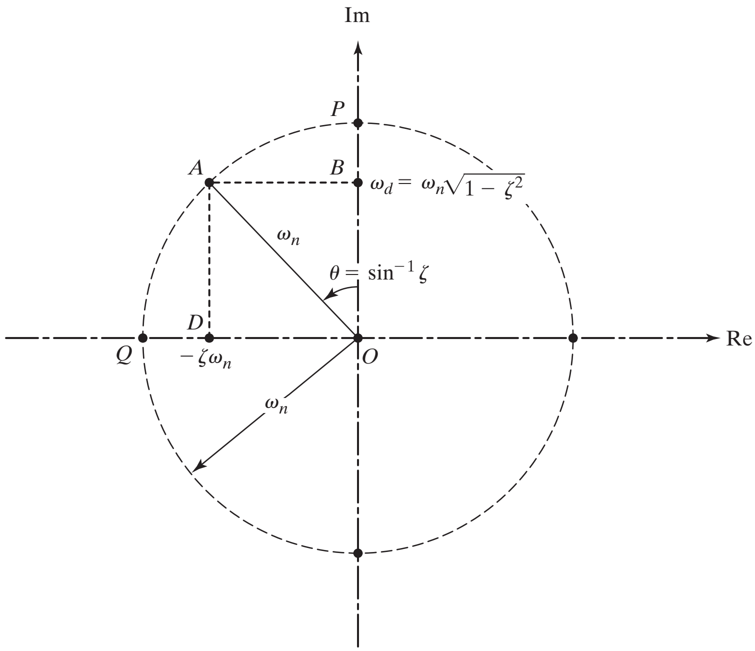


FIGURE 2.33 Interpretations of ω_n, ω_d, and ζ.

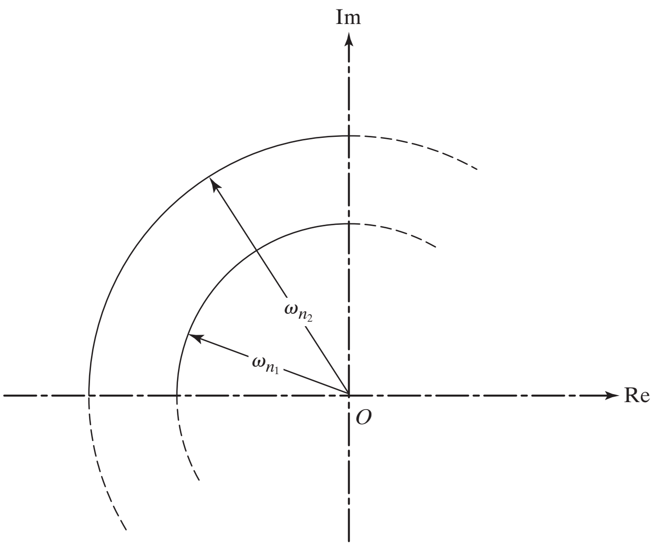


FIGURE 2.34 ω_n in s-plane.

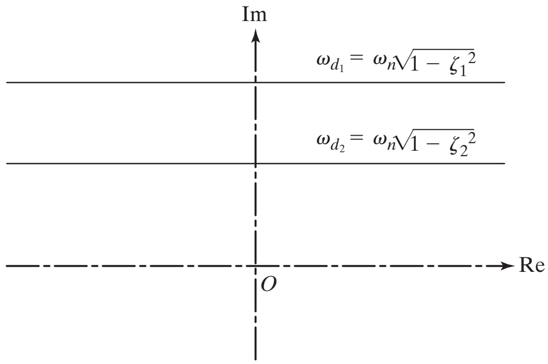


FIGURE 2.35 ω_d in s-plane.

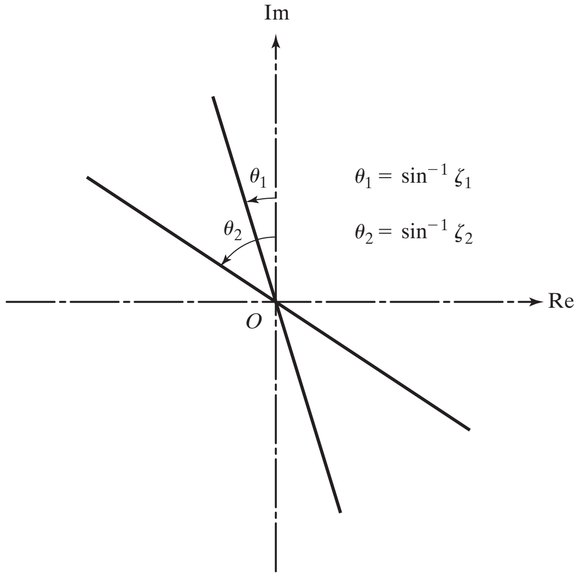


FIGURE 2.36 ζ in s-plane.

we have critical damping and the radical line lies along the negative real axis. The time constant of the system, τ, is defined as

$$\tau = \frac{1}{\zeta\omega_n} \tag{2.114}$$

and hence the distance DO or AB represents the reciprocal of the time constant, $\zeta\omega_n = \dfrac{1}{\tau}$. Hence different lines parallel to the imaginary axis denote reciprocals of different time constants (Fig. 2.37).

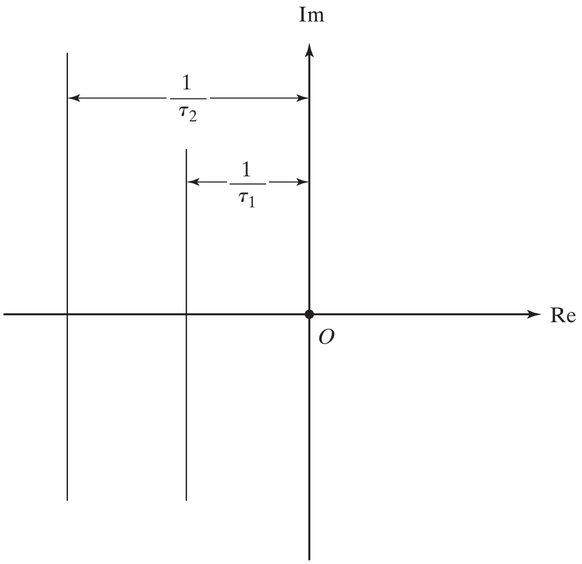


FIGURE 2.37 τ in s-plane.

2.8.2 Root Locus and Parameter Variations

A plot or graph that shows how changes in one of the parameters of the system will modify the roots of the characteristic equation of the system is known as the root locus plot. The root locus method is a powerful method of analysis and design for stability and transient response of a system. For a vibrating system, the root locus can be used to describe qualitatively the performance of the system as various parameters, such as the mass, damping constant, or spring constant, are changed. In the root locus method, the path or locus of the roots of the characteristic equation is plotted without actually finding the roots themselves. This is accomplished by using a set of rules which lead to a reasonably accurate plot in a relatively short time [2.8]. We study the behavior of the system by varying one parameter, among the damping ratio, spring constant, and mass, at a time in terms of the locations of its characteristic roots in the s-plane.

Variation of the damping ratio: We vary the damping constant from zero to infinity and study the migration of the characteristic roots in the s-plane. For this, we use Eq. (2.109). We notice that negative values of the damping constant $(c < 0)$ need not be considered, because they result in roots lying in the positive real half-plane that correspond to an unstable system. Thus we start with $c = 0$ to obtain, from Eq. (2.109),

$$s_{1,2} = \pm \frac{\sqrt{-4mk}}{2m} = \pm i \sqrt{\frac{k}{m}} = \pm i\omega_n \tag{2.115}$$

Thus the locations of the characteristic roots start on the imaginary axis. Because the roots appear in complex conjugate pairs, we concentrate on the upper imaginary half-plane and then locate the roots in the lower imaginary half-plane as mirror images. By keeping the undamped natural frequency (ω_n) constant, we vary the damping constant c. Noting that the real and imaginary parts of the roots in Eq. (2.109) can be expressed as

$$-\sigma = -\frac{c}{2m} = -\zeta\omega_n \quad \text{and} \quad \frac{\sqrt{4mk - c^2}}{2m} = \omega_n\sqrt{1 - \zeta^2} = \omega_d \tag{2.116}$$

for $0 < \zeta < 1$, we find that

$$\sigma^2 + \omega_d^2 = \omega_n^2 \tag{2.117}$$

Since ω_n is held fixed, Eq. (2.117) represents the equation of a circle with a radius $r = \omega_n$ in the σ (real) and ω_d (imaginary) plane. The radius vector $r = \omega_n$ will make an angle θ with the positive imaginary axis with

$$\sin\theta = \frac{\omega_d}{\omega_n} = \alpha \tag{2.118}$$

$$\cos\theta = \frac{\sigma}{\omega_n} = \frac{\zeta\omega_n}{\omega_n} = \zeta \tag{2.119}$$

with

$$\alpha = \sqrt{1 - \zeta^2} \tag{2.120}$$

Thus the two roots trace loci or paths in the form of circular arcs as the damping ratio is increased from zero to unity as shown in Fig. 2.38. The root with positive imaginary part moves in the counterclockwise direction while the root with negative imaginary part moves in the clockwise direction. When the damping ratio (ζ) is equal to one, the two loci meet, denoting that the two roots coincide—that is, the characteristic equation has repeated roots. As we increase the damping ratio beyond the value of unity, the system becomes overdamped and, as seen earlier in Section 2.6, both the roots will become real. From the properties of a quadratic equation, we find that the product of the two roots is equal to the coefficient of the lowest power of s (which is equal to ω_n^2 in Eq. (2.108)).

Since the value of ω_n is held constant in this study, the product of the two roots is a constant. With increasing values of the damping ratio (ζ), one root will increase and the other root will decrease, with the locus of each root remaining on the negative real axis. Thus one root will approach $-\infty$ and the other root will approach zero. The two loci will join or coincide at a point, known as the *breakaway point*, on the negative real axis. The two parts of the loci that lie on the negative real axis, one from point P to $-\infty$ and the other from point P to the origin, are known as *segments*.

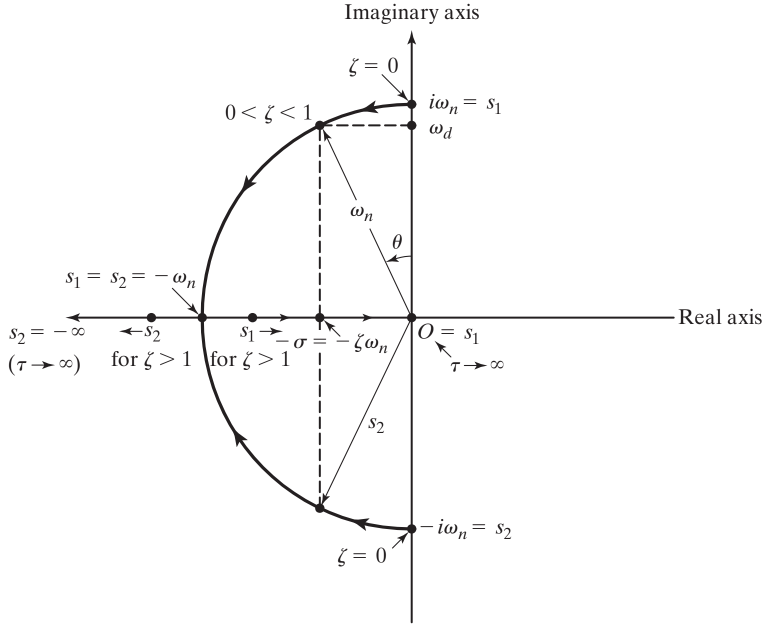


FIGURE 2.38 Root locus plot with variation of damping ratio ζ.

EXAMPLE 2.13 Study of Roots with Variation of c

Plot the root locus diagram of the system governed by the equation

$$3s^2 + cs + 27 = 0 \tag{E.1}$$

by varying the value of $c > 0$.

Solution: The roots of Eq. (E.1) are given by

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 324}}{6} \tag{E.2}$$

We start with a value of $c = 0$. At $c = 0$, the roots are given by $s_{1,2} = \pm 3i$. These roots are shown as dots on the imaginary axis in Fig. 2.39. By using an increasing sequence of values of c, Eq. (E.2) gives the roots as indicated in Table 2.1.

It can be seen that the roots remain complex conjugates as c is increased up to a value of $c = 18$. At $c = 18$, both the roots become real and identical with a value of -3.0. As c increases beyond a value of 18, the roots remain distinct with negative real values. One root becomes more and more negative and the other root becomes less and less negative. Thus, as $c \to \infty$, one root approaches $-\infty$ while the other root approaches 0. These trends of the roots are shown in Fig 2.39.

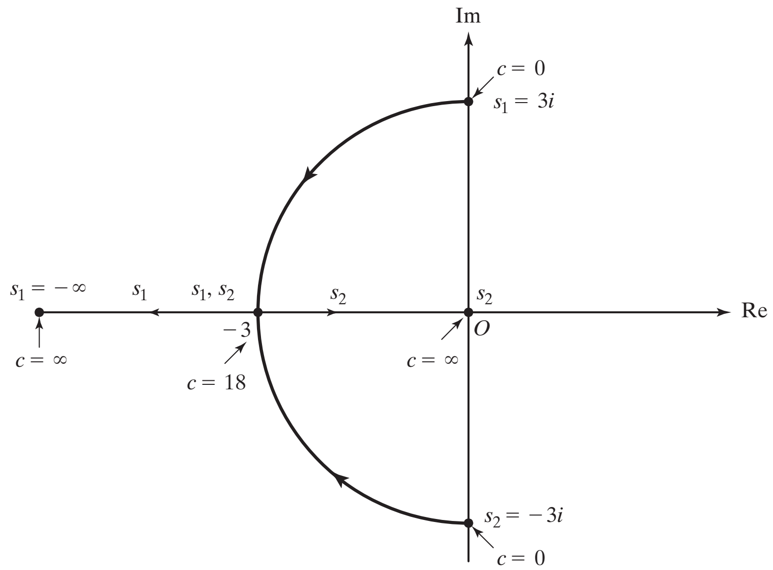


FIGURE 2.39 Root locus plot with variation of damping constant (c).

TABLE 2.1

Value of c	Value of s_1	Value of s_2
0	$+3i$	$-3i$
2	$-0.3333 + 2.9814i$	$-0.3333 - 2.9814i$
4	$-0.6667 + 2.9721i$	$-0.6667 - 2.9721i$
6	$-1.0000 + 2.8284i$	$-1.0000 - 2.8284i$
8	$-1.3333 + 2.6874i$	$-1.3333 - 2.6874i$
10	$-1.6667 + 2.4944i$	$-1.6667 - 2.4944i$
12	$-2.0000 + 2.2361i$	$-2.0000 - 2.2361i$
14	$-2.3333 + 1.8856i$	$-2.3333 - 1.8856i$
16	$-2.6667 + 1.3744i$	$-2.6667 - 1.3744i$
18	-3.0000	-3.0000
20	-1.8803	-4.7863
30	-1.0000	-9.0000
40	-0.7131	-12.6202
50	-5587	-16.1079
100	-0.2722	-33.0611
1000	-0.0270	-333.3063

■

Variation of the spring constant: Since the spring constant does not appear explicitly in Eq. (2.108), we consider a specific form of the characteristic equation (2.107) as:

$$s^2 + 16s + k = 0 \tag{2.121}$$

The roots of Eq. (2.121) are given by

$$s_{1,2} = \frac{-16 \pm \sqrt{256 - 4k}}{2} = -8 \pm \sqrt{64 - k} \tag{2.122}$$

Since the spring stiffness cannot be negative for real vibration systems, we consider the variation of the values of k from zero to infinity. Equation (2.122) shows that for $0 \le k < 64$, both the roots are real and identical. As k is made greater than 64, the roots become complex conjugates. The roots for different values of k are shown in Table 2.2. The variations of the two roots can be plotted (as dots), as shown in Fig. 2.40.

Variation of the mass: To find the migration of the roots with a variation of the mass m, we consider a specific form of the characteristic equation, Eq. (2.107), as

$$ms^2 + 14s + 20 = 0 \tag{2.123}$$

whose roots are given by

$$s_{1,2} = \frac{-14 \pm \sqrt{196 - 80m}}{2m} \tag{2.124}$$

Since negative values as well as zero value of mass need not be considered for physical systems, we vary the value of m in the range $1 \le m < \infty$. Some values of m and the corresponding roots given by Eq. (2.124) are shown in Table 2.3.

It can be seen that both the roots are negative with values $(-1.6148, -12.3852)$ for $m = 1$ and $(-2, -5)$ for $m = 2$. The larger root is observed to move to the left and the

TABLE 2.2

Value of k	Value of s_1	Value of s_2
0	0	−16
16	−1.0718	−14.9282
32	−2.3431	−13.6569
48	−4	−12
64	−8	−8
80	$-8 + 4i$	$-8 - 4i$
96	$-8 + 5.6569i$	$-8 - 5.6569i$
112	$-8 + 6.9282i$	$-8 - 6.9282i$
128	$-8 + 8i$	$-8 - 8i$

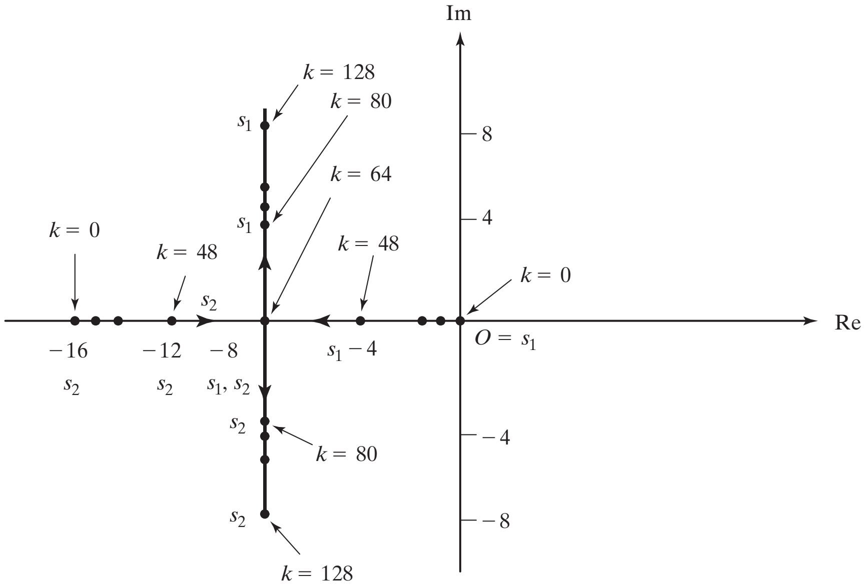


FIGURE 2.40 Root locus plot with variation of spring constant (k).

TABLE 2.3

Value of m	**Value of s_1**	**Value of s_2**
1	-1.6148	-12.3852
2	-2.0	-5.0
2.1	-2.0734	-4.5932
2.4	-2.5	-3.3333
2.45	-2.8571	-2.8571
2.5	$-2.8 + 0.4000i$	$-2.8 + 0.4000i$
3	$-2.3333 + 1.1055i$	$-2.3333 - 1.1055i$
5	$-1.4 + 1.4283i$	$-1.4 + 1.4283i$
8	$-0.8750 + 1.3169i$	$-0.8750 - 1.3169i$
10	$-0.7000 + 1.2288i$	$-0.7000 - 1.2288i$
14	$-0.5000 + 1.0856i$	$-0.5000 - 1.0856i$
20	$-0.3500 + 0.9367i$	$-0.3500 - 0.9367i$
30	$-0.2333 + 0.7824i$	$-0.2333 - 0.7824i$
40	$-0.1750 + 0.6851i$	$-0.1750 - 0.6851i$
50	$-0.1400 + 0.6167i$	$-0.1400 - 0.6167i$
100	$-0.0700 + 0.4417i$	$-0.0700 - 0.4417i$
1000	$-0.0070 + 0.1412i$	$-0.0070 - 0.1412i$

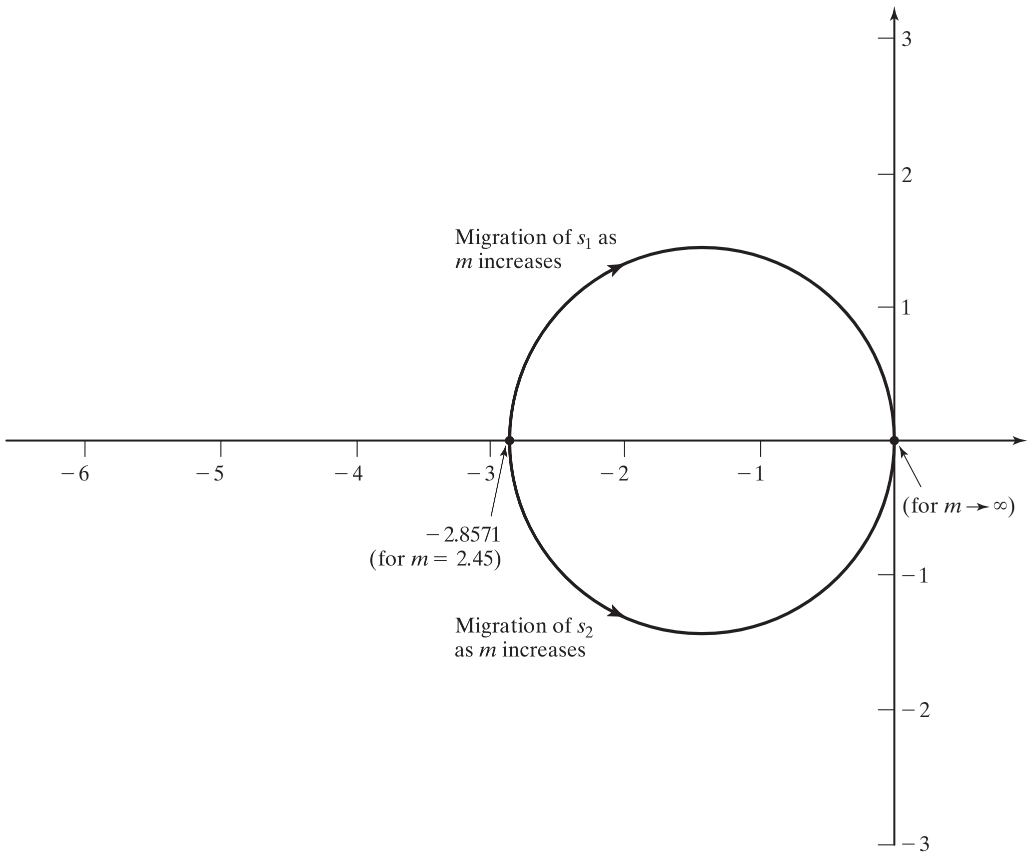


FIGURE 2.41 Root locus plot with variation of mass (m).

smaller root is found to move to the right, as shown in Fig. 2.41. The larger and smaller roots are found to converge to the value −2.8571 as m increases to a value of 2.45. Beyond this value of $m = 2.45$, the roots become complex conjugate. As the value of m increases from 2.45 to a large value ($\to \infty$), the loci of the two complex conjugates (roots) are shown by the curve (circle) shown in Fig. 2.41. For $m \to \infty$, both the complex conjugate roots converge to zero ($s_1, s_2 \to 0$).

2.9 Free Vibration with Coulomb Damping

In many mechanical systems, *Coulomb* or *dry-friction* dampers are used because of their mechanical simplicity and convenience [2.9]. Also, in vibrating structures, whenever the components slide relative to each other, dry-friction damping appears internally. As stated in Section 1.9, Coulomb damping arises when bodies slide on dry surfaces. Coulomb's law of dry friction states that, when two bodies are in contact, the force required to produce sliding is proportional to the normal force acting in the plane of contact. Thus the friction force F is given by

$$F = \mu N = \mu W = \mu m g \tag{2.125}$$

where N is the normal force, equal to the weight of the mass $(W = mg)$ and μ is the coefficient of sliding or kinetic friction. The value of the coefficient of friction (μ) depends on the materials in contact and the condition of the surfaces in contact. For example, $\mu \simeq 0.1$ for metal on metal (lubricated), 0.3 for metal on metal (unlubricated), and nearly 1.0 for rubber on metal. The friction force acts in a direction opposite to the direction of velocity. Coulomb damping is sometimes called *constant damping,* since the damping force is independent of the displacement and velocity; it depends only on the normal force N between the sliding surfaces.

2.9.1 Equation of Motion

Consider a single-degree-of-freedom system with dry friction as shown in Fig. 2.42(a). Since the friction force varies with the direction of velocity, we need to consider two cases, as indicated in Figs. 2.42(b) and (c).

Case 1. When x is positive and dx/dt is positive or when x is negative and dx/dt is positive (i.e., for the half cycle during which the mass moves from left to right), the equation of motion can be obtained using Newton's second law (see Fig. 2.42(b)):

$$m\ddot{x} = -kx - \mu N \quad \text{or} \quad m\ddot{x} + kx = -\mu N \tag{2.126}$$

This is a second-order nonhomogeneous differential equation. The solution can be verified by substituting Eq. (2.127) into Eq. (2.126):

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu N}{k} \tag{2.127}$$

where $\omega_n = \sqrt{k/m}$ is the frequency of vibration and A_1 and A_2 are constants whose values depend on the initial conditions of this half cycle.

Case 2. When x is positive and dx/dt is negative or when x is negative and dx/dt is negative (i.e., for the half cycle during which the mass moves from right to left), the equation of motion can be derived from Fig. 2.42(c) as

$$-kx + \mu N = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = \mu N \tag{2.128}$$

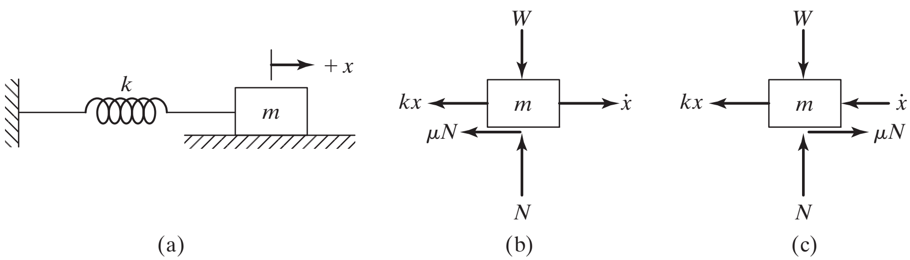


FIGURE 2.42 Spring-mass system with Coulomb damping.

The solution of Eq. (2.128) is given by

$$x(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu N}{k} \tag{2.129}$$

where A_3 and A_4 are constants to be found from the initial conditions of this half cycle. The term $\mu N/k$ appearing in Eqs. (2.127) and (2.129) is a constant representing the virtual displacement of the spring under the force μN, if it were applied as a static force. Equations (2.127) and (2.129) indicate that in each half cycle the motion is harmonic, with the equilibrium position changing from $\mu N/k$ to $-(\mu N/k)$ every half cycle, as shown in Fig. 2.43.

2.9.2 Solution

Equations (2.126) and (2.128) can be expressed as a single equation (using $N = mg$):

$$m\ddot{x} + \mu mg \operatorname{sgn}(\dot{x}) + kx = 0 \tag{2.130}$$

where sgn(y) is called the signum function, whose value is defined as 1 for $y > 0$, -1 for $y < 0$, and 0 for $y = 0$. Equation (2.130) can be seen to be a nonlinear differential equation for which a simple analytical solution does not exist. Numerical methods can be used to solve Eq. (2.130) conveniently (see Example 2.21). Equation (2.130), however, can be solved analytically if we break the time axis into segments separated by $\dot{x} = 0$ (i.e., time intervals with different directions of motion). To find the solution using this procedure, let us assume the initial conditions as

$$\begin{aligned} x(t = 0) &= x_0 \\ \dot{x}(t = 0) &= 0 \end{aligned} \tag{2.131}$$

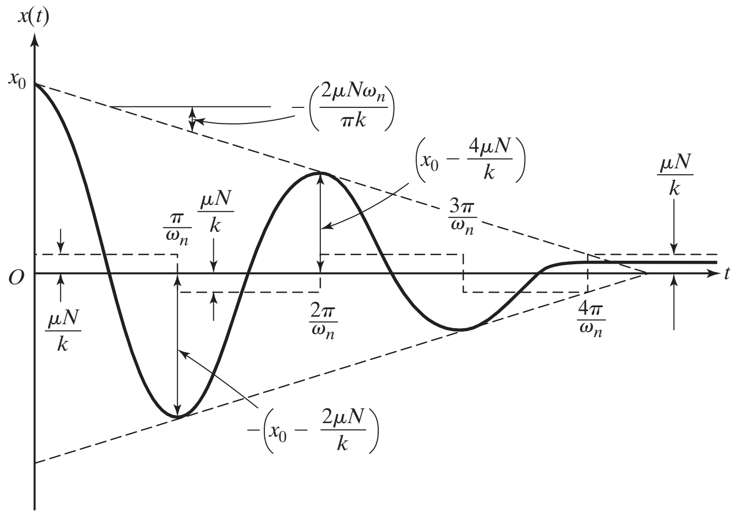


FIGURE 2.43 Motion of the mass with Coulomb damping.

That is, the system starts with zero velocity and displacement x_0 at $t = 0$. Since $x = x_0$ at $t = 0$, the motion starts from right to left. Let $x_0, x_1, x_2, \ldots$ denote the amplitudes of motion at successive half cycles. Using Eqs. (2.129) and (2.131), we can evaluate the constants A_3 and A_4:

$$A_3 = x_0 - \frac{\mu N}{k}, \qquad A_4 = 0$$

Thus Eq. (2.129) becomes

$$x(t) = \left(x_0 - \frac{\mu N}{k}\right)\cos \omega_n t + \frac{\mu N}{k} \tag{2.132}$$

This solution is valid for half the cycle only—that is, for $0 \leq t \leq \pi/\omega_n$. When $t = \pi/\omega_n$, the mass will be at its extreme left position and its displacement from equilibrium position can be found from Eq. (2.132):

$$-x_1 = x\left(t = \frac{\pi}{\omega_n}\right) = \left(x_0 - \frac{\mu N}{k}\right)\cos \pi + \frac{\mu N}{k} = -\left(x_0 - \frac{2\mu N}{k}\right)$$

Since the motion started with a displacement of $x = x_0$ and, in a half cycle, the value of x became $-[x_0 - (2\mu N/k)]$, the reduction in magnitude of x in time π/ω_n is $2\mu N/k$.

In the second half cycle, the mass moves from left to right, so Eq. (2.127) is to be used. The initial conditions for this half cycle are

$$x(t = 0) = \text{value of } x \text{ at } t = \frac{\pi}{\omega_n} \text{ in Eq. (2.132)} = -\left(x_0 - \frac{2\mu N}{k}\right)$$

and

$$\begin{aligned}\dot{x}(t = 0) &= \text{value of } \dot{x} \text{ at } t = \frac{\pi}{\omega_n} \text{ in Eq. (2.132)}\\ &= \left\{\text{value of } -\omega_n\left(x_0 - \frac{\mu N}{k}\right)\sin \omega_n t \text{ at } t = \frac{\pi}{\omega_n}\right\} = 0\end{aligned}$$

Thus the constants in Eq. (2.127) become

$$-A_1 = -x_0 + \frac{3\mu N}{k}, \qquad A_2 = 0$$

so that Eq. (2.127) can be written as

$$x(t) = \left(x_0 - \frac{3\mu N}{k}\right)\cos \omega_n t - \frac{\mu N}{k} \tag{2.133}$$

This equation is valid only for the second half cycle—that is, for $\pi/\omega_n \leq t \leq 2\pi/\omega_n$. At the end of this half cycle the value of $x(t)$ is

$$x_2 = x\left(t = \frac{\pi}{\omega_n}\right) \text{ in Eq. (2.133)} = x_0 - \frac{4\mu N}{k}$$

and

$$\dot{x}\left(t = \frac{\pi}{\omega_n}\right) \text{ in Eq. (2.133)} = 0$$

These become the initial conditions for the third half cycle, and the procedure can be continued until the motion stops. The motion stops when $x_n \leq \mu N/k$, since the restoring force exerted by the spring (kx) will then be less than the friction force μN. Thus the number of half cycles (r) that elapse before the motion ceases is given by

$$x_0 - r\frac{2\mu N}{k} \leq \frac{\mu N}{k}$$

—that is,

$$r \geq \left\{ \frac{x_0 - \dfrac{\mu N}{k}}{\dfrac{2\mu N}{k}} \right\} \tag{2.134}$$

Note the following characteristics of a system with Coulomb damping:

1. The equation of motion is nonlinear with Coulomb damping, while it is linear with viscous damping.
2. The natural frequency of the system is unaltered with the addition of Coulomb damping, while it is reduced with the addition of viscous damping.
3. The motion is periodic with Coulomb damping, while it can be nonperiodic in a viscously damped (overdamped) system.
4. The system comes to rest after some time with Coulomb damping, whereas the motion theoretically continues forever (perhaps with an infinitesimally small amplitude) with viscous and hysteresis damping.
5. The amplitude reduces linearly with Coulomb damping, whereas it reduces exponentially with viscous damping.
6. In each successive cycle, the amplitude of motion is reduced by the amount $4\mu N/k$, so the amplitudes at the end of any two consecutive cycles are related:

$$X_m = X_{m-1} - \frac{4\mu N}{k} \tag{2.135}$$

As the amplitude is reduced by an amount $4\mu N/k$ in one cycle (i.e., in time $2\pi/\omega_n$), the slope of the enveloping straight lines (shown dotted) in Fig. 2.43 is

$$-\left(\frac{4\mu N}{k}\right)\Big/\left(\frac{2\pi}{\omega_n}\right) = -\left(\frac{2\mu N\omega_n}{\pi k}\right)$$

The final position of the mass is usually displaced from equilibrium $(x = 0)$ position and represents a permanent displacement in which the friction force is locked. Slight tapping will usually make the mass come to its equilibrium position.

2.9.3 Torsional Systems with Coulomb Damping

If a constant frictional torque acts on a torsional system, the equation governing the angular oscillations of the system can be derived, similar to Eqs. (2.126) and (2.128), as

$$J_0\ddot{\theta} + k_t\theta = -T \tag{2.136}$$

and

$$J_0\ddot{\theta} + k_t\theta = T \tag{2.137}$$

where T denotes the constant damping torque (similar to μN for linear vibrations). The solutions of Eqs. (2.136) and (2.137) are similar to those for linear vibrations. In particular, the frequency of vibration is given by

$$\omega_n = \sqrt{\frac{k_t}{J_0}} \tag{2.138}$$

and the amplitude of motion at the end of the rth half cycle (θ_r) is given by

$$\theta_r = \theta_0 - r\frac{2T}{k_t} \tag{2.139}$$

where θ_0 is the initial angular displacement at $t = 0$ (with $\dot{\theta} = 0$ at $t = 0$). The motion ceases when

$$r \geq \left\{\frac{\theta_0 - \dfrac{T}{k_t}}{\dfrac{2T}{k_t}}\right\} \tag{2.140}$$

EXAMPLE 2.14 Coefficient of Friction from Measured Positions of Mass

A metal block, placed on a rough surface, is attached to a spring and is given an initial displacement of 10 cm from its equilibrium position. After five cycles of oscillation in 2 s, the final position of the metal block is found to be 1 cm from its equilibrium position. Find the coefficient of friction between the surface and the metal block.

Solution: Since five cycles of oscillation were observed to take place in 2 s, the period (τ_n) is $2/5 = 0.4$ s, and hence the frequency of oscillation is $\omega_n = \sqrt{\frac{k}{m}} = \frac{2\pi}{\tau_n} = \frac{2\pi}{0.4} = 15.708$ rad/s. Since the amplitude of oscillation reduces by

$$\frac{4\mu N}{k} = \frac{4\mu mg}{k}$$

in each cycle, the reduction in amplitude in five cycles is

$$5\left(\frac{4\mu mg}{k}\right) = 0.10 - 0.01 = 0.09 \text{ m}$$

or

$$\mu = \frac{0.09k}{20mg} = \frac{0.09\omega_n^2}{20g} = \frac{0.09(15.708)^2}{20(9.81)} = 0.1132$$

■

EXAMPLE 2.15 Pulley Subjected to Coulomb Damping

A steel shaft of length 1 m and diameter 50 mm is fixed at one end and carries a pulley of mass moment of inertia 25 kg-m^2 at the other end. A band brake exerts a constant frictional torque of 400 N-m around the circumference of the pulley. If the pulley is displaced by 6° and released, determine (1) the number of cycles before the pulley comes to rest and (2) the final settling position of the pulley.

Solution:

1. The number of half cycles that elapse before the angular motion of the pulley ceases is given by Eq. (2.140):

$$r \geq \left\{\frac{\theta_0 - \dfrac{T}{k_t}}{\dfrac{2T}{k_t}}\right\} \tag{E.1}$$

where θ_0 = initial angular displacement = 6° = 0.10472 rad, k_t = torsional spring constant of the shaft given by

$$k_t = \frac{GJ}{l} = \frac{(8 \times 10^{10})\left\{\dfrac{\pi}{32}(0.05)^4\right\}}{1} = 49{,}087.5 \text{ N-m/rad}$$

and T = constant friction torque applied to the pulley = 400 N-m. Equation (E.1) gives

$$r \geq \frac{0.10472 - \left(\dfrac{400}{49{,}087.5}\right)}{\left(\dfrac{800}{49{,}087.5}\right)} = 5.926$$

Thus the motion ceases after six half cycles.

2. The angular displacement after six half cycles is given by Eq. (2.120):

$$\theta = 0.10472 - 6 \times 2\left(\frac{400}{49{,}087.5}\right) = 0.006935 \text{ rad} = 0.39734°$$

Thus the pulley stops at 0.39734° from the equilibrium position on the same side of the initial displacement.

■

2.10 Free Vibration with Hysteretic Damping

Consider the spring-viscous-damper arrangement shown in Fig. 2.44(a). For this system, the force F needed to cause a displacement $x(t)$ is given by

$$F = kx + c\dot{x} \tag{2.141}$$

For a harmonic motion of frequency ω and amplitude X,

$$x(t) = X \sin \omega t \tag{2.142}$$

Equations (2.141) and (2.142) yield

$$\begin{aligned} F(t) &= kX \sin \omega t + cX\omega \cos \omega t \\ &= kx \pm c\omega\sqrt{X^2 - (X \sin \omega t)^2} \\ &= kx \pm c\omega\sqrt{X^2 - x^2} \end{aligned} \tag{2.143}$$

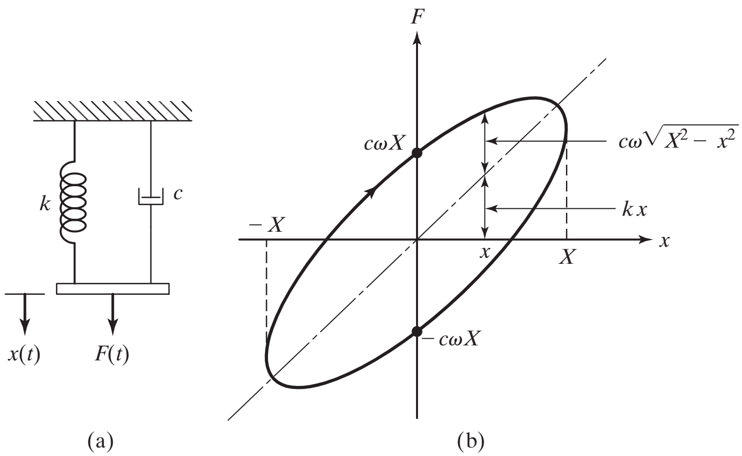


FIGURE 2.44 Spring-viscous-damper system.

When F versus x is plotted, Eq. (2.143) represents a closed loop, as shown in Fig. 2.44(b). The area of the loop denotes the energy dissipated by the damper in a cycle of motion and is given by

$$\Delta W = \oint F\, dx = \int_0^{2\pi/\omega} (kX \sin \omega t + cX\omega \cos \omega t)(\omega X \cos \omega t)\, dt$$
$$= \pi \omega c X^2 \tag{2.144}$$

Equation (2.144) has been derived in Section 2.6.4 also (see Eq. (2.98)).

As stated in Section 1.9, the damping caused by the friction between the internal planes that slip or slide as the material deforms is called hysteresis (or solid or structural) damping. This causes a hysteresis loop to be formed in the stress-strain or force-displacement curve (see Fig. 2.45(a)). The energy loss in one loading and unloading cycle is equal to the area enclosed by the hysteresis loop [2.11–2.13]. The similarity between Figs. 2.44(b) and 2.45(a) can be used to define a hysteresis damping constant. It was found experimentally that the energy loss per cycle due to internal friction is independent of the frequency but approximately proportional to the square of the amplitude. In order to achieve this observed behavior from Eq. (2.144), the damping coefficient c is assumed to be inversely proportional to the frequency as

$$c = \frac{h}{\omega} \tag{2.145}$$

where h is called the hysteresis damping constant. Equations (2.144) and (2.145) give

$$\Delta W = \pi h X^2 \tag{2.146}$$

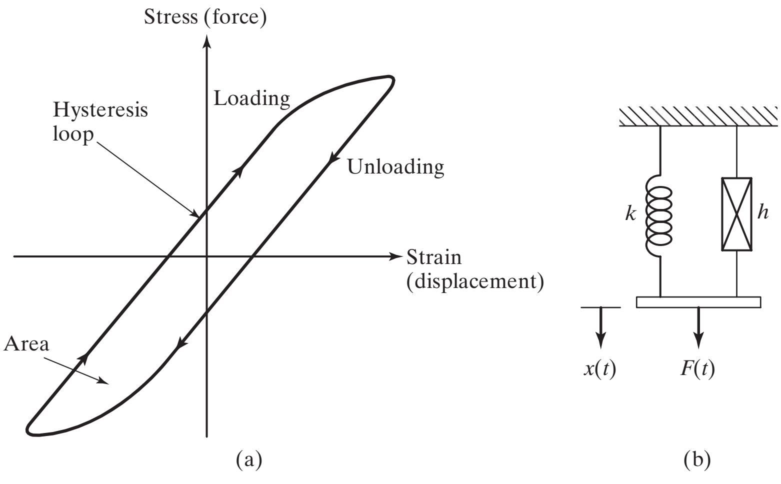


FIGURE 2.45 Hysteresis loop.

Complex Stiffness. In Fig. 2.44(a), the spring and the damper are connected in parallel, and for a general harmonic motion, $x = Xe^{i\omega t}$, the force is given by

$$F = kXe^{i\omega t} + c\omega iXe^{i\omega t} = (k + i\omega c)x \tag{2.147}$$

Similarly, if a spring and a hysteresis damper are connected in parallel, as shown in Fig. 2.45(b), the force-displacement relation can be expressed as

$$F = (k + ih)x \tag{2.148}$$

where

$$k + ih = k\left(1 + i\frac{h}{k}\right) = k(1 + i\beta) \tag{2.149}$$

is called the complex stiffness of the system and $\beta = h/k$ is a constant indicating a dimensionless measure of damping.

Response of the System. In terms of β, the energy loss per cycle can be expressed as

$$\Delta W = \pi k \beta X^2 \tag{2.150}$$

Under hysteresis damping, the motion can be considered to be nearly harmonic (since ΔW is small), and the decrease in amplitude per cycle can be determined using energy balance. For example, the energies at points P and Q (separated by half a cycle) in Fig. 2.46 are related as

$$\frac{kX_j^2}{2} - \frac{\pi k \beta X_j^2}{4} - \frac{\pi k \beta X_{j+0.5}^2}{4} = \frac{kX_{j+0.5}^2}{2}$$

or

$$\frac{X_j}{X_{j+0.5}} = \sqrt{\frac{2 + \pi\beta}{2 - \pi\beta}} \tag{2.151}$$

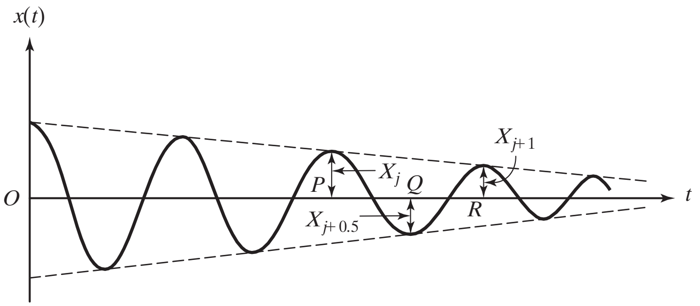


FIGURE 2.46 Response of a hysteretically damped system.

Similarly, the energies at points Q and R give

$$\frac{X_{j+0.5}}{X_{j+1}} = \sqrt{\frac{2 + \pi\beta}{2 - \pi\beta}} \tag{2.152}$$

Multiplication of Eqs. (2.151) and (2.152) gives

$$\frac{X_j}{X_{j+1}} = \frac{2 + \pi\beta}{2 - \pi\beta} = \frac{2 - \pi\beta + 2\pi\beta}{2 - \pi\beta} \simeq 1 + \pi\beta = \text{constant} \tag{2.153}$$

The hysteresis logarithmic decrement can be defined as

$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) \simeq \ln(1 + \pi\beta) \simeq \pi\beta \tag{2.154}$$

Since the motion is assumed to be approximately harmonic, the corresponding frequency is defined by [2.10]:

$$\omega = \sqrt{\frac{k}{m}} \tag{2.155}$$

The equivalent viscous damping ratio ζ_{eq} can be found by equating the relation for the logarithmic decrement δ:

$$\delta \simeq 2\pi\zeta_{\text{eq}} \simeq \pi\beta = \frac{\pi h}{k}$$

$$\zeta_{\text{eq}} = \frac{\beta}{2} = \frac{h}{2k} \tag{2.156}$$

Thus the equivalent damping constant c_{eq} is given by

$$c_{\text{eq}} = c_c \cdot \zeta_{\text{eq}} = 2\sqrt{mk} \cdot \frac{\beta}{2} = \beta\sqrt{mk} = \frac{\beta k}{\omega} = \frac{h}{\omega} \tag{2.157}$$

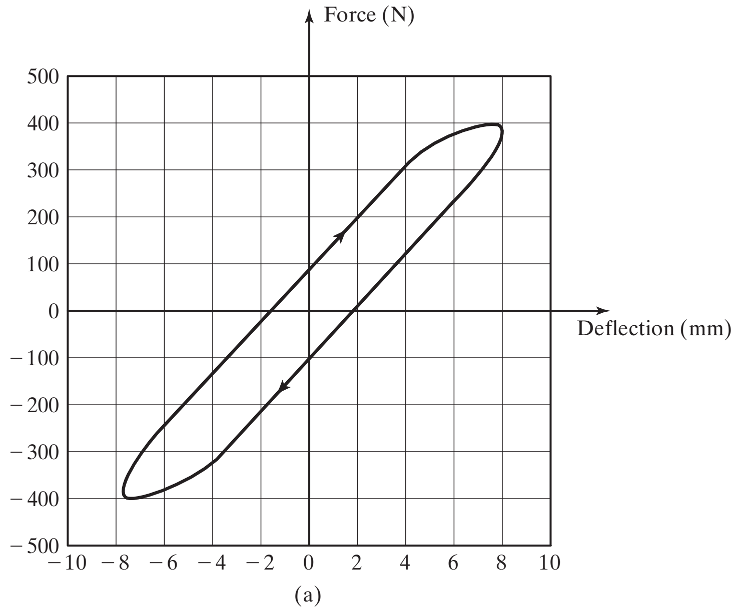
Note that the method of finding an equivalent viscous damping coefficient for a structurally damped system is valid only for harmonic excitation. The above analysis assumes that the system responds approximately harmonically at the frequency ω.

EXAMPLE 2.16 Estimation of Hysteretic Damping Constant

The experimental measurements on a structure gave the force-deflection data shown in Fig. 2.47. From this data, estimate the hysteretic damping constant β and the logarithmic decrement δ.

Solution

Approach: We equate the energy dissipated in a cycle (area enclosed by the hysteresis loop) to ΔW of Eq. (2.146).



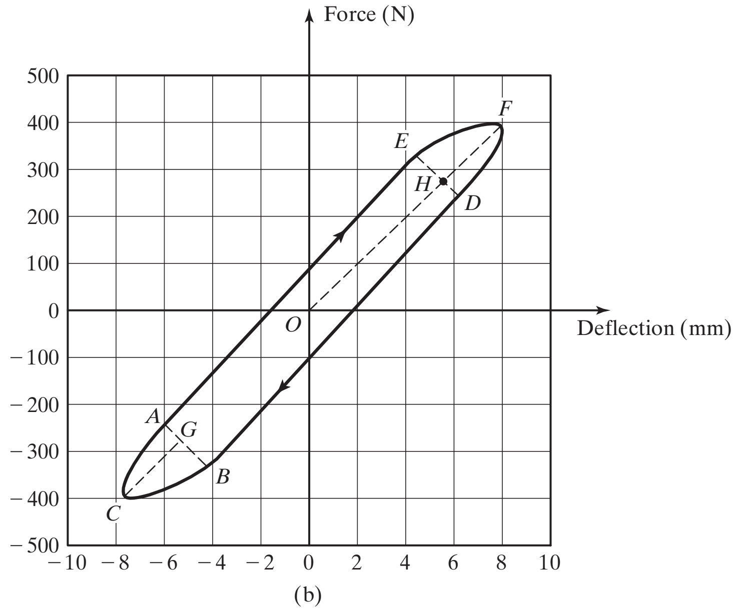


FIGURE 2.47 Force-deflection curve.

The energy dissipated in each full load cycle is given by the area enclosed by the hysteresis curve. Each square in Fig. 2.47 denotes $100 \times 2 = 200$ N-mm. The area enclosed by the loop can be found as area ACB + area $ABDE$ + area $DFE \simeq \frac{1}{2}(AB)(CG) + (AB)(AE) + \frac{1}{2}(DE)(FH) = \frac{1}{2}(1.25)(1.8) + (1.25)(8) + \frac{1}{2}(1.25)(1.8) = 12.25$ square units. This area represents an energy of $12.25 \times 200/1{,}000 = 2.5$ N-m. From Eq. (2.146), we have

$$\Delta W = \pi h X^2 = 2.5 \text{ N-m} \tag{E.1}$$

Since the maximum deflection X is 0.008 m and the slope of the force-deflection curve (given approximately by the slope of the line OF) is $k = 400/8 = 50$ N/mm $= 50{,}000$ N/m, the hysteretic damping constant h is given by

$$h = \frac{\Delta W}{\pi X^2} = \frac{2.5}{\pi(0.008)^2} = 12{,}433.95 \tag{E.2}$$

and hence

$$\beta = \frac{h}{k} = \frac{12{,}433.95}{50{,}000} = 0.248679$$

The logarithmic decrement can be found as

$$\delta \simeq \pi\beta = \pi(0.248679) = 0.78125 \tag{E.3}$$

■

EXAMPLE 2.17 Response of a Hysteretically Damped Bridge Structure

A bridge structure is modeled as a single-degree-of-freedom system with an equivalent mass of 5×10^5 kg and an equivalent stiffness of 25×10^6 N/m. During a free-vibration test, the ratio of successive amplitudes was found to be 1.04. Estimate the structural damping constant (β) and the approximate free-vibration response of the bridge.

Solution: Using the ratio of successive amplitudes, Eq. (2.154) yields the hysteresis logarithmic decrement (δ) as

$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) = \ln(1.04) = \ln(1 + \pi\beta)$$

or

$$1 + \pi\beta = 1.04 \quad \text{or} \quad \beta = \frac{0.04}{\pi} = 0.0127$$

The equivalent viscous damping coefficient (c_{eq}) can be determined from Eq. (2.157) as

$$c_{eq} = \frac{\beta k}{\omega} = \frac{\beta k}{\sqrt{\frac{k}{m}}} = \beta\sqrt{km} \tag{E.1}$$

Using the known values of the equivalent stiffness (k) and the equivalent mass (m) of the bridge, Eq. (E.1) yields

$$c_{\text{eq}} = (0.0127)\sqrt{(25 \times 10^6)(5 \times 10^5)} = 44.9013 \times 10^3 \text{ N-s/m}$$

The equivalent critical damping constant of the bridge can be computed using Eq. (2.65) as

$$c_c = 2\sqrt{km} = 2\sqrt{(25 \times 10^6)(5 \times 10^5)} = 7071.0678 \times 10^3 \text{ N-s/m}$$

Since $c_{\text{eq}} < c_c$, the bridge is underdamped, and hence its free-vibration response is given by Eq. (2.72) as

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos \sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \sin \sqrt{1-\zeta^2}\,\omega_n t\right\}$$

where

$$\zeta = \frac{c_{\text{eq}}}{c_c} = \frac{40.9013 \times 10^3}{7071.0678 \times 10^3} = 0.0063$$

and x_0 and $\dot{x}_0$ denote the initial displacement and initial velocity given to the bridge at the start of free vibration.

■

2.11 Stability of Systems

Stability is one of the most important characteristics for any vibrating system. Although many definitions can be given for the term *stability* depending on the kind of system or the point of view, we consider our definition for linear and time-invariant systems (i.e., systems for which the parameters m, c, and k do not change with time). A system is defined to be *asymptotically stable* (called *stable* in controls literature) if its free-vibration response approaches zero as time approaches infinity. A system is considered to be *unstable* if its free-vibration response grows without bound (approaches infinity) as time approaches infinity. Finally, a system is said to be *stable* (called *marginally stable* in controls literature) if its free-vibration response neither decays nor grows, but remains constant or oscillates as time approaches infinity. It is evident that an unstable system whose free-vibration response grows without bounds can cause damage to the system, adjacent property, or human life. Usually, dynamic systems are designed with limit stops to prevent their responses from growing with no limit.

As will be seen in Chapters 3 and 4, the total response of a vibrating system, subjected to external forces/excitations, is composed of two parts—one the forced response and the other the free-vibration response. For such systems, the definitions of asymptotically stable, unstable, and stable systems given above are still applicable. This implies that, for stable systems, only the forced response remains as the free-vibration response approaches zero as time approaches infinity.

Stability can be interpreted in terms of the roots of the characteristic roots of the system. As seen in Section 2.7, the roots lying in the left half-plane (LHP) yield either pure exponential decay or damped sinusoidal free-vibration responses. These responses decay to zero as time approaches infinity. Thus, systems whose characteristics roots lie in the left half of the s-plane (with a negative real part) will be asymptotically stable. The roots lying in the right half-plane yield either pure exponentially increasing or exponentially increasing sinusoidal free-vibration responses. These free-vibration responses approach infinity as time approaches infinity. Thus, systems whose characteristic roots lie in the right half of the s-plane (with positive real part) will be unstable. Finally, the roots lying on the imaginary axis of the s-plane yield pure sinusoidal oscillations as free-vibration response. These responses neither increase nor decrease in amplitude as time grows. Thus, systems whose characteristic roots lie on the imaginary axis of the s-plane (with zero real part) will be stable.[3]

Notes:

1. It is evident, from the definitions given, that the signs of the coefficients of the characteristic equation, Eq. (2.107), determine the stability behavior of a system. For example, from the theory of polynomial equations, if there is any number of negative terms or if any term in the polynomial in s is missing, then one of the roots will be positive, which results in an unstable behavior of the system. This aspect is considered further in Section 3.11 as well as in Section 5.8 in the form of the Routh-Hurwitz stability criterion.
2. In an unstable system, the free-vibration response may grow without bound with no oscillations or it may grow without bound with oscillations. The first behavior is called *divergent instability* and the second is called *flutter instability.* These cases are also known as *selfexcited vibration* (see Section 3.11).
3. If a linear model of a system is asymptotically stable, then it is not possible to find a set of initial conditions for which the response approaches infinity. On the other hand, if the linear model of the system is unstable, it is possible that certain initial conditions might make the response approach zero as time increases. As an example, consider a system governed by the equation of motion $\ddot{x} - x = 0$ with characteristic roots given by $s_{1,2} = \mp 1$. Thus the response is given by $x(t) = C_1 e^{-t} + C_2 e^{t}$, where C_1 and C_2 are constants. If the initial conditions are specified as $x(0) = 1$ and $\dot{x}(0) = -1$, we find that $C_1 = 1$ and $C_2 = 0$ and hence the response becomes $x(t) = e^{-t}$, which approaches zero as time increases to infinity.
4. Typical responses corresponding to different types of stability are shown in Figs. 2.48(a)–(d).
5. Stability of a system can also be explained in terms of its energy. According to this scheme, a system is considered to be asymptotically stable, stable, or unstable if its energy decreases, remains constant, or increases, respectively, with time. This idea forms the basis for Lyapunov stability criterion [2.14, 2.15].
6. Stability of a system can also be investigated based on how sensitive the response or motion is to small perturbations (or variations) in the parameters (m, c and k) and/or small perturbations in the initial conditions.

[3]Strictly speaking, the statement is true only if the roots that lie on the imaginary axis appear with multiplicity one. If such roots appear with multiplicity $n > 1$, the system will be unstable because the free-vibration response of such systems will be of the form $Ct^n \sin(\omega t + \phi)$.

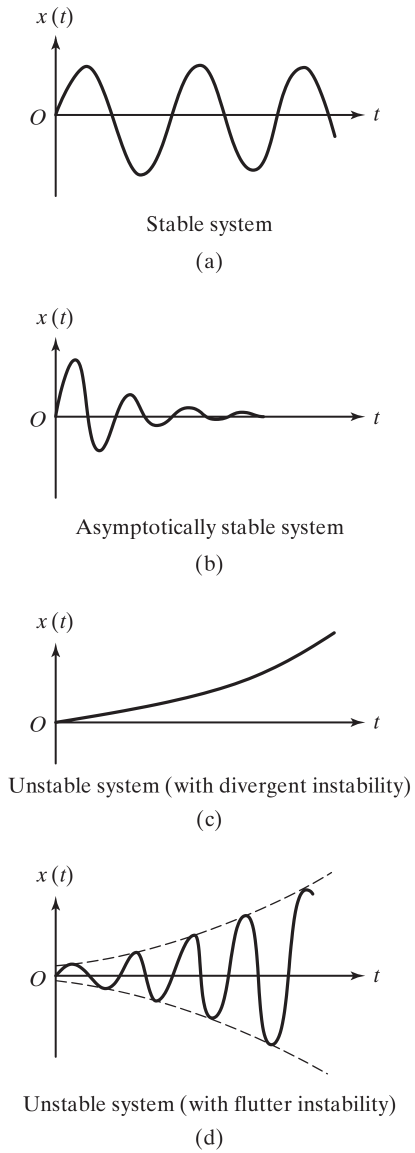


FIGURE 2.48 Different types of stability.

EXAMPLE 2.18 Stability of a System

Consider a uniform rigid bar, of mass m and length l, pivoted at one end and connected symmetrically by two springs at the other end, as shown in Fig. 2. 49. Assuming that the springs are unstretched when the bar is vertical, derive the equation of motion of the system for small angular displacements (θ) of the bar about the pivot point, and investigate the stability behavior of the system.

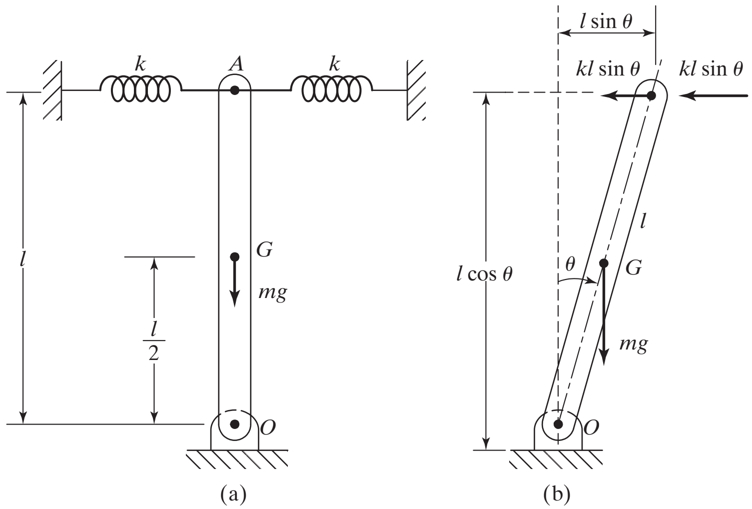


FIGURE 2.49 Stability of a rigid bar.

Solution: When the bar is displaced by an angle θ, the spring force in each spring is $kl \sin \theta$; the total spring force is $2kl \sin \theta$. The gravity force $W = mg$ acts vertically downward through the center of gravity, G. The moment about the point of rotation O due to the angular acceleration $\ddot{\theta}$ is $J_0 \ddot{\theta} = (ml^2/3)\ddot{\theta}$. Thus the equation of motion of the bar, for rotation about the point O, can be written as

$$\frac{ml^2}{3}\ddot{\theta} + (2kl \sin \theta) l \cos \theta - W \frac{l}{2} \sin \theta = 0 \tag{E.1}$$

For small oscillations, Eq. (E.1) reduces to

$$\frac{ml^2}{3}\ddot{\theta} + 2kl^2\theta - \frac{Wl}{2}\theta = 0 \tag{E.2}$$

or

$$\ddot{\theta} + \alpha^2 \theta = 0 \tag{E.3}$$

where

$$\alpha^2 = \left(\frac{12kl^2 - 3Wl}{2ml^2} \right) \tag{E.4}$$

The characteristic equation is given by

$$s^2 + \alpha^2 = 0 \tag{E.5}$$

and hence the solution of Eq. (E.2) depends on the sign of α^2 as indicated below.

Case 1. When $(12kl^2 - 3Wl)/2ml^2 > 0$, the solution of Eq. (E.2) represents a stable system with stable oscillations and can be expressed as

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{E.6}$$

where A_1 and A_2 are constants and

$$\omega_n = \left(\frac{12kl^2 - 3Wl}{2ml^2} \right)^{1/2} \tag{E.7}$$

Case 2. When $(12kl^2 - 3Wl)/2ml^2 = 0$, Eq. (E.2) reduces to $\ddot{\theta} = 0$ and the solution can be obtained directly by integrating twice as

$$\theta(t) = C_1 t + C_2 \tag{E.8}$$

For the initial conditions $\theta(t = 0) = \theta_0$ and $\dot{\theta}(t = 0) = \dot{\theta}_0$, the solution becomes

$$\theta(t) = \dot{\theta}_0 t + \theta_0 \tag{E.9}$$

Equation (E.9) shows that the system is unstable with the angular displacement increasing linearly at a constant velocity $\dot{\theta}_0$. However, if $\dot{\theta}_0 = 0$, Eq. (E.9) denotes a stable or static equilibrium position with $\theta = \theta_0$—that is, the pendulum remains in its original position, defined by $\theta = \theta_0$.

Case 3. When $(12kl^2 - 3Wl)/2ml^2 < 0$, the solution of Eq. (E.2) can be expressed as

$$\theta(t) = B_1 e^{\alpha t} + B_2 e^{-\alpha t} \tag{E.10}$$

where B_1 and B_2 are constants. For the initial conditions $\theta(t = 0) = \theta_0$ and $\dot{\theta}(t = 0) = \dot{\theta}_0$, Eq. (E.10) becomes

$$\theta(t) = \frac{1}{2\alpha}[(\alpha\theta_0 + \dot{\theta}_0)e^{\alpha t} + (\alpha\theta_0 - \dot{\theta}_0)e^{-\alpha t}] \tag{E.11}$$

Equation (E.11) shows that $\theta(t)$ increases exponentially with time; hence the motion is unstable. The physical reason for this is that the restoring moment due to the spring $(2kl^2\theta)$, which tries to bring the system to the equilibrium position, is less than the nonrestoring moment due to gravity $[-W(l/2)\theta]$, which tries to move the mass away from the equilibrium position.

■

2.12 Examples Using MATLAB

EXAMPLE 2.19 Variations of Natural Frequency and Period with Static Deflection

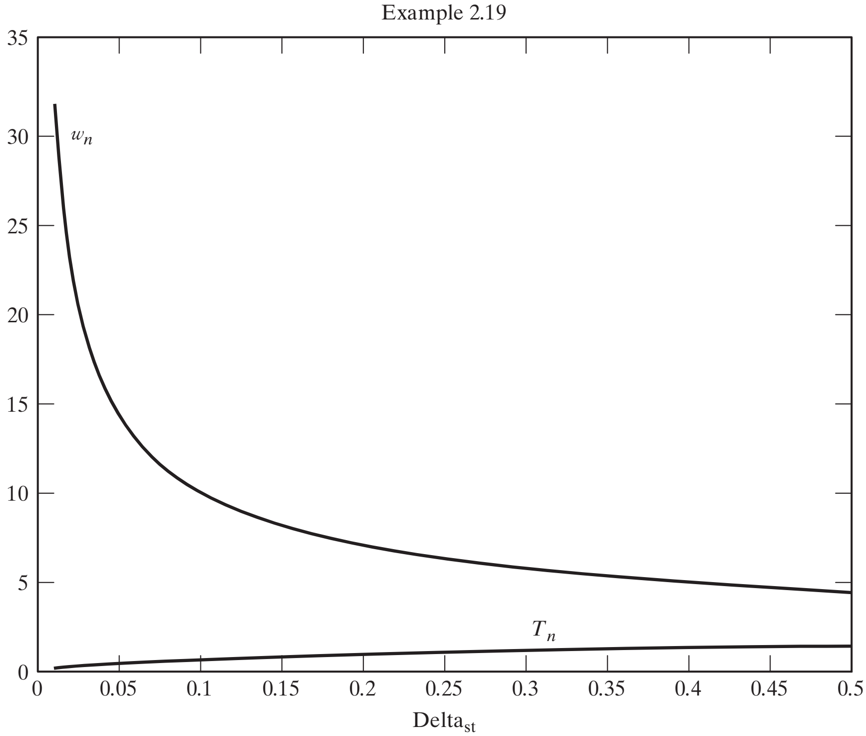
Plot the variations of the natural frequency and the time period with static deflection of an undamped system using MATLAB.

Solution: The natural frequency (ω_n) and the time period (τ_n) are given by Eqs. (2.28) and (2.30):

$$\omega_n = \left(\frac{g}{\delta_{st}} \right)^{1/2}, \qquad \tau_n = 2\pi \left(\frac{\delta_{st}}{g} \right)^{1/2}$$

Using $g = 9.81$ m/s^2, ω_n and τ_n are plotted over the range of $\delta_{st} = 0$ to 0.5 using a MATLAB program.

```
% Ex2_19.m
g = 9.81;
for i = 1: 101
    t(i) = 0.01 + (0.5-0.01) * (i-1)/100;
    w(i) = (g/t(i))^0.5;
    tao(i) = 2 * pi * (t(i)/g)^0.5;
end
plot(t,w);
gtext('w_n');
hold on;
plot(t, tao);
gtext('T_n');
xlabel('Delta_s_t');
title('Example 2.17');
```



Variations of natural frequency and time period.

■

EXAMPLE 2.20 Free-Vibration Response of a Spring-Mass System

A spring-mass system with a mass of 20 lb-sec^2/in. and stiffness 500 lb/in. is subject to an initial displacement of $x_0 = 3.0$ in. and an initial velocity of $\dot{x}_0 = 4.0$ in/sec. Plot the time variations of the mass's displacement, velocity, and acceleration using MATLAB.

Solution: The displacement of an undamped system can be expressed as (see Eq. (2.23)):

$$x(t) = A_0 \sin(\omega_n t + \phi_0) \tag{E.1}$$

where

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{500}{20}} = 5 \text{ rad/sec}$$

$$A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = \left[(3.0)^2 + \left(\frac{4.0}{5.0} \right)^2 \right]^{1/2} = 3.1048 \text{ in.}$$

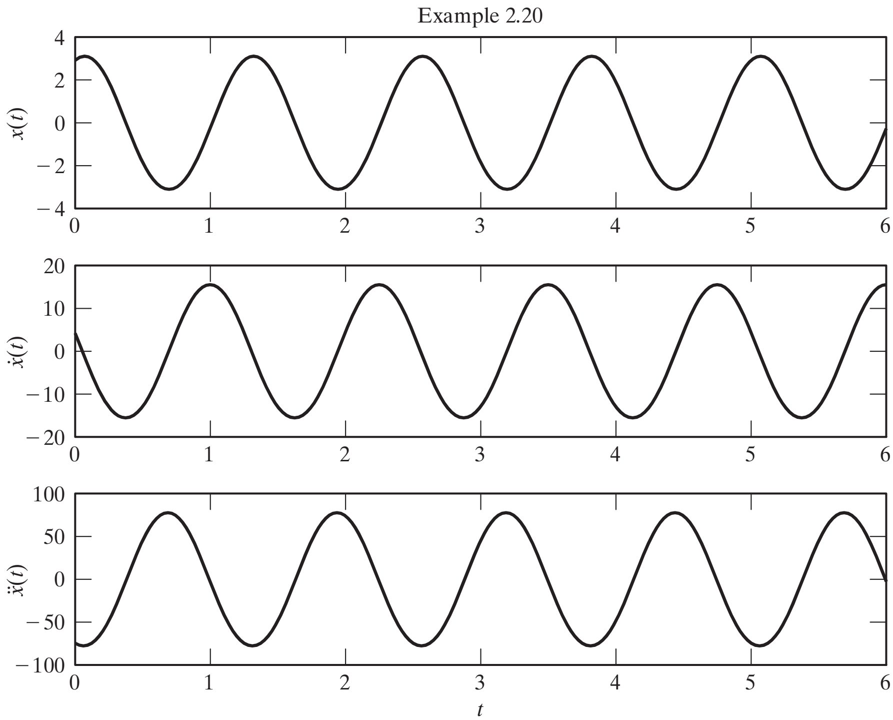
$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0} \right) = \tan^{-1}\left(\frac{(3.0)(5.0)}{4.0} \right) = 75.0686^\circ = 1.3102 \text{ rad}$$

Thus Eq. (E.1) yields

$$x(t) = 3.1048 \sin(5t + 1.3102) \text{ in.} \tag{E.2}$$

$$\dot{x}(t) = 15.524 \cos(5t + 1.3102) \text{ in./sec} \tag{E.3}$$

$$\ddot{x}(t) = -77.62 \sin(5t + 1.3102) \text{ in./sec}^2 \tag{E.4}$$



Response of an undamped system.

Equations (E.2)–(E.4) are plotted using MATLAB in the range $t = 0$ to 6 sec.

```
% Ex2_20.m
for i = 1: 101
    t(i) = 6 * (i-1)/100;
    x(i) = 3.1048 * sin(5 * t(i) + 1.3102);
    x1(i) = 15.524 * cos(5 * t(i) + 1.3102);
    x2(i) = -77.62 * sin(5 * t(i) + 1.3102);
end
subplot (311);
plot (t,x);
ylabel ('x(t)');
title ('Example 2.18');
subplot (312);
plot (t,x1);
ylabel ('x^.(t)');
subplot (313);
plot (t,x2);
xlabel ('t');
ylabel ('x^.^.(t)');
```

■

EXAMPLE 2.21 Free-Vibration Response of a System with Coulomb Damping

Find the free-vibration response of a spring-mass system subject to Coulomb damping for the following initial conditions: $x(0) = 0.5$ m, $\dot{x}(0) = 0$.

Data: $m = 10$ kg, $k = 200$ N/m, $\mu = 0.5$

Solution: The equation of motion can be expressed as

$$m\ddot{x} + \mu m g \operatorname{sgn}(\dot{x}) + kx = 0 \tag{E.1}$$

In order to solve the second-order differential equation, Eq. (E.1), using the Runge-Kutta method (see Appendix F), we rewrite Eq. (E.1) as a set of two first-order differential equations as follows:

$$x_1 = x, \qquad x_2 = \dot{x}_1 = \dot{x}$$

$$\dot{x}_1 = x_2 \equiv f_1(x_1, x_2) \tag{E.2}$$

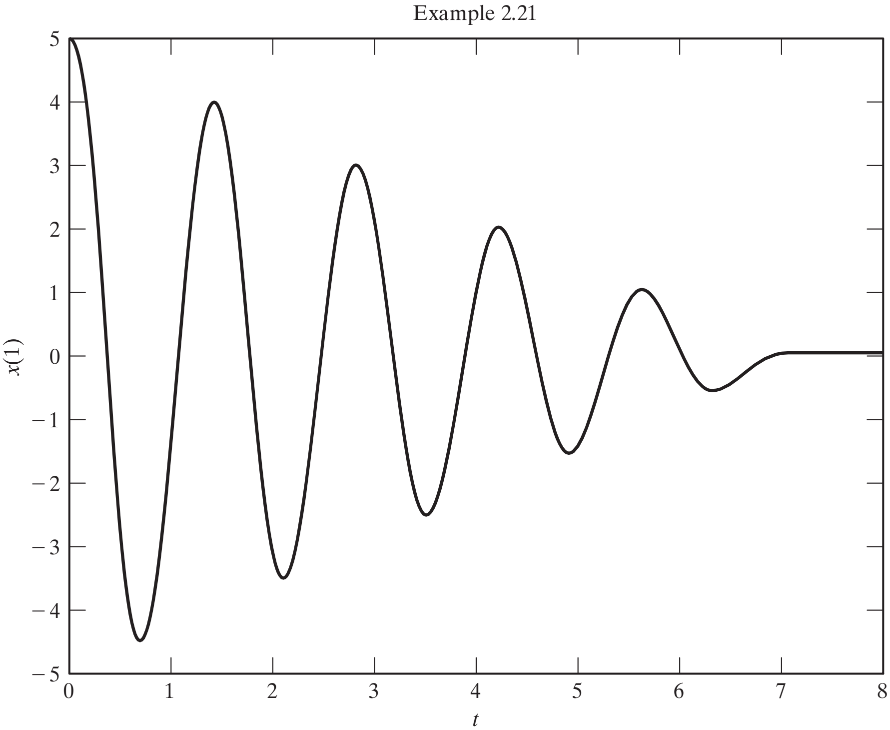
$$\dot{x}_2 = -\mu g \operatorname{sgn}(x_2) - \frac{k}{m}x_1 \equiv f_2(x_1, x_2) \tag{E.3}$$

Equations (E.2) and (E.3) can be expressed in matrix notation as

$$\dot{\vec{X}} = \vec{f}(\vec{X}) \tag{E.4}$$

where

$$\vec{X} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}, \qquad \vec{f} = \begin{Bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{Bmatrix}, \qquad \vec{X}(t = 0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix}$$



Solution of Eq. (E4):

The MATLAB program **ode23** is used to find the solution of Eq. (E.4) as shown below.

```
% Ex2_21.m
% This program will use dfunc1.m
tspan = [0: 0.05: 8];
x0 = [5.0; 0.0];
[t, x] = ode23 ('dfuncl', tspan, x0);
plot (t, x(:, 1));
xlabel ('t');
ylabel ('x(1)');
title ('Example 2.19');

% dfunc1.m
function f = dfuncl (t, x)
f = zeros (2, 1);
f(1) = x(2);
f(2) = -0.5 * 9.81 * sign(x(2)) - 200 * x(1) / 10;
```

■

EXAMPLE 2.22 Free Vibration Response of a Viscously Damped System Using MATLAB

Develop a general-purpose MATLAB program, called **Program2.m**, to find the free-vibration response of a viscously damped system. Use the program to find the response of a system with the following data:

$$m = 450.0, \quad k = 26519.2, \quad c = 1000.0, \quad x_0 = 0.539657, \quad \dot{x}_0 = 1.0$$

Solution: Program2.m is developed to accept the following input data:

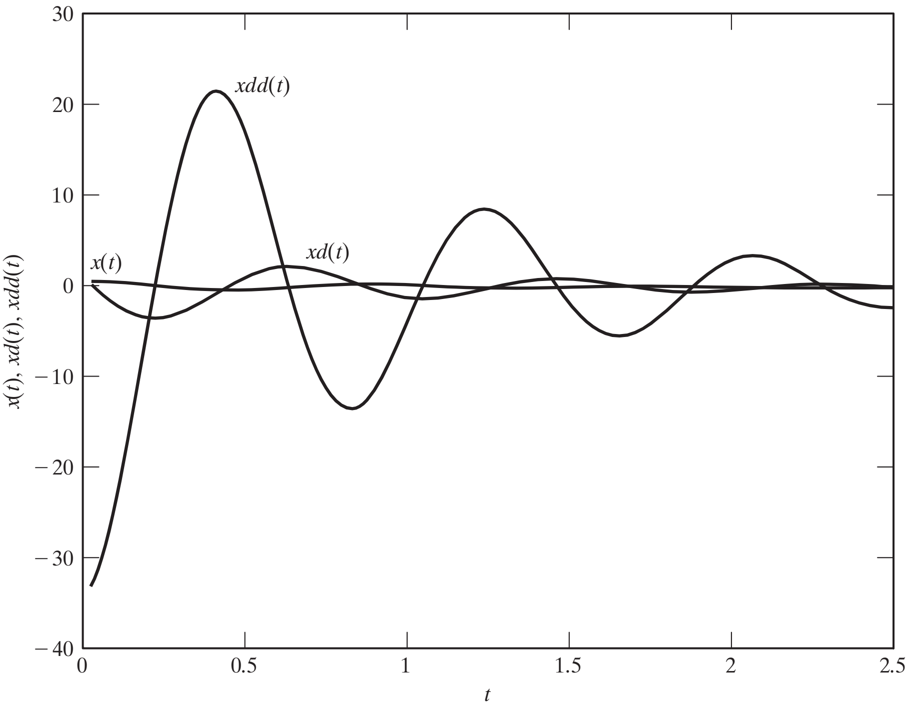
m = mass
k = spring stiffness
c = damping constant
x_0 = initial displacement
xd_0 = initial velocity
n = number of time steps at which values of $x(t)$ are to be found
delt = time interval between consecutive time steps (Δt)

The program gives the following output:

step number i, time (i), $x(i)$, $\dot{x}(i)$, $\ddot{x}(i)$

The program also plots the vatiations of x, $\dot{x}$, and $\ddot{x}$ with time.

```
 >> program2
Free vibration analysis of a single degree of freedom analysis

Data:

m=          4.50000000e+002
k=          2.65192000e+004
c=          1.00000000e+003
x0=         5.39657000e-001
xd0=        1.00000000e+000
n=          100
delt=       2.50000000e-002

system is under damped

Results:

 i  time(i)            x(i)              xd(i)             xdd(i)

  1 2.500000e-002    5.540992e-001     1.596159e-001    -3.300863e+001
  2 5.000000e-002    5.479696e-001    -6.410545e-001    -3.086813e+001
  3 7.500000e-002    5.225989e-001    -1.375559e+000    -2.774077e+001
  4 1.000000e-001    4.799331e-001    -2.021239e+000    -2.379156e+001
  5 1.250000e-001    4.224307e-001    -2.559831e+000    -1.920599e+001
  6 1.500000e-001    3.529474e-001    -2.977885e+000    -1.418222e+001
  .
  .
  .
 96 2.400000e+000    2.203271e-002    2.313895e-001    -1.812621e+000
 97 2.425000e+000    2.722809e-002    1.834092e-001    -2.012170e+000
 98 2.450000e+000    3.117018e-002    1.314707e-001    -2.129064e+000
 99 2.475000e+000    3.378590e-002    7.764312e-002    -2.163596e+000
100 2.500000e+000    3.505350e-002    2.395118e-002    -2.118982e+000
```



Variations of x, $\dot{x}$, and $\ddot{x}$.

■

CHAPTER SUMMARY

We considered the equations of motion and their solutions for the free vibration of undamped and damped single-degree-of-freedom systems. Four different methods—namely, Newton's second law of motion, D' Alembert's principle, the principle of virtual displacements, and the principle of conservation of energy—were presented for deriving the equation of motion of undamped systems. Both translational and torsional systems were considered. The free-vibration solutions have been presented for undamped systems. The equation of motion, in the form of a first-order differential equation, was considered for a mass-damper system (with no spring), and the idea of time constant was introduced.

The free-vibration solution of viscously damped systems was presented along with the concepts of underdamped, overdamped, and critically damped systems. The free-vibration solutions of systems with Coulomb and hysteretic damping were also considered. The graphical representation of characteristic roots in the complex plane and the corresponding solutions were explained. The effects of variation of the parameters m, c, and k on the characteristic roots and their representations using root locus plots were also considered. The identification of the stability status of a system was also explained.

Now that you have finished this chapter, you should be able to answer the review questions and solve the problems given below.

REFERENCES

2.1 R. W. Fitzgerald, *Mechanics of Materials* (2nd ed.), Addison-Wesley, Reading, MA, 1982.

2.2 R. F. Steidel, Jr., *An Introduction to Mechanical Vibrations* (4th ed.), Wiley, New York, 1989.

2.3 W. Zambrano, "A brief note on the determination of the natural frequencies of a spring-mass system," *International Journal of Mechanical Engineering Education,* Vol. 9, October 1981, pp. 331–334; Vol. 10, July 1982, p. 216.

2.4 R. D. Blevins, *Formulas for Natural Frequency and Mode Shape,* Van Nostrand Reinhold, New York, 1979.

2.5 A. D. Dimarogonas, *Vibration Engineering,* West Publishing, Saint Paul, MN, 1976.

2.6 E. Kreyszig, *Advanced Engineering Mathematics* (9th ed.), Wiley, New York, 2006.

2.7 S. H. Crandall, "The role of damping in vibration theory," *Journal of Sound and Vibration,* Vol. 11, 1970, pp. 3–18.

2.8 I. Cochin, *Analysis and Design of Dynamic Systems* (3rd ed.), Addison-Wesley, Reading, MA, 1997.

2.9 D. Sinclair, "Frictional vibrations," *Journal of Applied Mechanics,* Vol. 22, 1955, pp. 207–214.

2.10 T. K. Caughey and M. E. J. O'Kelly, "Effect of damping on the natural frequencies of linear dynamic systems," *Journal of the Acoustical Society of America,* Vol. 33, 1961, pp. 1458–1461.

2.11 E. E. Ungar, "The status of engineering knowledge concerning the damping of built-up structures," *Journal of Sound and Vibration,* Vol. 26, 1973, pp. 141–154.

2.12 W. Pinsker, "Structural damping," *Journal of the Aeronautical Sciences,* Vol. 16, 1949, p. 699.

2.13 R. H. Scanlan and A. Mendelson, "Structural damping," *AIAA Journal,* Vol. 1, 1963, pp. 938–939.

2.14 D. J. Inman, *Vibration with Control, Measurement, and Stability,* Prentice Hall, Englewood Cliffs, NJ, 1989.

2.15 G. F. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems* (5th ed.), Pearson Prentice Hall, Upper Saddle River, NJ, 2006.

2.16 N. S. Nise, *Control Systems Engineering* (3rd ed.), Wiley, New York, 2000.

2.17 K. Ogata, *System Dynamics* (4th ed.), Pearson Prentice Hall, Upper Saddle River, NJ, 2004.

REVIEW QUESTIONS

2.1 Give brief answers to the following:

1. Suggest a method for determining the damping constant of a highly damped vibrating system that uses viscous damping.
2. Can you apply the results of Section 2.2 to systems where the restoring force is not proportional to the displacement—that is, where k is not a constant?

3. State the parameters corresponding to m, c, k, and x for a torsional system.
4. What effect does a decrease in mass have on the frequency of a system?
5. What effect does a decrease in the stiffness of the system have on the natural period?
6. Why does the amplitude of free vibration gradually diminish in practical systems?
7. Why is it important to find the natural frequency of a vibrating system?
8. How many arbitrary constants must a general solution to a second-order differential equation have? How are these constants determined?
9. Can the energy method be used to find the differential equation of motion of all single-degree-of-freedom systems?
10. What assumptions are made in finding the natural frequency of a single-degree-of-freedom system using the energy method?
11. Is the frequency of a damped free vibration smaller or greater than the natural frequency of the system?
12. What is the use of the logarithmic decrement?
13. Is hysteresis damping a function of the maximum stress?
14. What is critical damping, and what is its importance?
15. What happens to the energy dissipated by damping?
16. What is equivalent viscous damping? Is the equivalent viscous-damping factor a constant?
17. What is the reason for studying the vibration of a single-degree-of-freedom system?
18. How can you find the natural frequency of a system by measuring its static deflection?
19. Give two practical applications of a torsional pendulum.
20. Define these terms: damping ratio, logarithmic decrement, loss coefficient, and specific damping capacity.
21. In what ways is the response of a system with Coulomb damping different from that of systems with other types of damping?
22. What is complex stiffness?
23. Define the hysteresis damping constant.
24. Give three practical applications of the concept of center of percussion.
25. What is the order of the equation of motion given by $m\dot{v} + cv = 0$?
26. Define the time constant.
27. What is a root locus plot?
28. What is the significance of $c < 0$?
29. What is a time-invariant system?

2.2 Indicate whether each of the following statements is true or false:

1. The amplitude of an undamped system will not change with time.
2. A system vibrating in air can be considered a damped system.
3. The equation of motion of a single-degree-of-freedom system will be the same whether the mass moves in a horizontal plane or an inclined plane.
4. When a mass vibrates in a vertical direction, its weight can always be ignored in deriving the equation of motion.
5. The principle of conservation of energy can be used to derive the equation of motion of both damped and undamped systems.
6. The damped frequency can in some cases be larger than the undamped natural frequency of the system.
7. The damped frequency can be zero in some cases.

8. The natural frequency of vibration of a torsional system is given by $\sqrt{\frac{k}{m}}$, where k and m denote the torsional spring constant and the polar mass moment of inertia, respectively.
9. Rayleigh's method is based on the principle of conservation of energy.
10. The final position of the mass is always the equilibrium position in the case of Coulomb damping.
11. The undamped natural frequency of a system is given by $\sqrt{g/\delta_{st}}$, where δ_{st} is the static deflection of the mass.
12. For an undamped system, the velocity leads the displacement by $\pi/2$.
13. For an undamped system, the velocity leads the acceleration by $\pi/2$.
14. Coulomb damping can be called constant damping.
15. The loss coefficient denotes the energy dissipated per radian per unit strain energy.
16. The motion diminishes to zero in both underdamped and overdamped cases.
17. The logarithmic decrement can be used to find the damping ratio.
18. The hysteresis loop of the stress-strain curve of a material causes damping.
19. The complex stiffness can be used to find the damping force in a system with hysteresis damping.
20. Motion in the case of hysteresis damping can be considered harmonic.
21. In the s-plane, the locus corresponding to constant natural frequency will be a circle.
22. The characteristic equation of a single-degree-of-freedom system can have one real root and one complex root.

2.3 Fill in the blanks with proper words:

1. The free vibration of an undamped system represents interchange of _____ and _____ energies.
2. A system undergoing simple harmonic motion is called a _____ oscillator.
3. The mechanical clock represents a _____ pendulum.
4. The center of _____ can be used advantageously in a baseball bat.
5. With viscous and hysteresis damping, the motion _____ forever, theoretically.
6. The damping force in Coulomb damping is given by _____.
7. The _____ coefficient can be used to compare the damping capacity of different engineering materials.
8. Torsional vibration occurs when a _____ body oscillates about an axis.
9. The property of _____ damping is used in many practical applications, such as large guns.
10. The logarithmic decrement denotes the rate at which the _____ of a free damped vibration decreases.
11. Rayleigh's method can be used to find the _____ frequency of a system directly.
12. Any two successive displacements of the system, separated by a cycle, can be used to find the _____ decrement.
13. The damped natural frequency (ω_d) can be expressed in terms of the undamped natural frequency (ω_n) as _____.
14. The time constant denotes the time at which the initial response reduces by _____ percent.
15. The term e^{-2t} decays _____ than the term e^{-t} as time t increases.
16. In the s-plane, lines parallel to real axis denote systems having different _____ frequencies.

2.4 Select the most appropriate answer out of the multiple choices given:

1. The natural frequency of a system with mass m and stiffness k is given by:
a. $\frac{k}{m}$ **b.** $\sqrt{\frac{k}{m}}$ **c.** $\sqrt{\frac{m}{k}}$

2. In Coulomb damping, the amplitude of motion is reduced in each cycle by:
a. $\frac{\mu N}{k}$ **b.** $\frac{2\mu N}{k}$ **c.** $\frac{4\mu N}{k}$

3. The amplitude of an undamped system subject to an initial displacement 0 and initial velocity $\dot{x}_0$ is given by:
a. $\dot{x}_0$ **b.** $\dot{x}_0\omega_n$ **c.** $\frac{\dot{x}_0}{\omega_n}$

4. The effect of the mass of the spring can be accounted for by adding the following fraction of its mass to the vibrating mass:
a. $\frac{1}{2}$ **b.** $\frac{1}{3}$ **c.** $\frac{4}{3}$

5. For a viscous damper with damping constant c, the damping force is:
a. $c\dot{x}$ **b.** cx **c.** $c\ddot{x}$

6. The relative sliding of components in a mechanical system causes:
a. dry-friction damping **b.** viscous damping **c.** hysteresis damping

7. In torsional vibration, the displacement is measured in terms of a:
a. linear coordinate **b.** angular coordinate **c.** force coordinate

8. The damping ratio, in terms of the damping constant c and critical damping constant (c_c), is given by:
a. $\frac{c_c}{c}$ **b.** $\frac{c}{c_c}$ **c.** $\sqrt{\frac{c}{c_c}}$

9. The amplitude of an underdamped system subject to an initial displacement x_0 and initial velocity 0 is given by:
a. x_0 **b.** $2x_0$ **c.** $x_0\omega_n$

10. The phase angle of an undamped system subject to an initial displacement x_0 and initial velocity 0 is given by:
a. x_0 **b.** $2x_0$ **c.** 0

11. The energy dissipated due to viscous damping is proportional to the following power of the amplitude of motion:
a. 1 **b.** 2 **c.** 3

12. For a critically damping system, the motion will be:
a. periodic **b.** aperiodic **c.** harmonic

13. The energy dissipated per cycle in viscous damping with damping constant c during the simple harmonic motion $x(t) = X \sin \omega_d t$, is given by:
a. $\pi c\omega_d X^2$ **b.** $\pi\omega_d X^2$ **c.** $\pi c\omega_d X$

14. For a vibrating system with a total energy W and a dissipated energy ΔW per cycle, the specific damping capacity is given by:
a. $\frac{W}{\Delta W}$ **b.** $\frac{\Delta W}{W}$ **c.** ΔW

15. If the characteristic roots have positive real values, the system response will be:
a. stable **b.** unstable **c.** asymptotically stable

16. The frequency of oscillation of the response of a system will be higher if the imaginary part of the roots is:
a. smaller **b.** zero **c.** larger

17. If the characteristic roots have a zero imaginary part, the response of the system will be:
a. oscillatory **b.** nonoscillatory **c.** steady
18. The shape of the root locus of a single-degree-of-freedom system for $0 \le \zeta \le 1$ is:
a. circular **b.** horizontal line **c.** radial line
19. The shape of the root locus of a single-degree-of-freedom system as k is varied is:
a. vertical and horizontal lines **b.** circular arc **c.** radial lines

2.5 Match the following for a single-degree-of-freedom system with $m = 1$, $k = 2$, and $c = 0.5$:

1. Natural frequency, ω_n	**a.** 1.3919
2. Linear frequency, f_n	**b.** 2.8284
3. Natural time period, τ_n	**c.** 2.2571
4. Damped frequency, ω_d	**d.** 0.2251
5. Critical damping constant, c_c	**e.** 0.1768
6. Damping ratio, ζ	**f.** 4.4429
7. Logarithmic decrement, δ	**g.** 1.4142

2.6 Match the following for a mass $m = 5$ kg moving with velocity $v = 10$ m/s:

Damping force	Type of damper
1. 20 N	**a.** Coulomb damping with a coefficient of friction of 0.3
2. 1.5 N	**b.** Viscous damping with a damping coefficient 1 N-s/m
3. 30 N	**c.** Viscous damping with a damping coefficient 2 N-s/m
4. 25 N	**d.** Hysteretic damping with a hysteretic damping coefficient of 12 N/m at a frequency of 4 rad/s
5. 10 N	**e.** Quadratic damping (force $= av^2$) with damping constant $a = 0.25$ N-s^2/m^2

2.7 Match the following characteristics of the s-plane:

Locus	Significance
1. Concentric circles	**a.** Different values of damped natural frequency
2. Lines parallel to real axis	**b.** Different values of reciprocals of time constant
3. Lines parallel to imaginary axis	**c.** Different values of damping ratio
4. Radial lines through origin	**d.** Different values of natural frequency

2.8 Match the following terms related to stability of systems:

Type of system	Nature of free-vibration response as time approaches infinity
1. Asymptotically stable	**a.** Neither decays nor grows
2. Unstable	**b.** Grows with oscillations
3. Stable	**c.** Grows without oscillations
4. Divergent instability	**d.** Approaches zero
5. Flutter instability	**e.** Grows without bound

PROBLEMS

Section 2.2 Free Vibration of an Undamped Translational System

2.1 An industrial press is mounted on a rubber pad to isolate it from its foundation. If the rubber pad is compressed 5 mm by the self weight of the press, find the natural frequency of the system.

2.2 A spring-mass system has a natural period of 0.21 sec. What will be the new period if the spring constant is (a) increased by 50 percent and (b) decreased by 50 percent?

2.3 A spring-mass system has a natural frequency of 10 Hz. When the spring constant is reduced by 800 N/m, the frequency is altered by 45 percent. Find the mass and spring constant of the original system.

2.4 A helical spring, when fixed at one end and loaded at the other, requires a force of 100 N to produce an elongation of 10 mm. The ends of the spring are now rigidly fixed, one end vertically above the other, and a mass of 10 kg is attached at the middle point of its length. Determine the time taken to complete one vibration cycle when the mass is set vibrating in the vertical direction.

2.5 An air-conditioning chiller unit weighing 2,000 lb is to be supported by four air springs (Fig. 2.50). Design the air springs such that the natural frequency of vibration of the unit lies between 5 rad/s and 10 rad/s.

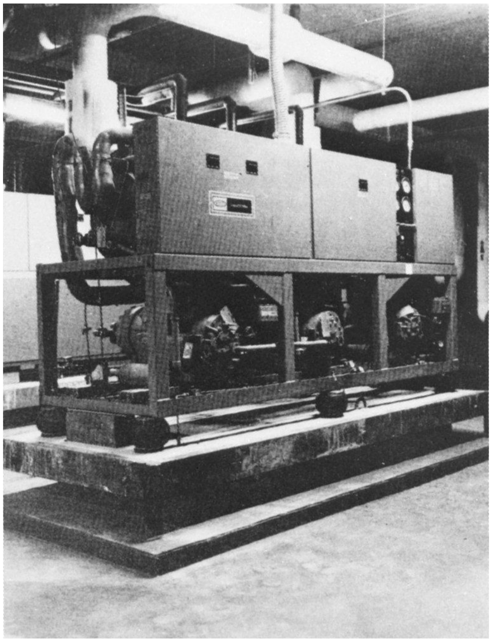


FIGURE 2.50 (Courtesy of *Sound and Vibration.*)

2.6 The maximum velocity attained by the mass of a simple harmonic oscillator is 10 cm/s, and the period of oscillation is 2 s. If the mass is released with an initial displacement of 2 cm, find (a) the amplitude, (b) the initial velocity, (c) the maximum acceleration, and (d) the phase angle.

2.7 Three springs and a mass are attached to a rigid, weightless bar *PQ* as shown in Fig. 2.51. Find the natural frequency of vibration of the system.

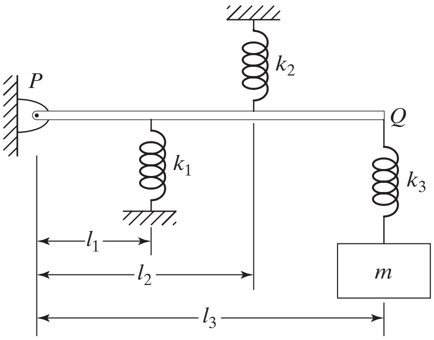


FIGURE 2.51

2.8 An automobile having a mass of 2,000 kg deflects its suspension springs 0.02 m under static conditions. Determine the natural frequency of the automobile in the vertical direction by assuming damping to be negligible.

2.9 Find the natural frequency of vibration of a spring-mass system arranged on an inclined plane, as shown in Fig. 2.52.

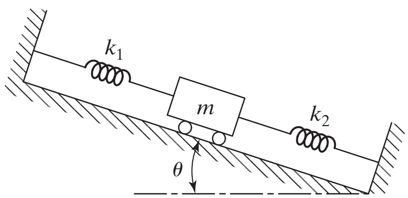


FIGURE 2.52

2.10 A loaded mine cart, weighing 5,000 lb, is being lifted by a frictionless pulley and a wire rope, as shown in Fig. 2.53. Find the natural frequency of vibration of the cart in the given position.

2.11 An electronic chassis weighing 500 N is isolated by supporting it on four helical springs, as shown in Fig. 2.54. Design the springs so that the unit can be used in an environment in which the vibratory frequency ranges from 0 to 5 Hz.

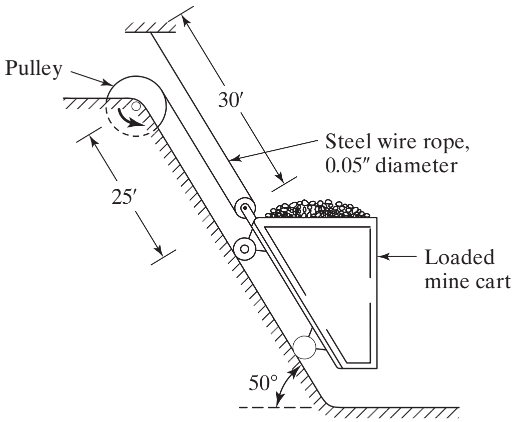


FIGURE 2.53

FIGURE 2.54 An electronic chassis mounted on vibration isolators. (Courtesy of Titan SESCO.)

2.12 Find the natural frequency of the system shown in Fig. 2.55 with and without the springs k_1 and k_2 in the middle of the elastic beam.

2.13 Find the natural frequency of the pulley system shown in Fig. 2.56 by neglecting the friction and the masses of the pulleys.

2.14 A weight W is supported by three frictionless and massless pulleys and a spring of stiffness k, as shown in Fig. 2.57. Find the natural frequency of vibration of weight W for small oscillations.

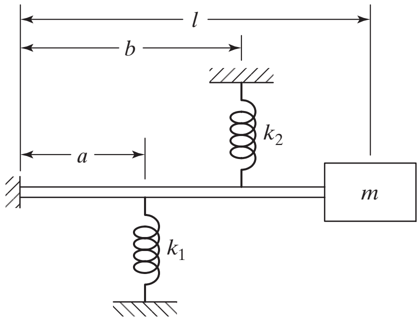


FIGURE 2.55

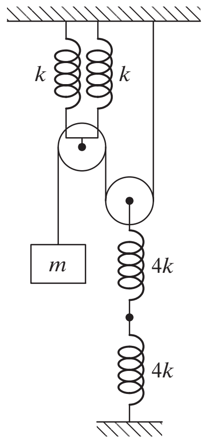


FIGURE 2.56

2.15 A rigid block of mass M is mounted on four elastic supports, as shown in Fig. 2.58. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each elastic support is k, find the natural frequency of vibration of the system (a) without the mass m, and (b) with the mass m. Also find the resulting motion of the system in case (b).

2.16 A sledgehammer strikes an anvil with a velocity of 50 ft/sec (Fig. 2.59). The hammer and the anvil weigh 12 lb and 100 lb, respectively. The anvil is supported on four springs, each of stiffness $k = 100$ lb/in. Find the resulting motion of the anvil (a) if the hammer remains in contact with the anvil and (b) if the hammer does not remain in contact with the anvil after the initial impact.

2.17 Derive the expression for the natural frequency of the system shown in Fig. 2.60. Note that the load W is applied at the tip of beam 1 and midpoint of beam 2.

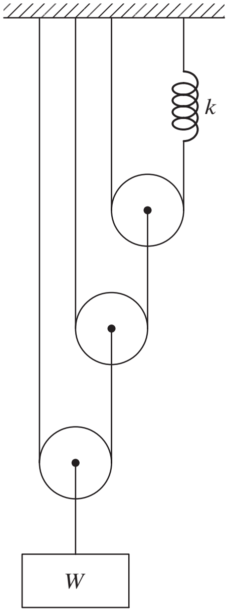


FIGURE 2.57

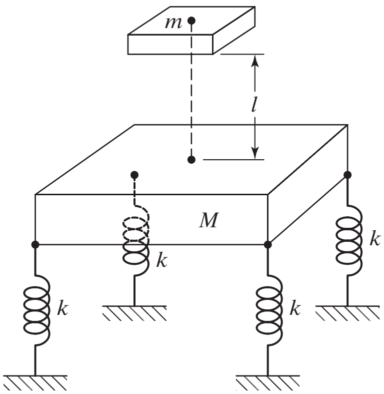


FIGURE 2.58

2.18 A heavy machine weighing 9,810 N is being lowered vertically down by a winch at a uniform velocity of 2 m/s. The steel cable supporting the machine has a diameter of 0.01 m. The winch is suddenly stopped when the steel cable's length is 20 m. Find the period and amplitude of the ensuing vibration of the machine.

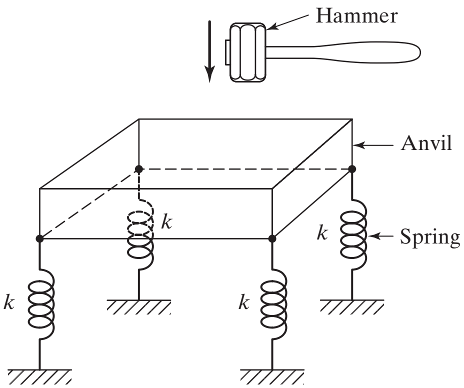


FIGURE 2.59

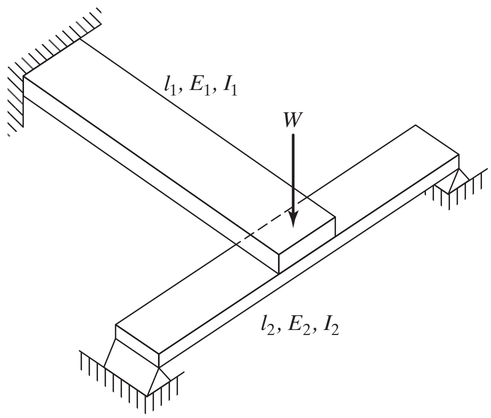


FIGURE 2.60

2.19 The natural frequency of a spring-mass system is found to be 2 Hz. When an additional mass of 1 kg is added to the original mass m, the natural frequency is reduced to 1 Hz. Find the spring constant k and the mass m.

2.20 An electrical switch gear is supported by a crane through a steel cable of length 4 m and diameter 0.01 m (Fig. 2.61). If the natural time period of axial vibration of the switch gear is found to be 0.1 s, find the mass of the switch gear.

2.21 Four weightless rigid links and a spring are arranged to support a weight W in two different ways, as shown in Fig. 2.62. Determine the natural frequencies of vibration of the two arrangements.

FIGURE 2.61 (Photo courtesy of the Institution of Electrical Engineers.)

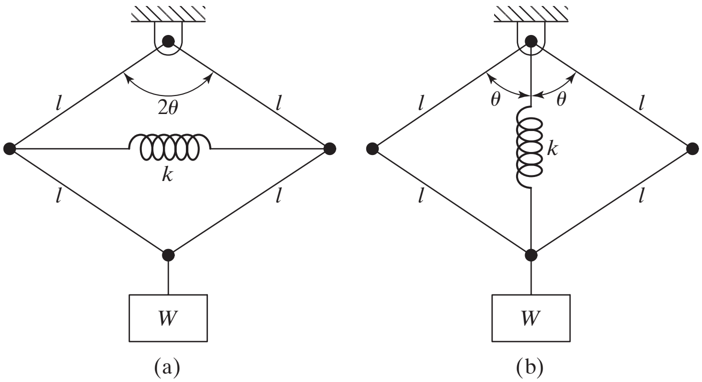


FIGURE 2.62

2.22 A scissors jack is used to lift a load W. The links of the jack are rigid and the collars can slide freely on the shaft against the springs of stiffnesses k_1 and k_2 (see Fig. 2.63). Find the natural frequency of vibration of the weight in the vertical direction.

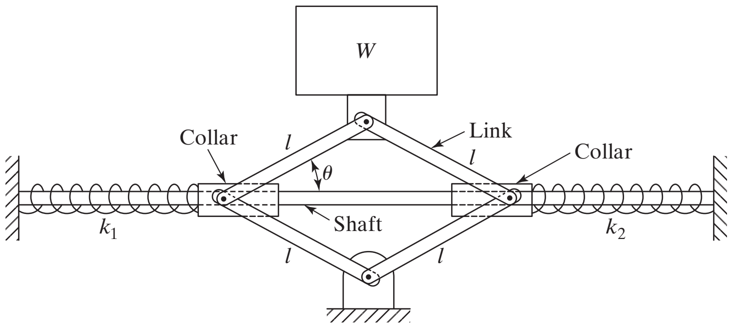


FIGURE 2.63

2.23 A weight is suspended using six rigid links and two springs in two different ways, as shown in Fig. 2.64. Find the natural frequencies of vibration of the two arrangements.

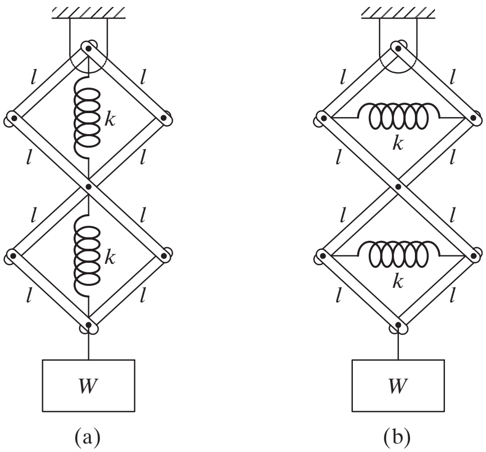


FIGURE 2.64

2.24 Figure 2.65 shows a small mass m restrained by four linearly elastic springs, each of which has an unstretched length l, and an angle of orientation of 45° with respect to the x-axis. Determine the equation of motion for small displacements of the mass in the x direction.

2.25 A mass m is supported by two sets of springs oriented at 30° and 120° with respect to the X-axis, as shown in Fig. 2.66. A third pair of springs, each with a stiffness of k_3, is to be designed so as to make the system have a constant natural frequency while vibrating in any direction x. Determine the necessary spring stiffness k_3 and the orientation of the springs with respect to the X-axis.

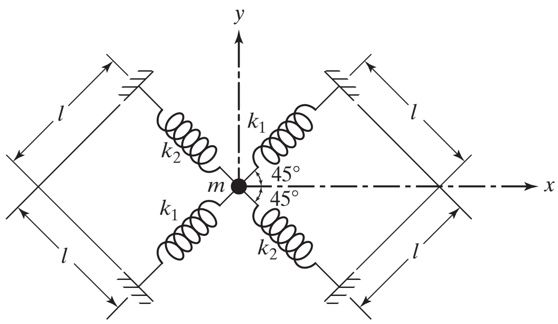


FIGURE 2.65

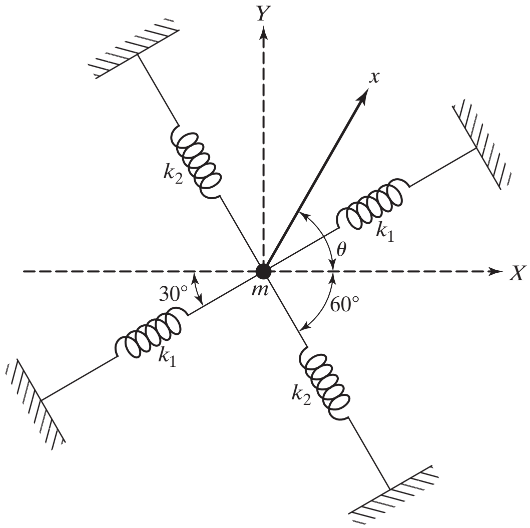


FIGURE 2.66

2.26 A mass m is attached to a cord that is under a tension T, as shown in Fig. 2.67. Assuming that T remains unchanged when the mass is displaced normal to the cord, (a) write the differential equation of motion for small transverse vibrations and (b) find the natural frequency of vibration.

2.27 A bungee jumper weighing 160 lb ties one end of an elastic rope of length 200 ft and stiffness 10 lb/in. to a bridge and the other end to himself and jumps from the bridge (Fig. 2.68). Assuming the bridge to be rigid, determine the vibratory motion of the jumper about his static equilibrium position.

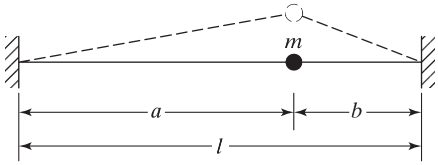


FIGURE 2.67

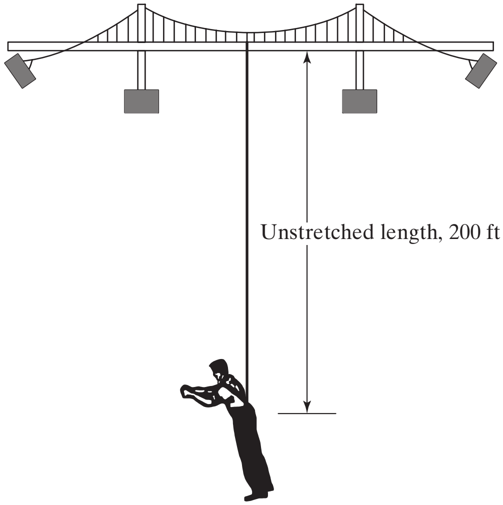


FIGURE 2.68

2.28 An acrobat weighing 120 lb walks on a tightrope, as shown in Fig. 2.69. If the natural frequency of vibration in the given position, in vertical direction, is 10 rad/sec, find the tension in the rope.

2.29 The schematic diagram of a centrifugal governor is shown in Fig. 2.70. The length of each rod is l, the mass of each ball is m, and the free length of the spring is h. If the shaft speed is ω, determine the equilibrium position and the frequency for small oscillations about this position.

2.30 In the Hartnell governor shown in Fig. 2.71, the stiffness of the spring is 10^4 N/m and the weight of each ball is 25 N. The length of the ball arm is 20 cm, and that of the sleeve arm is 12 cm. The distance between the axis of rotation and the pivot of the bell crank lever is 16 cm. The spring is compressed by 1 cm when the ball arm is vertical. Find (a) the speed of the governor at which the ball arm remains vertical and (b) the natural frequency of vibration for small displacements about the vertical position of the ball arms.

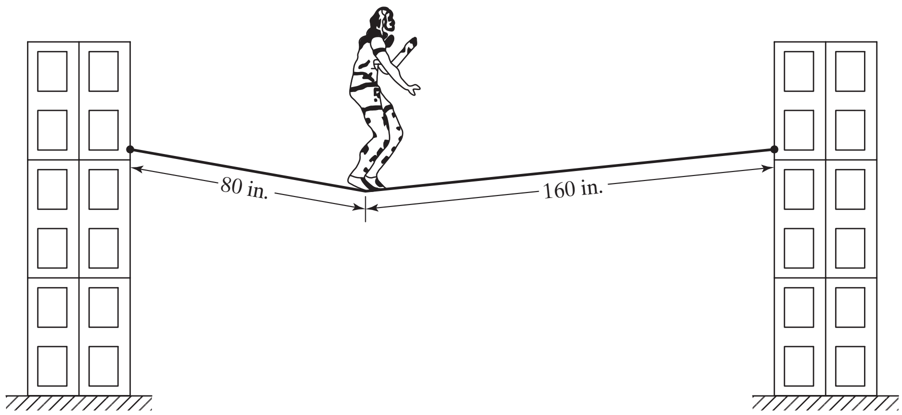


FIGURE 2.69

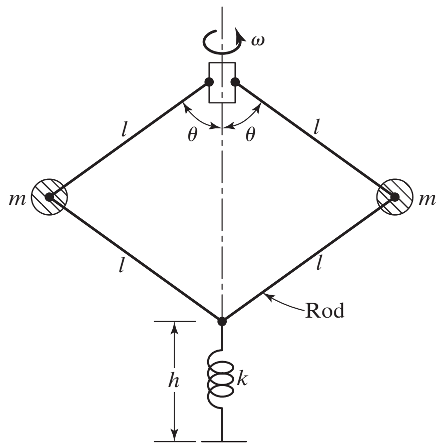


FIGURE 2.70

2.31 A square platform $PQRS$ and a car that it is supporting have a combined mass of M. The platform is suspended by four elastic wires from a fixed point O, as indicated in Fig. 2.72. The vertical distance between the point of suspension O and the horizontal equilibrium position of the platform is h. If the side of the platform is a and the stiffness of each wire is k, determine the period of vertical vibration of the platform.

2.32 The inclined manometer, shown in Fig. 2.73, is used to measure pressure. If the total length of mercury in the tube is L, find an expression for the natural frequency of oscillation of the mercury.

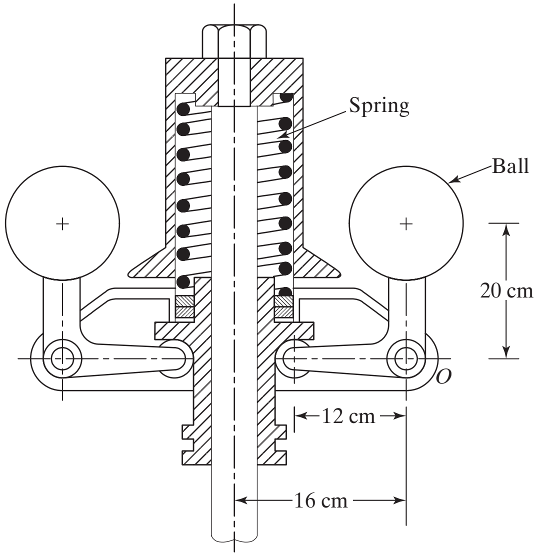


FIGURE 2.71 Hartnell governor.

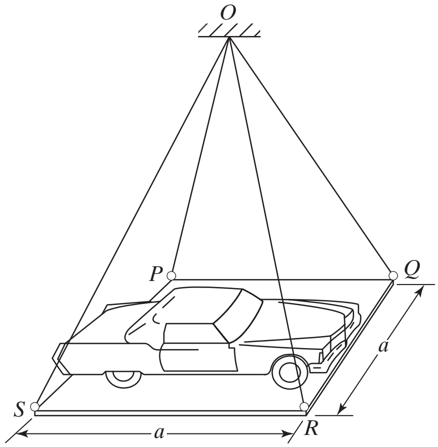


FIGURE 2.72

2.33 The crate, of mass 250 kg, hanging from a helicopter (shown in Fig. 2.74(a)) can be modeled as shown in Fig. 2.74(b). The rotor blades of the helicopter rotate at 300 rpm. Find the diameter of the steel cables so that the natural frequency of vibration of the crate is at least twice the frequency of the rotor blades.

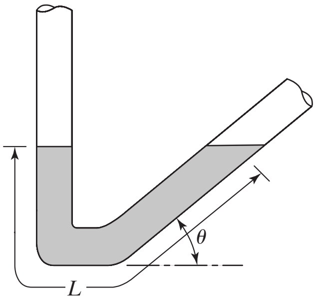


FIGURE 2.73

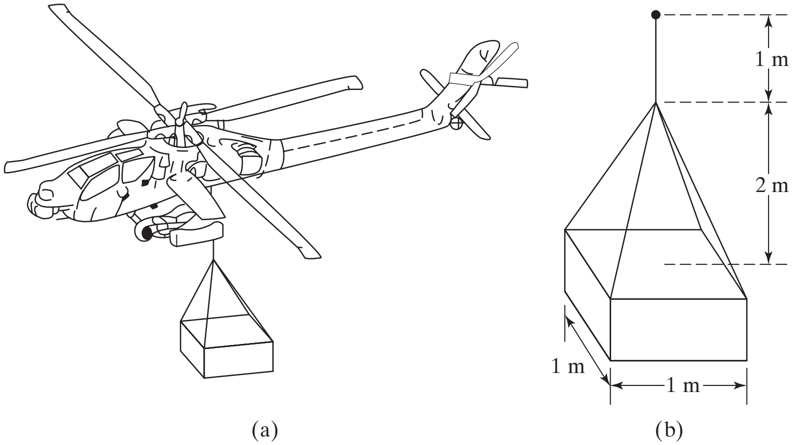


FIGURE 2.74

2.34 A pressure-vessel head is supported by a set of steel cables of length 2 m as shown in Fig. 2.75. The time period of axial vibration (in vertical direction) is found to vary from 5 s to 4.0825 s when an additional mass of 5,000 kg is added to the pressure-vessel head. Determine the equivalent cross-sectional area of the cables and the mass of the pressure-vessel head.

2.35 A flywheel is mounted on a vertical shaft, as shown in Fig. 2.76. The shaft has a diameter d and length l and is fixed at both ends. The flywheel has a weight of W and a radius of gyration of r. Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system.

FIGURE 2.75 (Photo courtesy of CBI Industries Inc.)

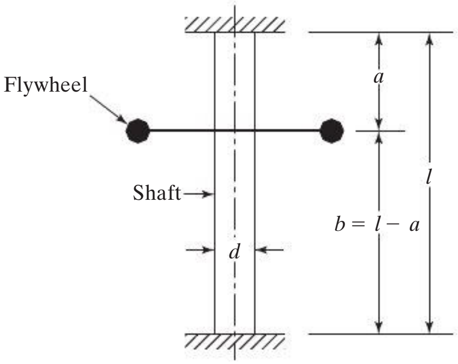


FIGURE 2.76

2.36 A TV antenna tower is braced by four cables, as shown in Fig. 2.77. Each cable is under tension and is made of steel with a cross-sectional area of 0.5 in.^{2}. The antenna tower can be modeled as a steel beam of square section of side 1 in. for estimating its mass and stiffness. Find the tower's natural frequency of bending vibration about the y-axis.

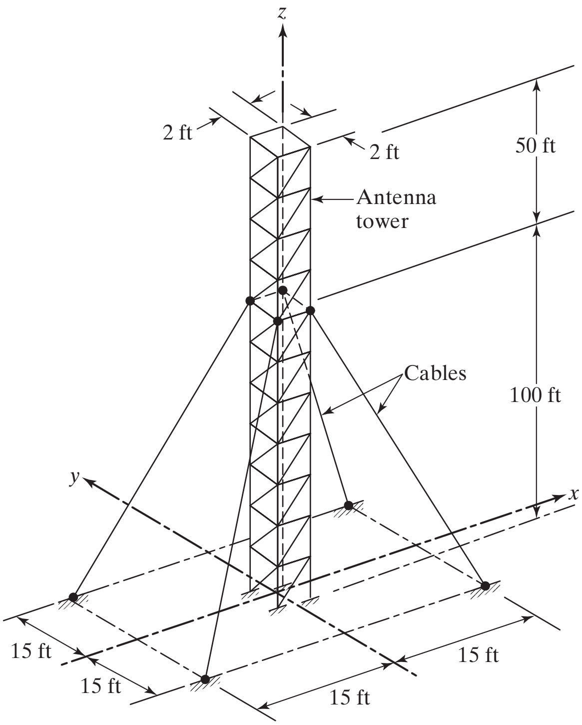


FIGURE 2.77

2.37 Figure 2.78(a) shows a steel traffic sign, of thickness $\frac{1}{8}$ in., fixed to a steel post. The post is 72 in. high with a cross section 2 in. $\times$ 1/4 in., and it can undergo torsional vibration (about the z-axis) or bending vibration (either in the zx-plane or the yz-plane). Determine the mode of vibration of the post in a storm during which the wind velocity has a frequency component of 1.25 Hz.

Hints:

1. Neglect the weight of the post in finding the natural frequencies of vibration.
2. Torsional stiffness of a shaft with a rectangular section (see Fig. 2.78(b)) is given by

$$k_t = 5.33\frac{ab^3G}{l}\left[1 - 0.63\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$$

where G is the shear modulus.

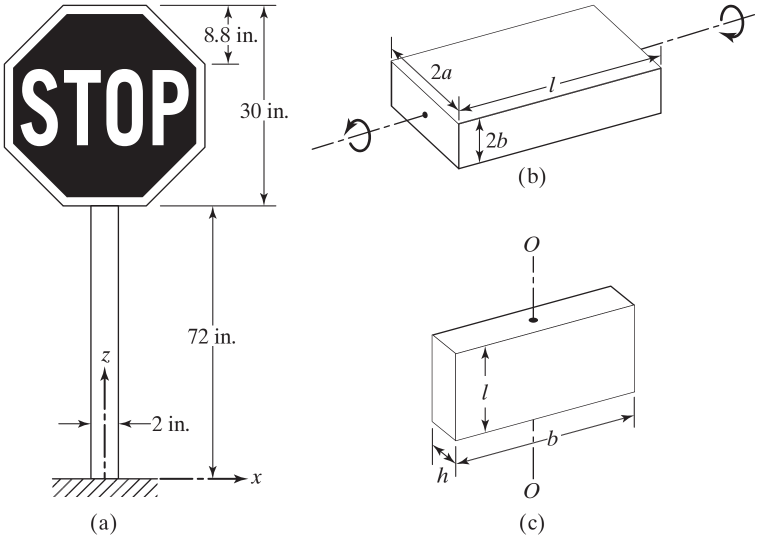


FIGURE 2.78

3. Mass moment of inertia of a rectangular block about axis OO (see Fig. 2.78(c)) is given by

$$I_{OO} = \frac{\rho l}{3}(b^3 h + h^3 b)$$

where ρ is the density of the block.

2.38 A building frame is modeled by four identical steel columns, each of weight w, and a rigid floor of weight W, as shown in Fig. 2.79. The columns are fixed at the ground and have a bending rigidity of EI each. Determine the natural frequency of horizontal vibration of the building frame by assuming the connection between the floor and the columns to be (a) pivoted as shown in Fig. 2.79(a) and (b) fixed against rotation as shown in Fig. 2.79(b). Include the effect of self weights of the columns.

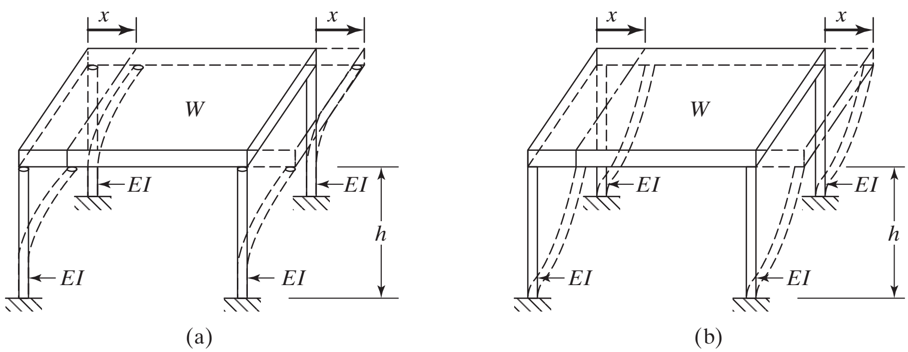


FIGURE 2.79

2.39 A pick-and-place robot arm, shown in Fig. 2.80, carries an object weighing 10 lb. Find the natural frequency of the robot arm in the axial direction for the following data: $l_1 = 12$ in., $l_2 = 10$ in., $l_3 = 8$ in.; $E_1 = E_2 = E_3 = 10^7$ psi; $D_1 = 2$ in., $D_2 = 1.5$ in., $D_3 = 1$ in.; $d_1 = 1.75$ in., $d_2 = 1.25$ in., $d_3 = 0.75$ in.

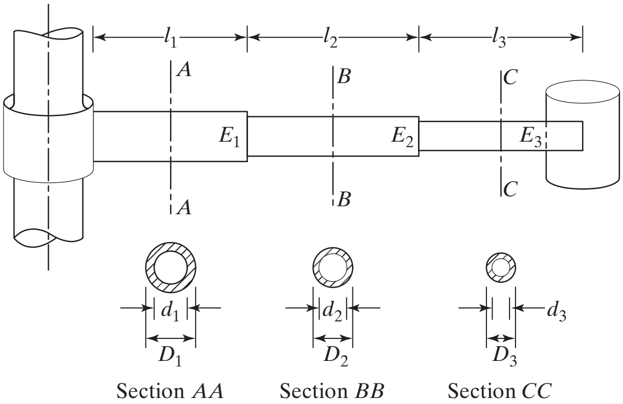


FIGURE 2.80

2.40 A helical spring of stiffness k is cut into two halves and a mass m is connected to the two halves as shown in Fig. 2.81(a). The natural time period of this system is found to be 0.5 s. If an identical spring is cut so that one part is one-fourth and the other part three-fourths of the original length, and the mass m is connected to the two parts as shown in Fig. 2.81(b), what would be the natural period of the system?

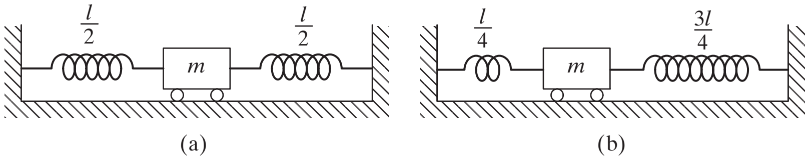


FIGURE 2.81

2.41* Figure 2.82 shows a metal block supported on two identical cylindrical rollers rotating in opposite directions at the same angular speed. When the center of gravity of the block is initially displaced by a distance x, the block will be set into simple harmonic motion. If the frequency of motion of the block is found to be ω, determine the coefficient of friction between the block and the rollers.

2.42* If two identical springs of stiffness k each are attached to the metal block of Problem 2.41 as shown in Fig. 2.83, determine the coefficient of friction between the block and the rollers.

*The asterisk denotes a design problem or a problem with no unique answer.

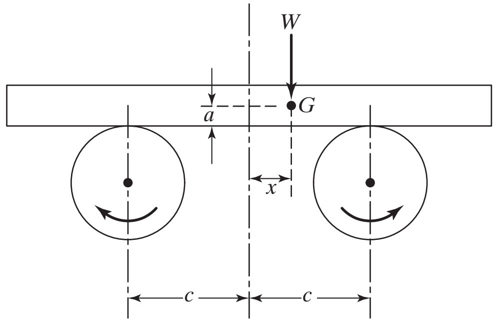


FIGURE 2.82

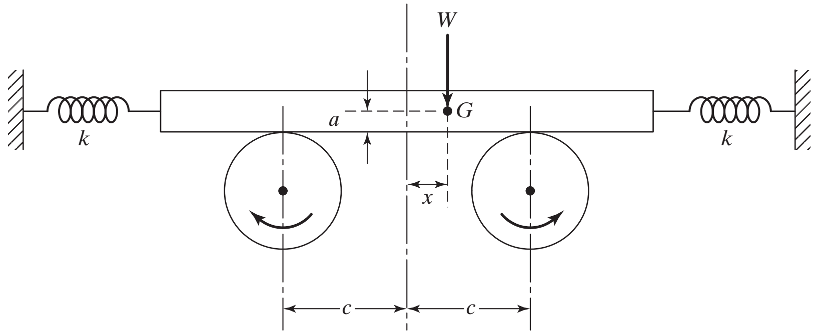


FIGURE 2.83

2.43 An electromagnet weighing 3,000 lb is at rest while holding an automobile of weight 2,000 lb in a junkyard. The electric current is turned off, and the automobile is dropped. Assuming that the crane and the supporting cable have an equivalent spring constant of 10,000 lb/in., find the following: (a) the natural frequency of vibration of the electromagnet, (b) the resulting motion of the electromagnet, and (c) the maximum tension developed in the cable during the motion.

2.44 Derive the equation of motion of the system shown in Fig. 2.84, using the following methods: (a) Newton's second law of motion, (b) D'Alembert's principle, (c) principle of virtual work, and (d) principle of conservation of energy.

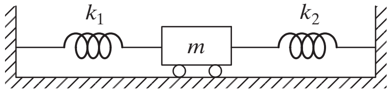


FIGURE 2.84

2.45–2.46 Draw the free-body diagram and derive the equation of motion using Newton's second law of motion for each of the systems shown in Figs. 2.85 and 2.86.

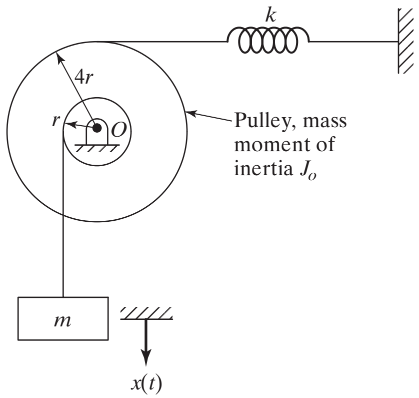


FIGURE 2.85

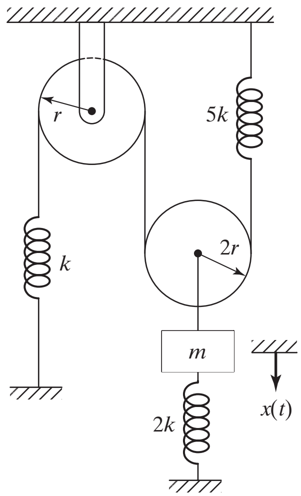


FIGURE 2.86

2.47–2.48 Derive the equation of motion using the principle of conservation of energy for each of the systems shown in Figs. 2.85 and 2.86.

2.49 A steel beam of length 1 m carries a mass of 50 kg at its free end, as shown in Fig. 2.87. Find the natural frequency of transverse vibration of the mass by modeling it as a single-degree-of-freedom system.

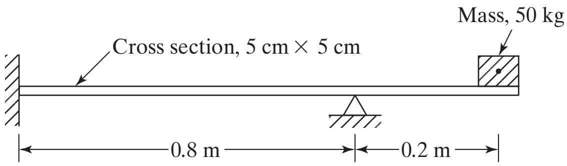


FIGURE 2.87

2.50 A steel beam of length 1 m carries a mass of 50 kg at its free end, as shown in Fig. 2.88. Find the natural frequency of transverse vibration of the system by modeling it as a single-degree-of-freedom system.

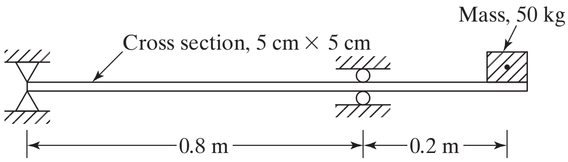


FIGURE 2.88

2.51 Determine the displacement, velocity, and acceleration of the mass of a spring-mass system with $k = 500$ N/m, $m = 2$ kg, $x_0 = 0.1$ m, and $\dot{x}_0 = 5$ m/s.

2.52 Determine the displacement (x), velocity ($\dot{x}$), and acceleration ($\ddot{x}$) of a spring-mass system with $\omega_n = 10$ rad/s for the initial conditions $x_0 = 0.05$ m and $\dot{x}_0 = 1$ m/s. Plot $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ from $t = 0$ to 5 s.

2.53 The free-vibration response of a spring-mass system is observed to have a frequency of 2 rad/s, an amplitude of 10 mm, and a phase shift of 1 rad from $t = 0$. Determine the initial conditions that caused the free vibration. Assume the damping ratio of the system as 0.1.

2.54 An automobile is found to have a natural frequency of 20 rad/s without passengers and 17.32 rad/s with passengers of mass 500 kg. Find the mass and stiffness of the automobile by treating it as a single-degree-of-freedom system.

2.55 A spring-mass system with mass 2 kg and stiffness 3,200 N/m has an initial displacement of $x_0 = 0$. What is the maximum initial velocity that can be given to the mass without the amplitude of free vibration exceeding a value of 0.1 m?

2.56 A helical spring, made of music wire of diameter d, has a mean coil diameter (D) of 0.5625 in. and N active coils (turns). It is found to have a frequency of vibration (f) of 193 Hz and a spring rate k of 26.4 lb/in. Determine the wire diameter d and the number of coils N, assuming the shear modulus G is 11.5×10^6 psi and weight density ρ is 0.282 lb/in.^{3}. The spring rate (k) and frequency (f) are given by

$$k = \frac{d^4 G}{8D^3 N}, \qquad f = \frac{1}{2}\sqrt{\frac{kg}{W}}$$

where W is the weight of the helical spring and g is the acceleration due to gravity.

2.57 Solve Problem 2.56 if the material of the helical spring is changed from music wire to aluminum with $G = 4 \times 10^6$ psi and $\rho = 0.1$ lb/in.^{3}.

2.58 A steel cantilever beam is used to carry a machine at its free end. To save weight, it is proposed to replace the steel beam by an aluminum beam of identical dimensions. Find the expected change in the natural frequency of the beam-machine system.

2.59 An oil drum of diameter 1 m and a mass of 500 kg floats in a bath of salt water of density $\rho_w = 1050$ kg/m^3. Considering small displacements of the drum in the vertical direction (x), determine the natural frequency of vibration of the system.

2.60 The equation of motion of a spring-mass system is given by (units: SI system)

$$500\ddot{x} + 1000\left(\frac{x}{0.025}\right)^3 = 0$$

a. Determine the static equilibrium position of the system.
b. Derive the linearized equation of motion for small displacements (x) about the static equilibrium position.
c. Find the natural frequency of vibration of the system for small displacements.
d. Find the natural frequency of vibration of the system for small displacements when the mass is 600 (instead of 500).

2.61 A deceleration of 10 m/s^2 is caused when brakes are applied to a vehicle traveling at a speed of 100 km/hour. Determine the time taken and the distance traveled before the vehicle comes to a complete stop.

2.62 A steel hollow cylindrical post is welded to a steel rectangular traffic sign as shown in Fig. 2.89 with the following data:
Dimensions: $l = 2$ m, $r_0 = 0.050$ m, $r_i = 0.045$ m, $b = 0.75$ m, $d = 0.40$ m, $t = 0.005$ m; material properties: ρ (specific weight) $= 76.50$ kN/m^3, $E = 207$ GPa, $G = 79.3$ GPa

Find the natural frequencies of the system in transverse vibration in the yz- and xz-planes by considering the masses of both the post and the sign.

Hint: Consider the post as a cantilever beam in transverse vibration in the appropriate plane.

2.63 Solve Problem 2.62 by changing the material from steel to bronze for both the post and the sign. Material properties of bronze: ρ (specific weight) $= 80.1$ kN/m^3, $E = 111.0$ GPa, $G = 41.4$ GPa.

Section 2.3 Free Vibration of an Undamped Torsional System

2.64 A simple pendulum is set into oscillation from its rest position by giving it an angular velocity of 1 rad/s. It is found to oscillate with an amplitude of 0.5 rad. Find the natural frequency and length of the pendulum.

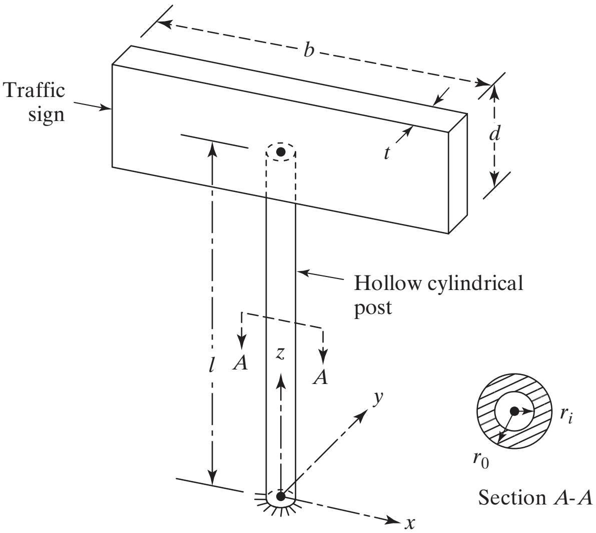


FIGURE 2.89

2.65 A pulley 250 mm in diameter drives a second pulley 1,000 mm in diameter by means of a belt (see Fig. 2.90). The moment of inertia of the driven pulley is 0.2 kg-m^2. The belt connecting these pulleys is represented by two springs, each of stiffness k. For what value of k will the natural frequency be 6 Hz?

2.66 Derive an expression for the natural frequency of the simple pendulum shown in Fig. 1.10. Determine the period of oscillation of a simple pendulum having a mass $m = 5$ kg and a length $l = 0.5$ m.

2.67 A mass m is attached at the end of a bar of negligible mass and is made to vibrate in three different configurations, as indicated in Fig. 2.91. Find the configuration corresponding to the highest natural frequency.

2.68 Figure 2.92 shows a spacecraft with four solar panels. Each panel has the dimensions 5 ft $\times$ 3 ft $\times$ 1 ft with a weight density of 0.1 lb/in.^{3} and is connected to the body of the spacecraft by aluminum rods of length 12 in. and diameter 1 in. Assuming that the body of the spacecraft is very large (rigid), determine the natural frequency of vibration of each panel about the axis of the connecting aluminum rod.

2.69 One of the blades of an electric fan is removed (as shown by dotted lines in Fig. 2.93). The steel shaft AB, on which the blades are mounted, is equivalent to a uniform shaft of diameter 1 in. and length 6 in. Each blade can be modeled as a uniform slender rod of weight 2 lb and length 12 in. Determine the natural frequency of vibration of the remaining three blades about the y-axis.

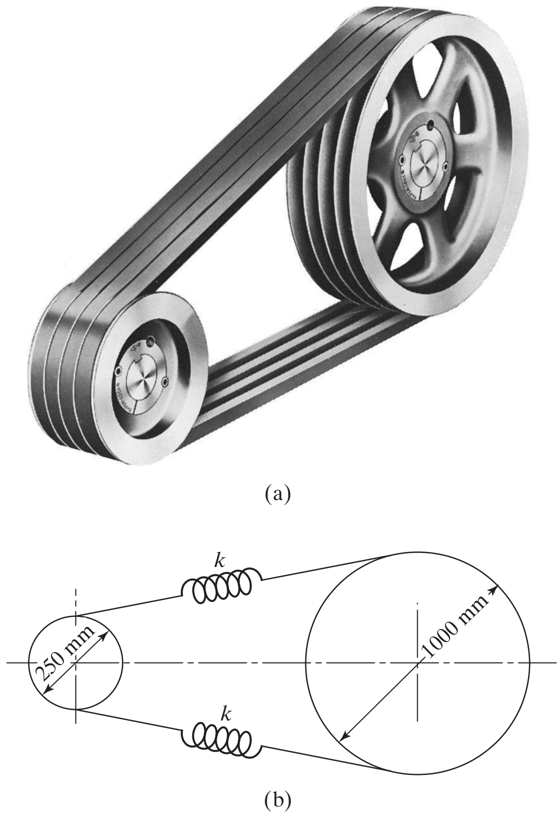


FIGURE 2.90 (Photo courtesy of Reliance Electric Company.)

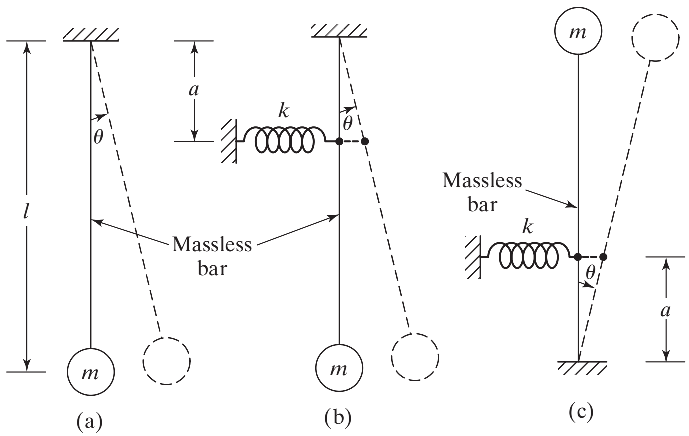


FIGURE 2.91

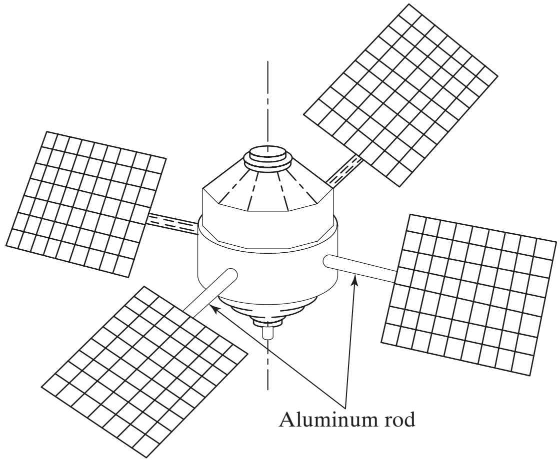


FIGURE 2.92

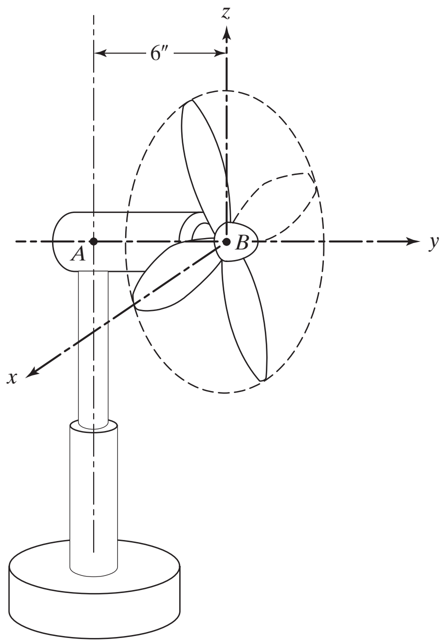


FIGURE 2.93

2.70 A heavy ring of mass moment of inertia 1.0 kg-m^2 is attached at the end of a two-layered hollow shaft of length 2 m (Fig. 2.94). If the two layers of the shaft are made of steel and brass, determine the natural time period of torsional vibration of the heavy ring.

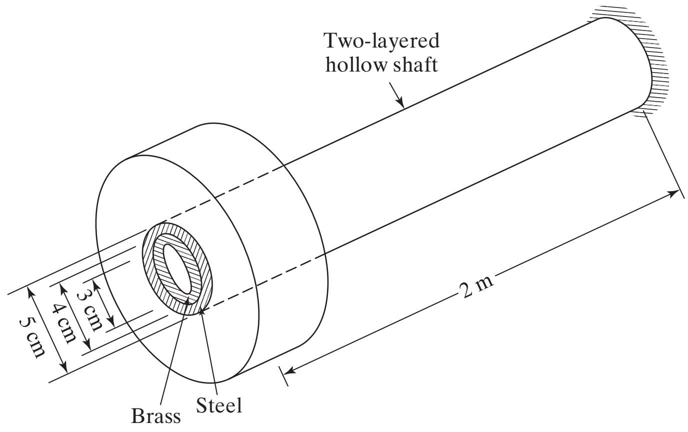


FIGURE 2.94

2.71 Find the natural frequency of the pendulum shown in Fig. 2.95 when the mass of the connecting bar is not negligible compared to the mass of the pendulum bob.

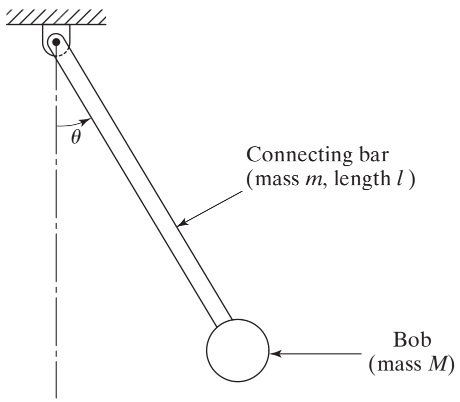


FIGURE 2.95

2.72 A steel shaft of 0.05 m diameter and 2 m length is fixed at one end and carries at the other end a steel disc of 1 m diameter and 0.1 m thickness, as shown in Fig. 2.14. Find the system's natural frequency of torsional vibration.

2.73 A uniform slender rod of mass m and length l is hinged at point A and is attached to four linear springs and one torsional spring, as shown in Fig. 2.96. Find the natural frequency of the system if $k = 2000$ N/m, $k_t = 1000$ N-m/rad, $m = 10$ kg, and $l = 5$ m.

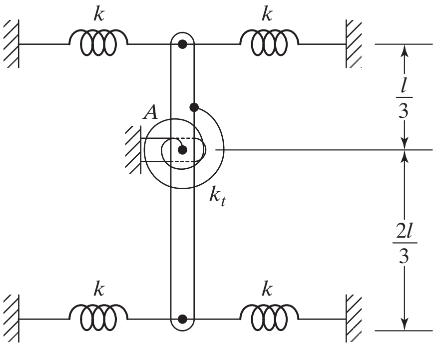


FIGURE 2.96

2.74 A cylinder of mass m and mass moment of inertia J_0 is free to roll without slipping but is restrained by two springs of stiffnesses k_1 and k_2, as shown in Fig. 2.97. Find its natural frequency of vibration. Also find the value of a that maximizes the natural frequency of vibration.

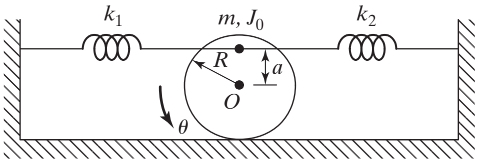


FIGURE 2.97

2.75 If the pendulum of Problem 2.66 is placed in a rocket moving vertically with an acceleration of 5 m/s^2, what will be its period of oscillation?

2.76 Find the equation of motion of the uniform rigid bar OA of length l and mass m shown in Fig. 2.98. Also find its natural frequency.

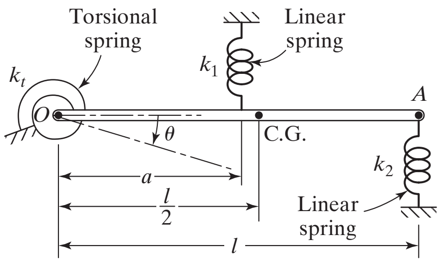


FIGURE 2.98

2.77 A uniform circular disc is pivoted at point O, as shown in Fig. 2.99. Find the natural frequency of the system. Also find the maximum frequency of the system by varying the value of b.

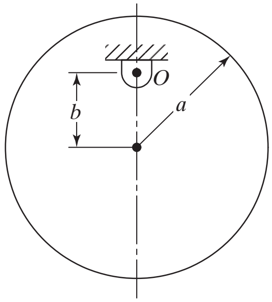


FIGURE 2.99

2.78 Derive the equation of motion of the system shown in Fig. 2.100, using the following methods: (a) Newton's second law of motion, (b) D'Alembert's principle, and (c) principle of virtual work.

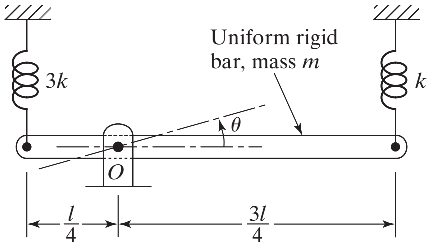


FIGURE 2.100

2.79 Find the natural frequency of the traffic sign system described in Problem 2.62 in torsional vibration about the z-axis by considering the masses of both the post and the sign.
Hint: The spring stiffness of the post in torsional vibration about the z-axis is given by $k_t = \dfrac{\pi G}{2l}(r_0^4 - r_i^4)$. The mass moment of inertia of the sign about the z-axis is given by $I_0 = \dfrac{1}{12} m_0 (d^2 + b^2)$, where m_0 is the mass of the sign.

2.80 Solve Problem 2.79 by changing the material from steel to bronze for both the post and the sign. Material properties of bronze: ρ (specific weight) $= 80.1$ kN/m^3, $E = 111.0$ GPa, $G = 41.4$ GPa.

2.81 A mass m_1 is attached at one end of a uniform bar of mass m_2 whose other end is pivoted at point O as shown in Fig. 2.101. Determine the natural frequency of vibration of the resulting pendulum for small angular displacements.

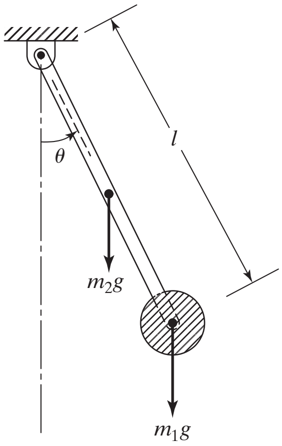


FIGURE 2.101

2.82 The angular motion of the forearm of a human hand carrying a mass m_0 is shown in Fig. 2.102. During motion, the forearm can be considered to rotate about the joint (pivot point) O with muscle forces modeled in the form of a force by triceps $(c_1\dot{x})$ and a force in biceps $(-c_2\theta)$, where c_1 and c_2 are constants and $\dot{x}$ is the velocity with which triceps are stretched (or contracted). Approximating the forearm as a uniform bar of mass m and length l, derive the equation of motion of the forearm for small angular displacements θ. Also find the natural frequency of the forearm.

Section 2.4 Response of First-Order Systems and Time Constant

2.83 Find the free-vibration response and the time constant, where applicable, of systems governed by the following equations of motion:

a. $100\dot{v} + 20v = 0, \quad v(0) = v(t = 0) = 10$
b. $100\dot{v} + 20v = 10, \quad v(0) = v(t = 0) = 10$
c. $100\dot{v} - 20v = 0, \quad v(0) = v(t = 0) = 10$
d. $500\dot{\omega} + 50\omega = 0, \quad \omega(0) = \omega(t = 0) = 0.5$

Hint: The time constant can also be defined as the value of time at which the step response of a system rises to 63.2% $(100.0\% - 36.8\%)$ of its final value.

2.84 A viscous damper, with damping constant c, and a spring, with spring stiffness k, are connected to a massless bar AB as shown in Fig. 2.103. The bar AB is displaced by a distance of $x = 0.1$ m when a constant force $F = 500$ N is applied. The applied force F is then abruptly released from its displaced position. If the displacement of the bar AB is reduced from its initial value of 0.1 m at $t = 0$ to 0.01 m at $t = 10$, see, find the values of c and k.

2.85 The equation of motion of a rocket, of mass m, traveling vertically under a thrust F and air resistance or drag D is given by

$$m\dot{v} = F - D - mg$$

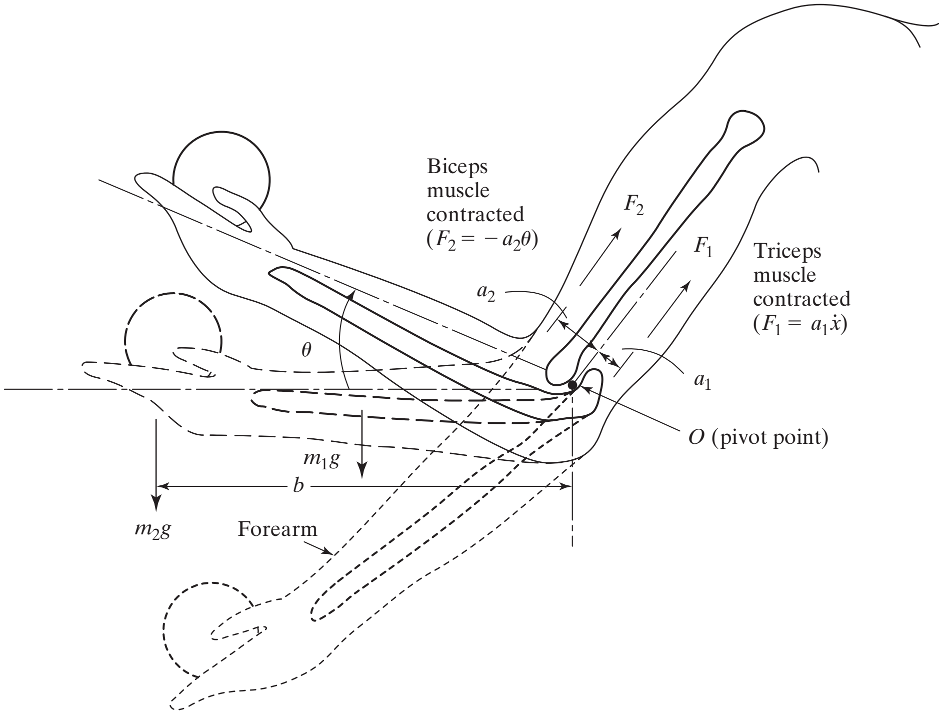


FIGURE 2.102 Motion of arm.

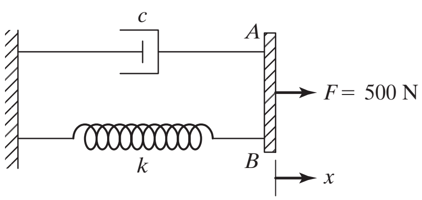


FIGURE 2.103

If $m = 1000$ kg, $F = 50{,}000$ N, $D = 2{,}000\ v$, and $g = 9.81$ m/s^2, find the time variation of the velocity of the rocket, $v(t) = \dfrac{dx(t)}{dt}$, using the initial conditions $x(0) = 0$ and $v(0) = 0$, where $x(t)$ is the distance traveled by the rocket in time t.

Section 2.5 Rayleigh's Energy Method

2.86 Determine the effect of self weight on the natural frequency of vibration of the pinned-pinned beam shown in Fig. 2.104.

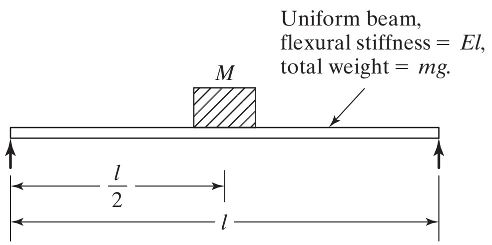


FIGURE 2.104

2.87 Use Rayleigh's method to solve Problem 2.7.

2.88 Use Rayleigh's method to solve Problem 2.13.

2.89 Find the natural frequency of the system shown in Fig. 2.54.

2.90 Use Rayleigh's method to solve Problem 2.26.

2.91 Use Rayleigh's method to solve Problem 2.73.

2.92 Use Rayleigh's method to solve Problem 2.76.

2.93 A wooden rectangular prism of density ρ_w, height h, and cross section $a \times b$ is initially depressed in an oil tub and made to vibrate freely in the vertical direction (see Fig. 2.105).

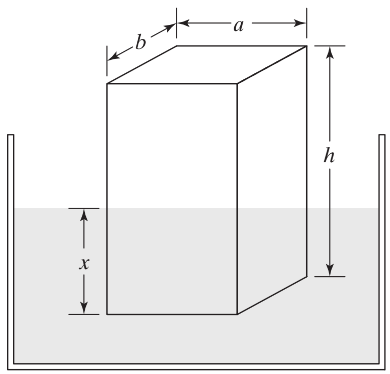


FIGURE 2.105

Use Rayleigh's method to find the natural frequency of vibration of the prism. Assume the density of oil is ρ_0. If the rectangular prism is replaced by a uniform circular cylinder of radius r, height h, and density ρ_w, will there be any change in the natural frequency?

2.94 Use the energy method to find the natural frequency of the system shown in Fig. 2.97.

2.95 Use the energy method to find the natural frequency of vibration of the system shown in Fig. 2.85.

2.96 A cylinder of mass m and mass moment of inertia J is connected to a spring of stiffness k and rolls on a rough surface as shown in Fig. 2.106. If the translational and angular displacements of the cylinder are x and θ from its equilibrium position, determine the following:

a. Equation of motion of the system for small displacements in terms of x using the energy method.

b. Equation of motion of the system for small displacements in terms of θ using the energy method.

c. Find the natural frequencies of the system using the equation of motion derived in parts (a) and (b). Are the resulting natural frequencies same?

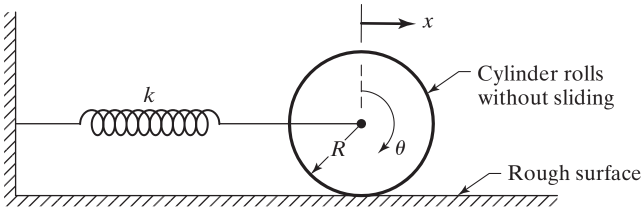


FIGURE 2.106

Section 2.6 Free Vibration with Viscous Damping

2.97 A simple pendulum is found to vibrate at a frequency of 0.5 Hz in a vacuum and 0.45 Hz in a viscous fluid medium. Find the damping constant, assuming the mass of the bob of the pendulum is 1 kg.

2.98 The ratio of successive amplitudes of a viscously damped single-degree-of-freedom system is found to be 18:1. Determine the ratio of successive amplitudes if the amount of damping is (a) doubled, and (b) halved.

2.99 Assuming that the phase angle is zero, show that the response $x(t)$ of an underdamped single-degree-of-freedom system reaches a maximum value when

$$\sin \omega_d t = \sqrt{1 - \zeta^2}$$

and a minimum value when

$$\sin \omega_d t = -\sqrt{1 - \zeta^2}$$

Also show that the equations of the curves passing through the maximum and minimum values of $x(t)$ are given, respectively, by

$$x = \sqrt{1 - \zeta^2} X e^{-\zeta \omega_n t}$$

and

$$x = -\sqrt{1 - \zeta^2} X e^{-\zeta \omega_n t}$$

2.100 Derive an expression for the time at which the response of a critically damped system will attain its maximum value. Also find the expression for the maximum response.

2.101 A shock absorber is to be designed to limit its overshoot to 15 percent of its initial displacement when released. Find the damping ratio ζ_0 required. What will be the overshoot if ζ is made equal to (a) $\frac{3}{4}\zeta_0$, and (b) $\frac{5}{4}\zeta_0$?

2.102 The free-vibration responses of an electric motor of weight 500 N mounted on different types of foundations are shown in Figs. 2.107(a) and (b). Identify the following in each case: (i) the nature of damping provided by the foundation, (ii) the spring constant and damping coefficient of the foundation, and (iii) the undamped and damped natural frequencies of the electric motor.

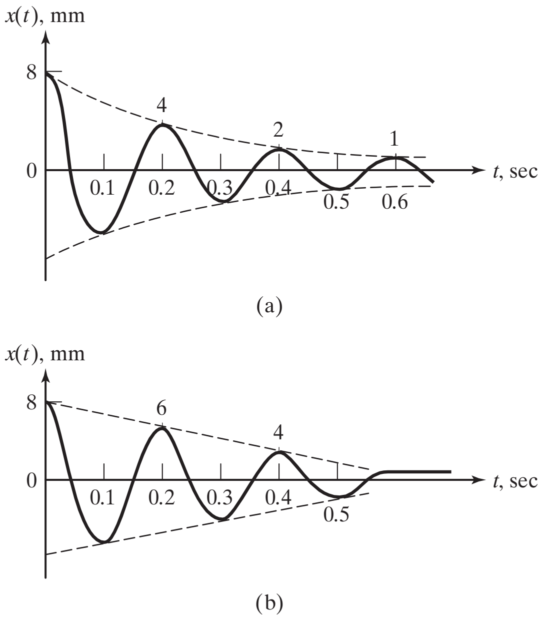


FIGURE 2.107

2.103 For a spring-mass-damper system, $m = 50$ kg and $k = 5{,}000$ N/m. Find the following: (a) critical damping constant c_c, (b) damped natural frequency when $c = c_c/2$, and (c) logarithmic decrement.

2.104 A railroad car of mass 2,000 kg traveling at a velocity $v = 10$ m/s is stopped at the end of the tracks by a spring-damper system, as shown in Fig. 2.108. If the stiffness of the spring is $k = 80$ N/mm and the damping constant is $c = 20$ N-s/mm, determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

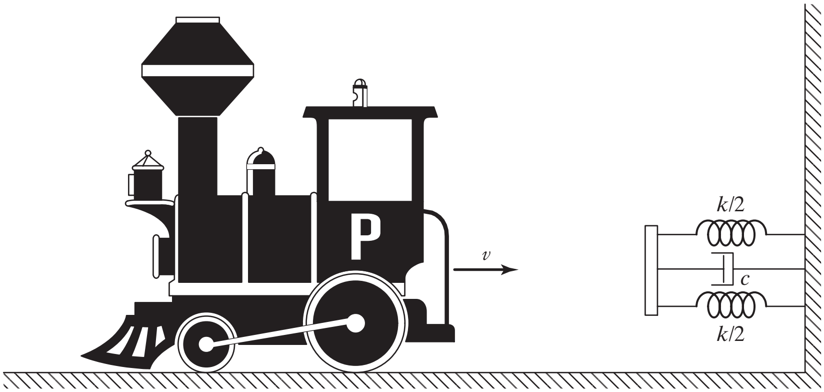


FIGURE 2.108

2.105 A torsional pendulum has a natural frequency of 200 cycles/min when vibrating in a vacuum. The mass moment of inertia of the disc is 0.2 kg-m^2. It is then immersed in oil and its natural frequency is found to be 180 cycles/min. Determine the damping constant. If the disc, when placed in oil, is given an initial displacement of 2°, find its displacement at the end of the first cycle.

2.106 A boy riding a bicycle can be modeled as a spring-mass-damper system with an equivalent weight, stiffness, and damping constant of 800 N, 50,000 N/m, and 1,000 N-s/m, respectively. The differential setting of the concrete blocks on the road caused the level surface to decrease suddenly, as indicated in Fig. 2.109. If the speed of the bicycle is 5 m/s (18 km/hr), determine the displacement of the boy in the vertical direction. Assume that the bicycle is free of vertical vibration before encountering the step change in the vertical displacement.

2.107 A wooden rectangular prism of weight 20 lb, height 3 ft, and cross section 1 ft $\times$ 2 ft floats and remains vertical in a tub of oil. The frictional resistance of the oil can be assumed to be equivalent to a viscous damping coefficient ζ. When the prism is depressed by a distance of 6 in. from its equilibrium and released, it is found to reach a depth of 5.5 in. at the end of its first cycle of oscillation. Determine the value of the damping coefficient of the oil.

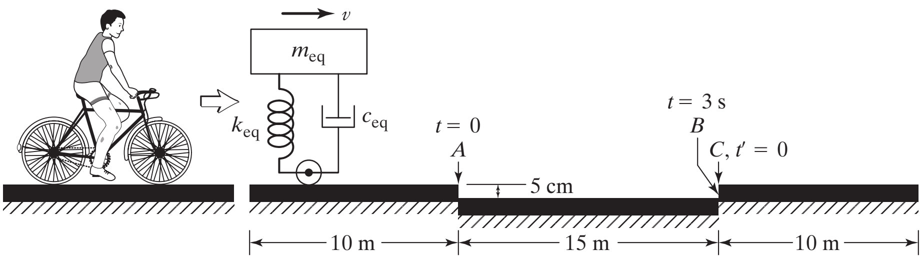


FIGURE 2.109

2.108 A body vibrating with viscous damping makes five complete oscillations per second, and in 50 cycles its amplitude diminishes to 10 percent. Determine the logarithmic decrement and the damping ratio. In what proportion will the period of vibration be decreased if damping is removed?

2.109 The maximum permissible recoil distance of a gun is specified as 0.5 m. If the initial recoil velocity is to be between 8 m/s and 10 m/s, find the mass of the gun and the spring stiffness of the recoil mechanism. Assume that a critically damped dashpot is used in the recoil mechanism and the mass of the gun has to be at least 500 kg.

2.110 A viscously damped system has a stiffness of 5,000 N/m, critical damping constant of 0.2 N-s/mm, and a logarithmic decrement of 2.0. If the system is given an initial velocity of 1 m/s, determine the maximum displacement of the system.

2.111 Explain why an overdamped system never passes through the static equilibrium position when it is given (a) an initial displacement only and (b) an initial velocity only.

2.112–2.114 Derive the equation of motion and find the natural frequency of vibration of each of the systems shown in Figs. 2.110 to 2.112.

2.115–2.117 Using the principle of virtual work, derive the equation of motion for each of the systems shown in Figs. 2.110 to 2.112.

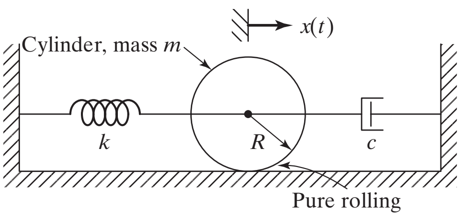


FIGURE 2.110

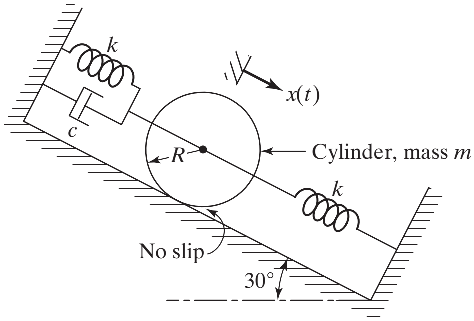


FIGURE 2.111

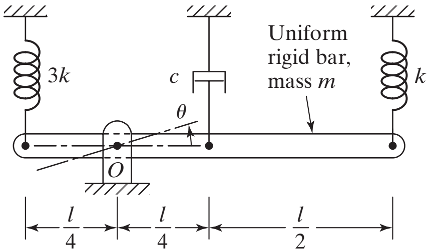


FIGURE 2.112

2.118 A wooden rectangular prism of cross section 40 cm × 60 cm, height 120 cm, and mass 40 kg floats in a fluid as shown in Fig. 2.105. When disturbed, it is observed to vibrate freely with a natural period of 0.5 s. Determine the density of the fluid.

2.119 The system shown in Fig. 2.113 has a natural frequency of 5 Hz for the following data: $m = 10$ kg, $J_0 = 5$ kg-m^2, $r_1 = 10$ cm, $r_2 = 25$ cm. When the system is disturbed by giving it an initial displacement, the amplitude of free vibration is reduced by 80 percent in 10 cycles. Determine the values of k and c.

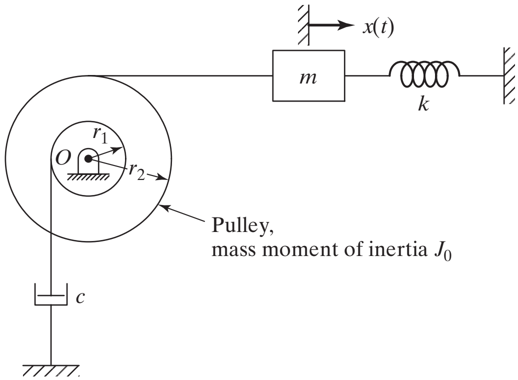


FIGURE 2.113

2.120 The rotor of a dial indicator is connected to a torsional spring and a torsional viscous damper to form a single-degree-of-freedom torsional system. The scale is graduated in equal divisions, and the equilibrium position of the rotor corresponds to zero on the scale. When a torque of 2×10^{-3} N-m is applied, the angular displacement of the rotor is found to be 50° with the pointer showing 80 divisions on the scale. When the rotor is released from this position, the pointer swings first to -20 divisions in one second and then to 5 divisions in another second. Find (a) the mass moment of inertia of the rotor, (b) the undamped natural time period of the rotor, (c) the torsional damping constant, and (d) the torsional spring stiffness.

2.121 Determine the values of ζ and ω_d for the following viscously damped systems:

a. $m = 10$ kg, $c = 150$ N-s/m, $k = 1000$ N/m

b. $m = 10$ kg, $c = 200$ N-s/m, $k = 1000$ N/m

c. $m = 10$ kg, $c = 250$ N-s/m, $k = 1000$ N/m

2.122 Determine the free-vibration response of the viscously damped systems described in Problem 2.121 when $x_0 = 0.1$ m and $\dot{x}_0 = 10$ m/s.

2.123 Find the energy dissipated during a cycle of simple harmonic motion given by $x(t) = 0.2 \sin \omega_d t$ m by a viscously damped single-degree-of-freedom system with the following parameters:

a. $m = 10$ kg, $c = 50$ N-s/m, $k = 1000$ N/m

b. $m = 10$ kg, $c = 150$ N-s/m, $k = 1000$ N/m

2.124 The equation of motion of a spring-mass-damper system, with a hardening-type spring, is given by (in SI units)

$$100\ddot{x} + 500\dot{x} + 10000x + 400x^3 = 0$$

a. Determine the static equilibrium position of the system.
b. Derive the linearized equation of motion for small displacements (x) about the static equilibrium position.
c. Find the natural frequency of vibration of the system for small displacements.

2.125 The equation of motion of a spring-mass-damper system, with a softening-type spring, is given by (in SI units)

$$100\ddot{x} + 500\dot{x} + 10000x - 400x^3 = 0$$

a. Determine the static equilibrium position of the system.
b. Derive the linearized equation of motion for small displacements (x) about the static equilibrium position.
c. Find the natural frequency of vibration of the system for small displacements.

2.126 The needle indicator of an electronic instrument is connected to a torsional viscous damper and a torsional spring. If the rotary inertia of the needle indicator about its pivot point is 25 kg-m^2 and the spring constant of the torsional spring is 100 N-m/rad, determine the damping constant of the torsional damper if the instrument is to be critically damped.

2.127 Find the responses of systems governed by the following equations of motion for the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$:

a. $2\ddot{x} + 8\dot{x} + 16x = 0$
b. $3\ddot{x} + 12\dot{x} + 9x = 0$
c. $2\ddot{x} + 8\dot{x} + 8x = 0$

2.128 Find the responses of systems governed by the following equations of motion for the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$:

a. $2\ddot{x} + 8\dot{x} + 16x = 0$
b. $3\ddot{x} + 12\dot{x} + 9x = 0$
c. $2\ddot{x} + 8\dot{x} + 8x = 0$

2.129 Find the responses of systems governed by the following equations of motion for the initial conditions $x(0) = 1$, $\dot{x}(0) = -1$:

a. $2\ddot{x} + 8\dot{x} + 16x = 0$
b. $3\ddot{x} + 12\dot{x} + 9x = 0$
c. $2\ddot{x} + 8\dot{x} + 8x = 0$

2.130 A spring-mass system is found to vibrate with a frequency of 120 cycles per minute in air and 100 cycles per minute in a liquid. Find the spring constant k, the damping constant c, and the damping ratio ζ when vibrating in the liquid. Assume $m = 10$ kg.

2.131 Find the frequency of oscillation and time constant for the systems governed by the following equations:

a. $\ddot{x} + 2\dot{x} + 9x = 0$
b. $\ddot{x} + 8\dot{x} + 9x = 0$
c. $\ddot{x} + 6\dot{x} + 9x = 0$

2.132 The mass moment of inertia of a nonhomogeneous and/or complex-shaped body of revolution about the axis of rotation can be determined by first finding its natural frequency of torsional vibration about its axis of rotation. In the torsional system shown in Fig. 2.114, the body of revolution (or rotor), of rotary inertia J, is supported on two frictionless bearings and

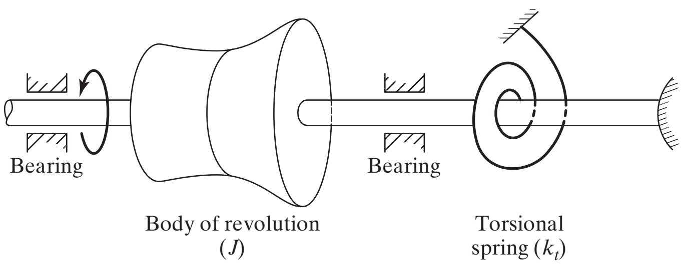


FIGURE 2.114

connected to a torsional spring of stiffness k_t. By giving an initial twist (angular displacement) of θ_0 and releasing the rotor, the period of the resulting vibration is measured as τ.

a. Find an expression for the mass moment of inertia of the rotor (J) in terms of τ and k_t.
b. Determine the value of J if $\tau = 0.5$ s and $k_t = 5000$ N-m/rad.

Section 2.7 Graphical Representation of Characteristic Roots and Corresponding Solutions

2.133 The characteristic roots of a single-degree-of-freedom system are given below. Find all the applicable features of the system among the characteristic equation, time constant, undamped natural frequency, damped frequency, and damping ratio.

a. $s_{1,2} = -4 \pm 5i$
b. $s_{1,2} = 4 \pm 5i$
c. $s_{1,2} = -4, -5$
d. $s_{1,2} = -4, -4$

2.134 Show the characteristic roots indicated in Problem 2.133 (a)–(d) in the s-plane and describe the nature of the response of the system in each case.

2.135 The characteristic equation of a single-degree-of-freedom system, given by Eq. (2.107), can be rewritten as

$$s^2 + as + b = 0 \tag{E.1}$$

where $a = c/m$ and $b = k/m$ can be considered as the parameters of the system. Identify regions that represent a stable, unstable, and marginally stable system in the parameter plane—i.e., the plane in which a and b are denoted along the vertical and horizontal axes, respectively.

Section 2.8 Parameter Variations and Root Locus Representations

2.136 Consider the characteristic equation: $2s^2 + cs + 18 = 0$. Draw the root locus of the system for $c \geq 0$.

2.137 Consider the characteristic equation: $2s^2 + 12s + k = 0$. Draw the root locus of the system for $k \geq 0$.

2.138 Consider the characteristic equation: $ms^2 + 12s + 4 = 0$. Draw the root locus of the system for $m \geq 0$.

Section 2.9 Free Vibration with Coulomb Damping

2.139 A single-degree-of-freedom system consists of a mass of 20 kg and a spring of stiffness 4,000 N/m. The amplitudes of successive cycles are found to be 50, 45, 40, 35, ... mm. Determine the nature and magnitude of the damping force and the frequency of the damped vibration.

2.140 A mass of 20 kg slides back and forth on a dry surface due to the action of a spring having a stiffness of 10 N/mm. After four complete cycles, the amplitude has been found to be 100 mm. What is the average coefficient of friction between the two surfaces if the original amplitude was 150 mm? How much time has elapsed during the four cycles?

2.141 A 10-kg mass is connected to a spring of stiffness 3,000 N/m and is released after giving an initial displacement of 100 mm. Assuming that the mass moves on a horizontal surface, as shown in Fig. 2.42(a), determine the position at which the mass comes to rest. Assume the coefficient of friction between the mass and the surface to be 0.12.

2.142 A weight of 25 N is suspended from a spring that has a stiffness of 1,000 N/m. The weight vibrates in the vertical direction under a constant damping force. When the weight is initially pulled downward a distance of 10 cm from its static equilibrium position and released, it comes to rest after exactly two complete cycles. Find the magnitude of the damping force.

2.143 A mass of 20 kg is suspended from a spring of stiffness 10,000 N/m. The vertical motion of the mass is subject to Coulomb friction of magnitude 50 N. If the spring is initially displaced downward by 5 cm from its static equilibrium position, determine (a) the number of half cycles elapsed before the mass comes to rest, (b) the time elapsed before the mass comes to rest, and (c) the final extension of the spring.

2.144 The Charpy impact test is a dynamic test in which a specimen is struck and broken by a pendulum (or hammer) and the energy absorbed in breaking the specimen is measured. The energy values serve as a useful guide for comparing the impact strengths of different materials. As shown in Fig. 2.115, the pendulum is suspended from a shaft, is released from a particular position, and is allowed to fall and break the specimen. If the pendulum is made to oscillate freely (with no specimen), find (a) an expression for the decrease in the angle of swing for each cycle caused by friction, (b) the solution for $\theta(t)$ if the pendulum is released from an angle θ_0, and (c) the number of cycles after which the motion ceases. Assume the mass of the pendulum is m and the coefficient of friction between the shaft and the bearing of the pendulum is μ.

2.145 Find the equivalent viscous-damping constant for Coulomb damping for sinusoidal vibration.

2.146 A single-degree-of-freedom system consists of a mass, a spring, and a damper in which both dry friction and viscous damping act simultaneously. The free-vibration amplitude is found to decrease by 1 percent per cycle when the amplitude is 20 mm and by 2 percent per cycle when the amplitude is 10 mm. Find the value of $(\mu N/k)$ for the dry-friction component of the damping.

2.147 A metal block, placed on a rough surface, is attached to a spring and is given an initial displacement of 10 cm from its equilibrium position. It is found that the natural time period of motion is 1.0 s and that the amplitude reduces by 0.5 cm in each cycle. Find (a) the kinetic coefficient of friction between the metal block and the surface and (b) the number of cycles of motion executed by the block before it stops.

2.148 The mass of a spring-mass system with $k = 10{,}000$ N/m and $m = 5$ kg is made to vibrate on a rough surface. If the friction force is $F = 20$ N and the amplitude of the mass is observed to decrease by 50 mm in 10 cycles, determine the time taken to complete the 10 cycles.

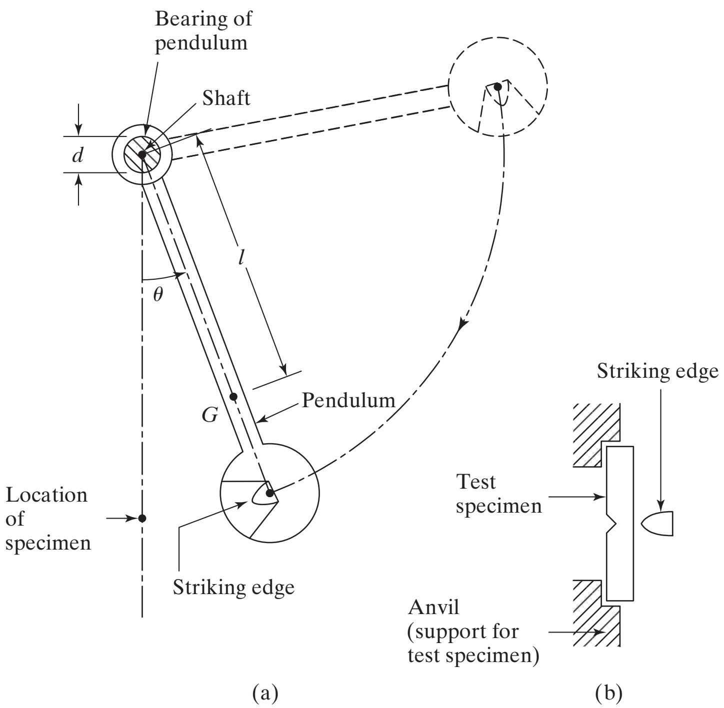


FIGURE 2.115

2.149 The mass of a spring-mass system vibrates on a dry surface inclined at 30° to the horizontal as shown in Fig. 2.116.

a. Derive the equation of motion.
b. Find the response of the system for the following data:

$$m = 20 \text{ kg}, \quad k = 1{,}000 \text{ N/m}, \quad \mu = 0.1, \quad x_0 = 0.1 \text{ m}, \quad \dot{x}_0 = 5 \text{ m/s}.$$

2.150 The mass of a spring-mass system is initially displaced by 10 cm from its unstressed position by applying a force of 25 N, which is equal to five times the weight of the mass. If the mass is released from this position, how long will the mass vibrate and at what distance will it stop from the unstressed position? Assume a coefficient of friction of 0.2.

Section 2.10 Free Vibration with Hysteretic Damping

2.151 The experimentally observed force-deflection curve for a composite structure is shown in Fig. 2.117. Find the hysteresis damping constant, the logarithmic decrement, and the equivalent viscous-damping ratio corresponding to this curve.

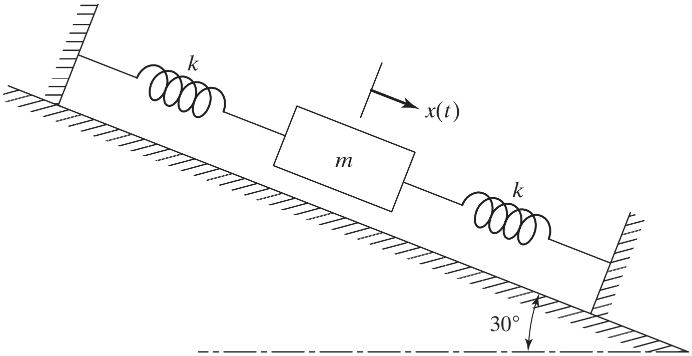


FIGURE 2.116

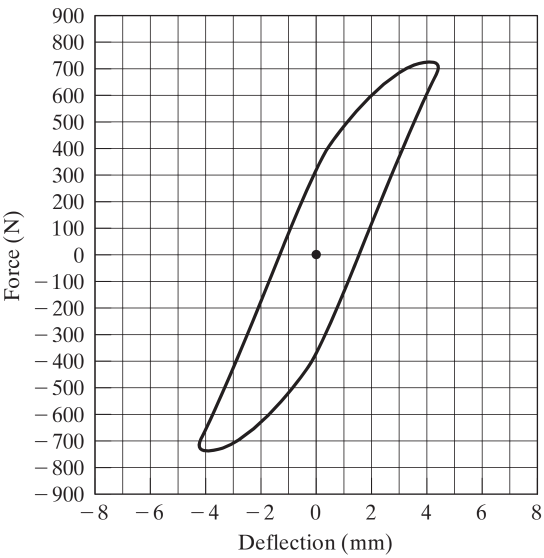


FIGURE 2.117

2.152 A panel made of fiber-reinforced composite material is observed to behave as a single-degree-of-freedom system of mass 1 kg and stiffness 2 N/m. The ratio of successive amplitudes is found to be 1.1. Determine the value of the hysteresis-damping constant β, the equivalent viscous-damping constant c_{eq}, and the energy loss per cycle for an amplitude of 10 mm.

2.153 A built-up cantilever beam having a bending stiffness of 200 N/m supports a mass of 2 kg at its free end. The mass is displaced initially by 30 mm and released. If the amplitude is found to be 20 mm after 100 cycles of motion, estimate the hysteresis-damping constant β of the beam.

2.154 A mass of 5 kg is attached to the top of a helical spring, and the system is made to vibrate by giving to the mass an initial deflection of 25 mm. The amplitude of the mass is found to reduce to 10 mm after 100 cycles of vibration. Assuming a spring rate of 200 N/m for the helical spring, find the value of the hysteretis-damping coefficient (h) of the spring.

Section 2.11 Stability of Systems

2.155 Consider the equation of motion of a simple pendulum:

$$\dddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{E.1}$$

a. Linearize Eq. (E.1) about an arbitrary angular displacement θ_0 of the pendulum.
b. Investigate the stability of the pendulum about $\theta_0 = 0$ and $\theta_0 = \pi$ using the linearized equation of motion.

2.156 Figure 2.118 shows a uniform rigid bar of mass m and length l, pivoted at one end (point O) and carrying a circular disk of mass M and mass moment of inertia J (about its rotational axis) at the other end (point P). The circular disk is connected to a spring of stiffness k and a viscous damper of damping constant c as indicated.

a. Derive the equation of motion of the system for small angular displacements of the rigid bar about the pivot point O and express it in the form:

$$m_0\dddot{\theta} + c_0\ddot{\theta} + k_0\theta = 0$$

b. Derive conditions corresponding to the stable, unstable, and marginally stable behavior of the system.

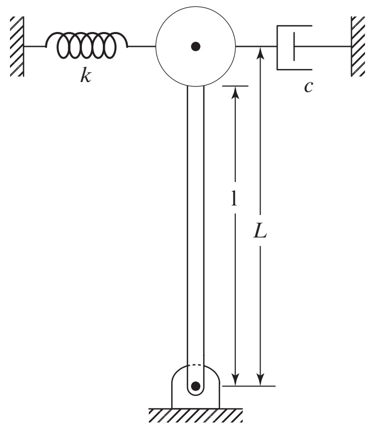


FIGURE 2.118

Section 2.12 Examples Using MATLAB

2.157 Find the free-vibration response of a spring-mass system subject to Coulomb damping using MATLAB for the following data:

$$m = 5 \text{ kg}, \quad k = 100 \text{ N/m}, \quad \mu = 0.5, \quad x_0 = 0.4 \text{ m}, \quad \dot{x}_0 = 0.$$

2.158 Plot the response of a critically damped system (Eq. 2.80) for the following data using MATLAB:

a. $x_0 = 10$ mm, 50 mm, 100 mm; $\dot{x}_0 = 0$, $\omega_n = 10$ rad/s.
b. $x_0 = 0$, $\dot{x}_0 = 10$ mm/s, 50 mm/s, 100 mm/s; $\omega_n = 10$ rad/s.

2.159 Plot Eq. (2.81) as well as each of the two terms of Eq. (2.81) as functions of t using MATLAB for the following data:

$$\omega_n = 10 \text{ rad/s}, \quad \zeta = 2.0, \quad x_0 = 20 \text{ mm}, \quad \dot{x}_0 = 50 \text{ mm/s}$$

2.160–2.163 Using the MATLAB Program2.m, plot the free-vibration response of a viscously damped system with $m = 4$ kg, $k = 2{,}500$ N/m, $x_0 = 100$ mm, $\dot{x}_0 = -10$ m/s, $\Delta t = 0.01$ s, $n = 50$ for the following values of the damping constant:

a. $c = 0$
b. $c = 100$ N-s/m
c. $c = 200$ N-s/m
d. $c = 400$ N-s/m

2.164 Find the response of the system described in Problem 2.149 using MATLAB.

DESIGN PROJECTS

2.165* A water turbine of mass 1,000 kg and mass moment of inertia 500 kg-m^2 is mounted on a steel shaft, as shown in Fig. 2.119. The operational speed of the turbine is 2,400 rpm. Assuming the ends of the shaft to be fixed, find the values of l, a, and d, such that the natural frequency of vibration of the turbine in each of the axial, transverse, and circumferential directions is greater than the operational speed of the turbine.

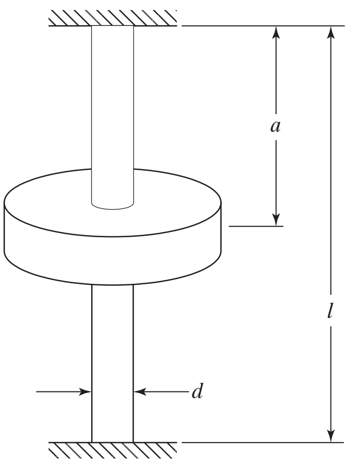


FIGURE 2.119

2.166* Design the columns for each of the building frames shown in Figs. 2.79(a) and (b) for minimum weight such that the natural frequency of vibration is greater than 50 Hz. The weight of the floor (W) is 4,000 lb and the length of the columns (l) is 96 in. Assume that the columns are made of steel and have a tubular cross section with outer diameter d and wall thickness t.

2.167* One end of a uniform rigid bar of mass m is connected to a wall by a hinge joint O, and the other end carries a concentrated mass M, as shown in Fig. 2.120. The bar rotates about the hinge point O against a torsional spring and a torsional damper. It is proposed to use this mechanism, in conjunction with a mechanical counter, to control entrance to an amusement park. Find the masses m and M, the stiffness of the torsional spring (k_t), and the damping force (F_d) necessary to satisfy the following specifications: (1) A viscous damper or a Coulomb damper can be used. (2) The bar has to return to within 5° of closing in less than 2 sec when released from an initial position of $\theta = 75°$.

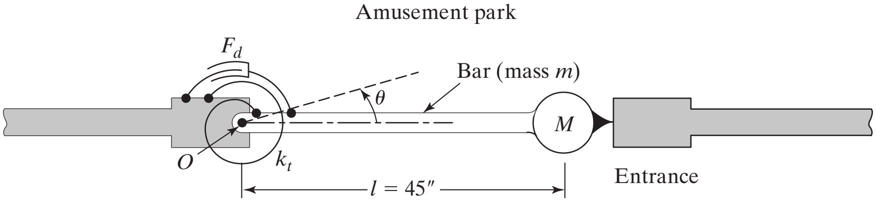


FIGURE 2.120

2.168* The lunar excursion module has been modeled as a mass supported by four symmetrically located legs, each of which can be approximated as a spring-damper system with negligible mass (see Fig. 2.121). Design the springs and dampers of the system in order to have the damped period of vibration between 1 s and 2 s.

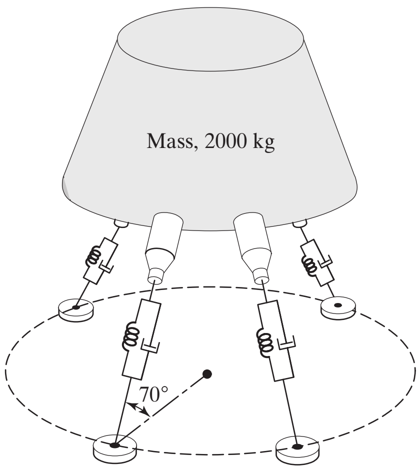


FIGURE 2.121

2.169* Consider the telescoping boom and cockpit of the firetruck shown in Fig. 2.12(a). Assume that the telescoping boom $PQRS$ is supported by a strut QT, as shown in Fig. 2.122. Determine the cross section of the strut QT so that the natural time period of vibration of the cockpit with the fireperson is equal to 1 s for the following data. Assume that each segment of the telescoping boom and the strut is hollow circular in cross section. In addition, assume that the strut acts as a spring that deforms only in the axial direction.

Data:

Lengths of segments: $PQ = 12$ ft, $QR = 10$ ft, $RS = 8$ ft, $TP = 3$ ft

Young's modulus of the telescoping arm and strut $= 30 \times 10^6$ psi

Outer diameters of sections: $PQ = 2.0$ in., $QR = 1.5$ in., $RS = 1.0$ in.

Inner diameters of sections: $PQ = 1.75$ in., $QR = 1.25$ in., $RS = 0.75$ in.

Weight of the cockpit $= 100$ lb

Weight of fireperson $= 200$ lb

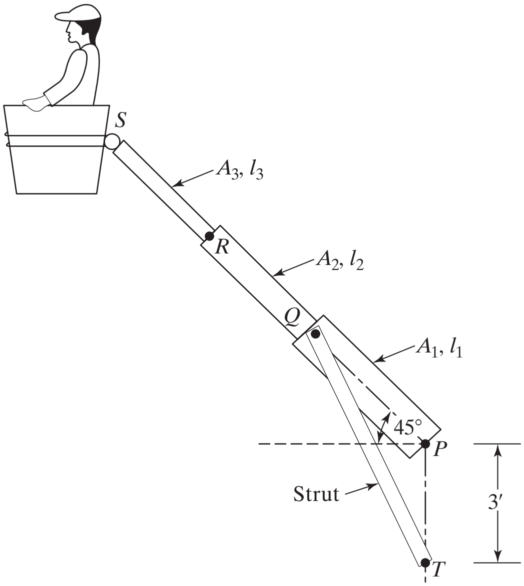


FIGURE 2.122

Charles Augustin de Coulomb (1736–1806) was a French military engineer and physicist. His early work on statics and mechanics was presented in 1779 in his great memoir *The Theory of Simple Machines*, which describes the effect of resistance and the so-called "Coulomb's law of proportionality" between friction and normal pressure. In 1784, he obtained the correct solution to the problem of the small oscillations of a body subjected to torsion. He is well known for his laws of force for electrostatic and magnetic charges. His name has been given to the unit of electric charge. (Courtesy of *Applied Mechanics Reviews*.)

CHAPTER 3

Harmonically Excited Vibration

Chapter Outline

This chapter deals with the response of single-degree-of-freedom systems subjected to harmonic excitations. First, it presents the derivation of the equation of motion and its solution when a single degree of freedom system is subjected to harmonic excitation. Both undamped and damped systems are considered. The magnification or amplification factor, and the phenomena of resonance and beating are introduced in the context of an undamped spring-mass system. The total solution of the governing nonhomogeneous second-order differential equation is presented as a sum of the homogeneous equation (free-vibration solution) and the particular integral (forced-vibration solution). The known initial conditions of the system are used to evaluate the constants in the total solution. The important characteristics of the magnification factor and the phase angle for a viscously damped system are presented in detail. Quality factor, bandwidth, and half-power point are defined and the use of quality factor in estimating the viscous damping factor in a mechanical system is outlined. The response of the spring-mass-damper system with the harmonic forcing function in complex form is presented and the concept of complex frequency response is introduced. The response of a damped system under the harmonic motion of the base and the ideas of displacement transmissibility and force transmissibility are introduced. The applications of this problem include vibration of airplanes caused by runway roughness during taxiing and landing, vibration of ground vehicles due to unevenness and potholes in roads, and vibration of buildings caused by ground motion during earthquakes. The response of a damped system under rotating unbalance is also presented. The applications of this problem include a variety of rotating machines with unbalance in the rotors. The forced vibration of a spring-mass system under Coulomb, hysteresis, and other types of damping is also presented. Self-excitation and dynamic stability analysis of a single-degree-of-freedom system along with applications are presented. The general transfer-function approach, the Laplace transform approach, and the harmonic transfer-function approach for the solution of harmonically excited systems are outlined. Finally, the solution of different types of harmonically excited undamped and damped vibration problems using MATLAB is presented.

Learning Objectives

After completing this chapter, you should be able to do the following:

- Find the responses of undamped and viscously damped single-degree-of-freedom systems subjected to different types of harmonic force, including base excitation and rotating unbalance.
- Distinguish between transient, steady-state, and total solutions.
- Understand the variations of magnification factor and phase angles with the frequency of excitation and the phenomena of resonance and beats.
- Find the response of systems involving Coulomb, hysteresis, and other types of damping.
- Identify self-excited problems and investigate their stability aspects.
- Derive transfer functions of systems governed by linear differential equations with constant coefficients.
- Solve harmonically excited single-degree-of-freedom vibration problems using Laplace transforms.

- Derive frequency transfer function from the general transfer function and represent frequency-response characteristics using Bode diagrams.
- Solve harmonically excited vibration response using MATLAB.

3.1 Introduction

A mechanical or structural system is said to undergo forced vibration whenever external energy is supplied to the system during vibration. External energy can be supplied through either an applied force or an imposed displacement excitation. The applied force or displacement excitation may be harmonic, nonharmonic but periodic, nonperiodic, or random in nature. The response of a system to a harmonic excitation is called *harmonic response*. The nonperiodic excitation may have a long or short duration. The response of a dynamic system to suddenly applied nonperiodic excitations is called *transient response*.

In this chapter, we shall consider the dynamic response of a single-degree-of-freedom system under harmonic excitations of the form $F(t) = F_0 e^{i(\omega t + \phi)}$ or $F(t) = F_0 \cos(\omega t + \phi)$ or $F(t) = F_0 \sin(\omega t + \phi)$, where F_0 is the amplitude, ω is the frequency, and ϕ is the phase angle of the harmonic excitation. The value of ϕ depends on the value of $F(t)$ at $t = 0$ and is usually taken to be zero. Under a harmonic excitation, the response of the system will also be harmonic. If the frequency of excitation coincides with the natural frequency of the system, the response will be very large. This condition, known as *resonance*, is to be avoided to prevent failure of the system. The vibration produced by an unbalanced rotating machine, the oscillations of a tall chimney due to vortex shedding in a steady wind, and the vertical motion of an automobile on a sinusoidal road surface are examples of harmonically excited vibration.

The applications of transfer-function, Laplace transform, and frequency-function approaches in the solution of harmonically excited systems are also discussed in this chapter.

3.2 Equation of Motion

If a force $F(t)$ acts on a viscously damped spring-mass system as shown in Fig. 3.1, the equation of motion can be obtained using Newton's second law:

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{3.1}$$

Since this equation is nonhomogeneous, its general solution $x(t)$ is given by the sum of the homogeneous solution, $x_h(t)$, and the particular solution, $x_p(t)$. The homogeneous solution, which is the solution of the homogeneous equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{3.2}$$

represents the free vibration of the system and was discussed in Chapter 2. As seen in Section 2.6.2, this free vibration dies out with time under each of the three possible conditions of damping (underdamping, critical damping, and overdamping) and under all possible initial

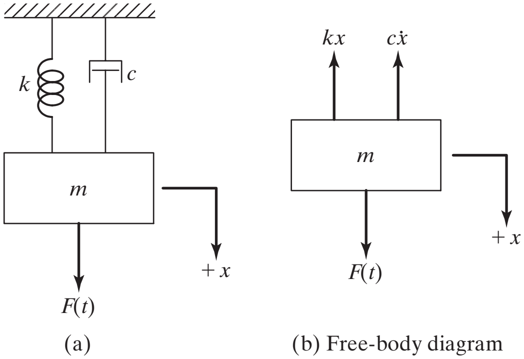


FIGURE 3.1 A spring-mass-damper system.

conditions. Thus the general solution of Eq. (3.1) eventually reduces to the particular solution $x_p(t)$, which represents the steady-state vibration. The steady-state motion is present as long as the forcing function is present. The variations of homogeneous, particular, and general solutions with time for a typical case are shown in Fig. 3.2. It can be seen that $x_h(t)$ dies out and $x(t)$ becomes $x_p(t)$ after some time (τ in Fig. 3.2). The part of the motion that dies out due to damping (the free-vibration part) is called *transient*. The rate at which the transient motion decays depends on the values of the system parameters *k, c*, and *m*. In this chapter, except in Section 3.3, we ignore the transient motion and derive only the particular solution of Eq. (3.1), which represents the steady-state response, under harmonic forcing functions.

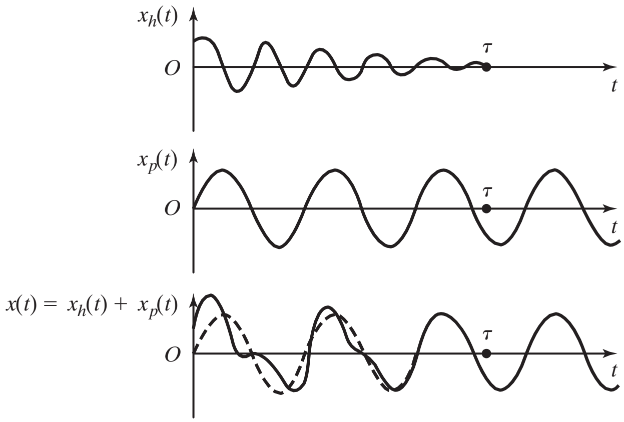


FIGURE 3.2 Homogenous, particular, and general solutions of Eq. (3.1) for an underdamped case.

3.3 Response of an Undamped System Under Harmonic Force

Before studying the response of a damped system, we consider an undamped system subjected to a harmonic force, for the sake of simplicity. If a force $F(t) = F_0 \cos \omega t$ acts on the mass m of an undamped system, the equation of motion, Eq. (3.1), reduces to

$$m\ddot{x} + kx = F_0 \cos \omega t \tag{3.3}$$

The homogeneous solution of this equation is given by

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \tag{3.4}$$

where $\omega_n = (k/m)^{1/2}$ is the natural frequency of the system. Because the exciting force $F(t)$ is harmonic, the particular solution $x_p(t)$ is also harmonic and has the same frequency ω. Thus we assume a solution in the form

$$x_p(t) = X \cos \omega t \tag{3.5}$$

where X is an constant that denotes the maximum amplitude of $x_p(t)$. By substituting Eq. (3.5) into Eq. (3.3) and solving for X, we obtain

$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \tag{3.6}$$

where $\delta_{st} = F_0/k$ denotes the deflection of the mass under a force F_0 and is sometimes called *static deflection* because F_0 is a constant (static) force. Thus the total solution of Eq. (3.3) becomes

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t \tag{3.7}$$

Using the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, we find that

$$C_1 = x_0 - \frac{F_0}{k - m\omega^2}, \qquad C_2 = \frac{\dot{x}_0}{\omega_n} \tag{3.8}$$

and hence

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right) \sin \omega_n t$$
$$+ \left(\frac{F_0}{k - m\omega^2}\right) \cos \omega t \tag{3.9}$$

The maximum amplitude X in Eq. (3.6) can be expressed as

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{3.10}$$

The quantity X/δ_{st} represents the ratio of the dynamic to the static amplitude of motion and is called the *magnification factor, amplification factor*, or *amplitude ratio*. The variation of the amplitude ratio, X/δ_{st}, with the frequency ratio $r = \omega/\omega_n$ (Eq. 3.10) is shown in Fig. 3.3. From this figure, the response of the system can be identified to be of three types.

Case 1. When $0 < \omega/\omega_n < 1$, the denominator in Eq. (3.10) is positive and the response is given by Eq. (3.5) without change. The harmonic response of the system $x_p(t)$ is said to be in phase with the external force as shown in Fig. 3.4.

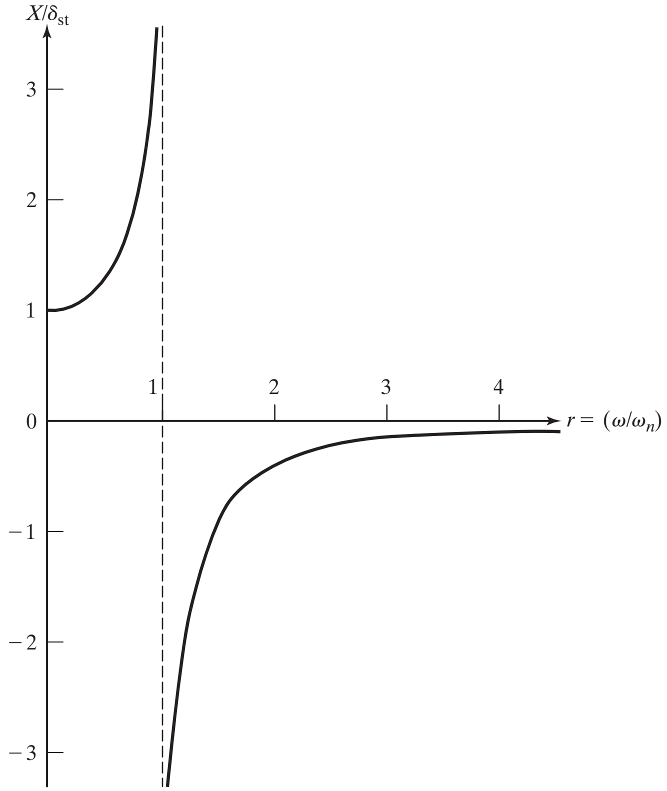


FIGURE 3.3 Magnification factor of an undamped system, Eq. (3.10).

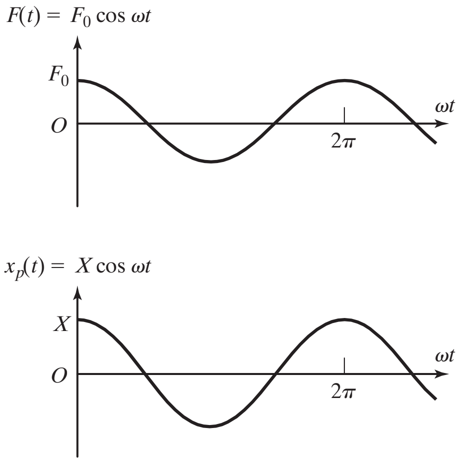


FIGURE 3.4 Harmonic response when $0 < \omega/\omega_n < 1$.

Case 2. When $\omega/\omega_n > 1$, the denominator in Eq. (3.10) is negative, and the steady-state solution can be expressed as

$$x_p(t) = -X \cos \omega t \tag{3.11}$$

where the amplitude of motion X is redefined to be a positive quantity as

$$X = \frac{\delta_{st}}{\left(\dfrac{\omega}{\omega_n}\right)^2 - 1} \tag{3.12}$$

The variations of $F(t)$ and $x_p(t)$ with time are shown in Fig. 3.5. Since $x_p(t)$ and $F(t)$ have opposite signs, the response is said to be 180° out of phase with the external force. Further, as $\omega/\omega_n \to \infty$, $X \to 0$. Thus the response of the system to a harmonic force of very high frequency is close to zero.

Case 3. When $\omega/\omega_n = 1$, the amplitude X given by Eq. (3.10) or (3.12) becomes infinite. This condition, for which the forcing frequency ω is equal to the natural frequency of the system ω_n, is called *resonance*. To find the response for this condition, we rewrite Eq. (3.9) as

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \right] \tag{3.13}$$

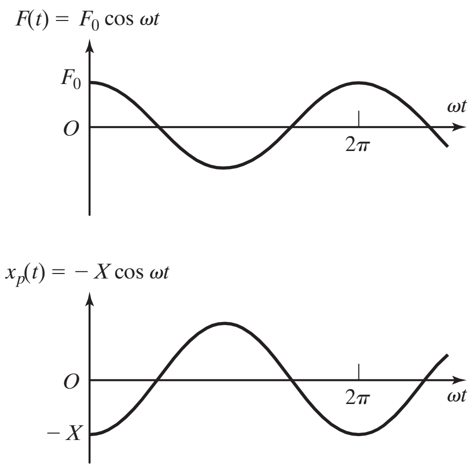


FIGURE 3.5 Harmonic response when $\omega/\omega_n > 1$.

Since the last term of this equation takes an indefinite form for $\omega = \omega_n$, we apply L'Hospital's rule [3.1] to evaluate the limit of this term:

$$\lim_{\omega \to \omega_n} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\dfrac{\omega}{\omega_n} \right)^2} \right] = \lim_{\omega \to \omega_n} \left[\frac{\dfrac{d}{d\omega}(\cos \omega t - \cos \omega_n t)}{\dfrac{d}{d\omega}\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)} \right]$$

$$= \lim_{\omega \to \omega_n} \left[\frac{t \sin \omega t}{2 \dfrac{\omega}{\omega_n^2}} \right] = \frac{\omega_n t}{2} \sin \omega_n t \tag{3.14}$$

Thus the response of the system at resonance becomes

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t \tag{3.15}$$

It can be seen from Eq. (3.15) that at resonance, $x(t)$ increases indefinitely. The last term of Eq. (3.15) is shown in Fig. 3.6, from which the amplitude of the response can be seen to increase linearly with time.

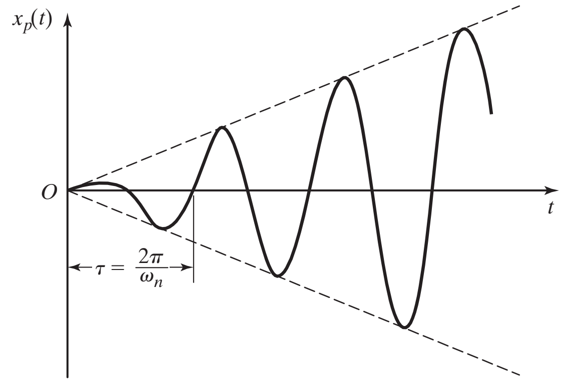


FIGURE 3.6 Response when $\omega/\omega_n = 1$.

3.3.1 Total Response

The total response of the system, Eq. (3.7) or Eq. (3.9), can also be expressed as

$$x(t) = A\cos(\omega_n t - \phi) + \frac{\delta_{st}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}\cos\omega t; \quad \text{for } \frac{\omega}{\omega_n} < 1 \tag{3.16}$$

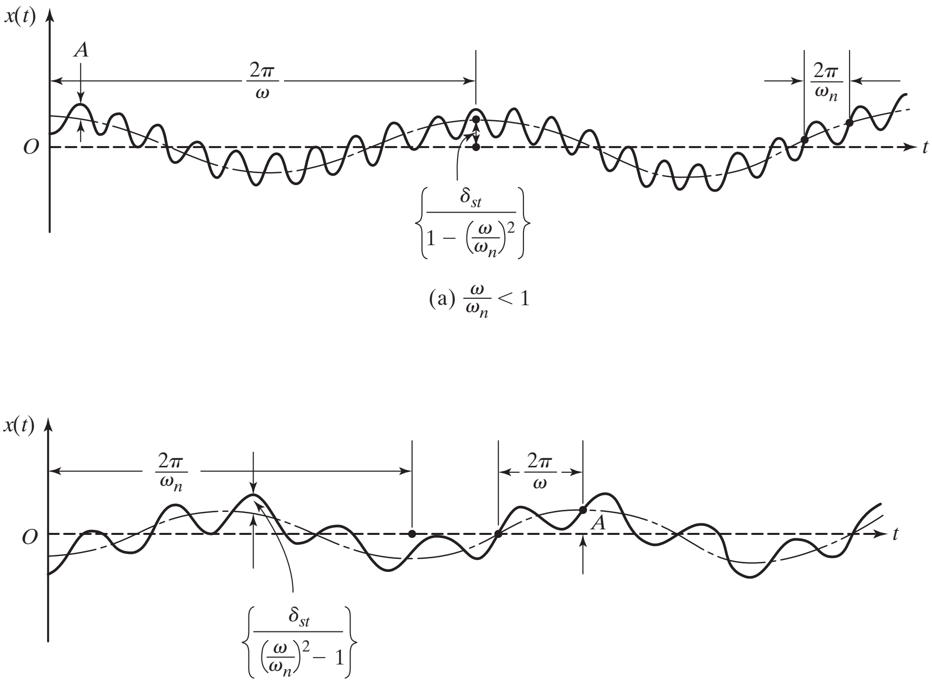
$$x(t) = A\cos(\omega_n t - \phi) - \frac{\delta_{st}}{-1 + \left(\dfrac{\omega}{\omega_n}\right)^2}\cos\omega t; \quad \text{for } \frac{\omega}{\omega_n} > 1 \tag{3.17}$$

where A and ϕ can be determined as in the case of Eq. (2.21). Thus the complete motion can be expressed as the sum of two cosine curves of different frequencies. In Eq. (3.16), the forcing frequency ω is smaller than the natural frequency, and the total response is shown in Fig. 3.7(a). In Eq. (3.17), the forcing frequency is greater than the natural frequency, and the total response appears as shown in Fig. 3.7(b).

3.3.2 Beating Phenomenon

If the forcing frequency is close to, but not exactly equal to, the natural frequency of the system, a phenomenon known as *beating* may occur. In this kind of vibration, the amplitude builds up and then diminishes in a regular pattern (see Section 1.10.5). The phenomenon of beating can be explained by considering the solution given by Eq. (3.9). If the initial conditions are taken as $x_0 = \dot{x}_0 = 0$, Eq. (3.9) reduces to

$$\begin{aligned} x(t) &= \frac{(F_0/m)}{\omega_n^2 - \omega^2}(\cos\omega t - \cos\omega_n t) \\ &= \frac{(F_0/m)}{\omega_n^2 - \omega^2}\left[2\sin\frac{\omega + \omega_n}{2}t \cdot \sin\frac{\omega_n - \omega}{2}t\right] \end{aligned} \tag{3.18}$$



(a) $\frac{\omega}{\omega_n} < 1$

(b) $\frac{\omega}{\omega_n} > 1$

FIGURE 3.7 Total response.

Let the forcing frequency ω be slightly less than the natural frequency:

$$\omega_n - \omega = 2\varepsilon \tag{3.19}$$

where ε is a small positive quantity. Then $\omega_n \approx \omega$ and

$$\omega + \omega_n \simeq 2\omega \tag{3.20}$$

Multiplication of Eqs. (3.19) and (3.20) gives

$$\omega_n^2 - \omega^2 = 4\varepsilon\omega \tag{3.21}$$

The use of Eqs. (3.19) to (3.21) in Eq. (3.18) gives

$$x(t) = \left(\frac{F_0/m}{2\varepsilon\omega} \sin \varepsilon t\right) \sin \omega t \tag{3.22}$$

Since ε is small, the function $\sin \varepsilon t$ varies slowly; its period, equal to $2\pi/\varepsilon$, is large. Thus Eq. (3.22) may be seen as representing vibration with period $2\pi/\omega$ and of variable amplitude equal to

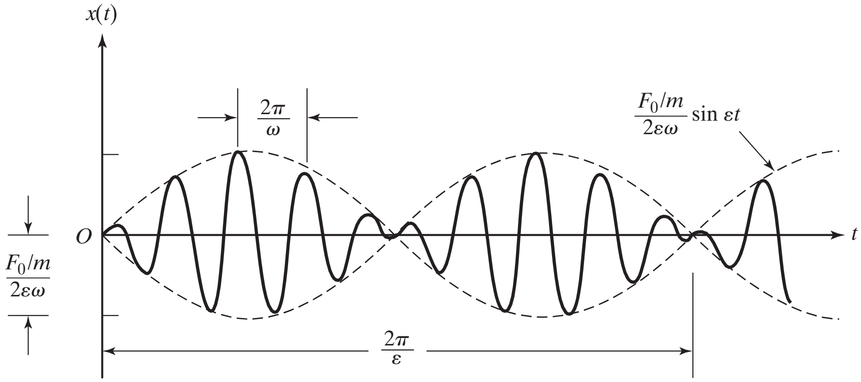


FIGURE 3.8 Phenomenon of beats.

$$\left(\frac{F_0/m}{2\varepsilon\omega}\right)\sin \varepsilon t$$

It can also be observed that the sin ωt curve will go through several cycles, while the sin εt wave goes through a single cycle, as shown in Fig. 3.8. Thus the amplitude builds up and dies down continuously. The time between the points of zero amplitude or the points of maximum amplitude is called the *period ofbeating* (τ_b) and is given by

$$\tau_b = \frac{2\pi}{2\varepsilon} = \frac{2\pi}{\omega_n - \omega} \tag{3.23}$$

with the frequency of beating defined as

$$\omega_b = 2\varepsilon = \omega_n - \omega$$

EXAMPLE 3.1

Plate Supporting a Pump

A reciprocating pump, weighing 150 lb, is mounted at the middle of a steel plate of thickness 0.5 in., width 20 in., and length 100 in., clamped along two edges as shown in Fig. 3.9. During operation of the pump, the plate is subjected to a harmonic force, $F(t) = 50 \cos 62.832t$ lb. Find the amplitude of vibration of the plate.

Solution: The plate can be modeled as a fixed-fixed beam having Young's modulus $(E) = 30 \times 10^6$ psi, length $(l) = 100$ in., and area moment of inertia $(I) = \frac{1}{12}(20)(0.5)^3 = 0.2083$ in^4. The bending stiffness of the beam is given by

$$k = \frac{192EI}{l^3} = \frac{192(30 \times 10^6)(0.2083)}{(100)^3} = 1200.0 \text{ lb/in.} \tag{E.1}$$

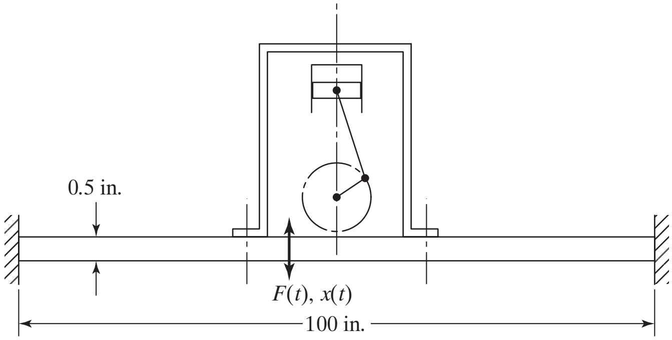


FIGURE 3.9 Plate supporting an unbalanced pump.

The amplitude of harmonic response is given by Eq. (3.6) with $F_0 = 50$ lb, $m = 150/386.4$ lb-sec^2/in. (neglecting the weight of the steel plate), $k = 1200.0$ lb/in., and $\omega = 62.832$ rad/s. Thus Eq. (3.6) gives

$$X = \frac{F_0}{k - m\omega^2} = \frac{50}{1200.0 - (150/386.4)(62.832)^2} = -0.1504 \text{ in.} \tag{E.2}$$

The negative sign indicates that the response $x(t)$ of the plate is out of phase with the excitation $F(t)$.

■

EXAMPLE 3.2

Determination of Mass from Known Harmonic Response

A spring-mass system, with a spring stiffness of 5,000 N/m, is subjected to a harmonic force of magnitude 30 N and frequency 20 Hz. The mass is found to vibrate with an amplitude of 0.2 m. Assuming that vibration starts from rest $(x_0 = \dot{x}_0 = 0)$, determine the mass of the system.

Solution: The vibration response of the system can be found from Eq. (3.9) with $x_0 = \dot{x}_0 = 0$:

$$x(t) = \frac{F_0}{k - m\omega^2}(\cos \omega t - \cos \omega_n t) \tag{E.1}$$

which can be rewritten as

$$x(t) = \frac{2F_0}{k - m\omega^2} \sin \frac{\omega_n + \omega}{2} t \sin \frac{\omega_n - \omega}{2} t \tag{E.2}$$

Since the amplitude of vibration is known to be 0.2 m, Eq. (E.2) gives

$$\frac{2F_0}{k - m\omega^2} = 0.2 \tag{E.3}$$

Using the known values of $F_0 = 30$ N, $\omega = 20$ Hz $= 125.665$ rad/s, and $k = 5{,}000$ N/m, Eq. (E.3) yields

$$\frac{2(30)}{5000 - m(125.664)^2} = 0.2 \tag{E.4}$$

Equation (E.4) can be solved to find $m = 0.2976$ kg.

■

3.4 Response of a Damped System Under Harmonic Force

If the forcing function is given by $F(t) = F_0 \cos \omega t$, the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \tag{3.24}$$

The particular solution of Eq. (3.24) is also expected to be harmonic; we assume it in the form[1]

$$x_p(t) = X \cos(\omega t - \phi) \tag{3.25}$$

where X and ϕ are constants to be determined. X and ϕ denote the amplitude and phase angle of the response, respectively. By substituting Eq. (3.25) into Eq. (3.24), we arrive at

$$X[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t \tag{3.26}$$

Using the trigonometric relations

$$\cos(\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi$$
$$\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$$

in Eq. (3.26) and equating the coefficients of $\cos \omega t$ and $\sin \omega t$ on both sides of the resulting equation, we obtain

$$\begin{aligned} X[(k - m\omega^2)\cos \phi + c\omega \sin \phi] &= F_0 \\ X[(k - m\omega^2)\sin \phi - c\omega \cos \phi] &= 0 \end{aligned} \tag{3.27}$$

Solution of Eq. (3.27) gives

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}} \tag{3.28}$$

---

[1] Alternatively, we can assume $x_p(t)$ to be of the form $x_p(t) = C_1 \cos \omega t + C_2 \sin \omega t$, which also involves two constants C_1 and C_2. But the final result will be the same in both cases.

and

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \tag{3.29}$$

By inserting the expressions of X and ϕ from Eqs. (3.28) and (3.29) into Eq. (3.25), we obtain the particular solution of Eq. (3.24). Figure 3.10(a) shows typical plots of the forcing function and (steady-state) response. The various terms of Eq. (3.26) are shown vectorially in Fig. 3.10(b). Dividing both the numerator and denominator of Eq. (3.28) by k and making the following substitutions

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency,}$$

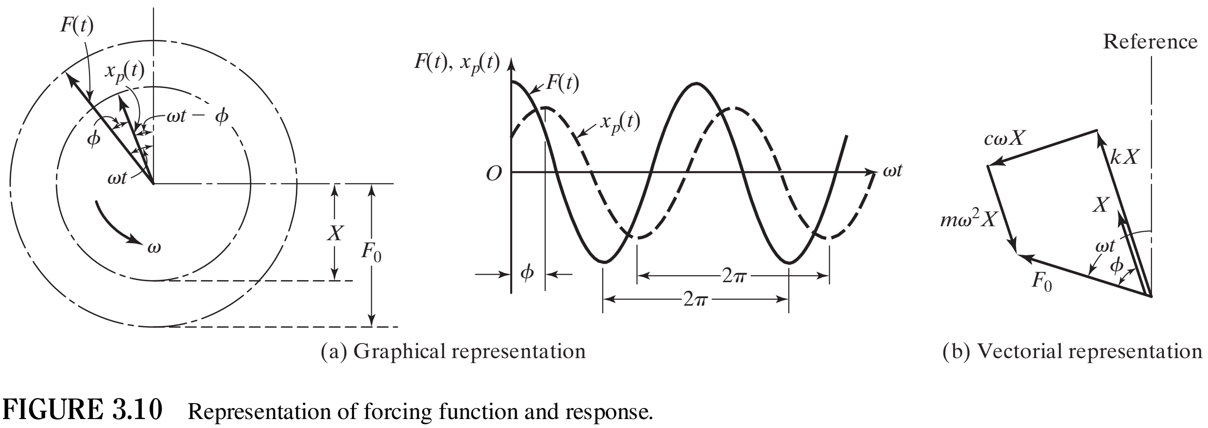
$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}}; \qquad \frac{c}{m} = 2\zeta\omega_n,$$

$$\delta_{st} = \frac{F_0}{k} = \text{deflection under the static force } F_0\text{, and}$$

$$r = \frac{\omega}{\omega_n} = \text{frequency ratio}$$

we obtain

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{3.30}$$



(a) Graphical representation

(b) Vectorial representation

FIGURE 3.10 Representation of forcing function and response.

and

$$\phi = \tan^{-1}\left\{\frac{2\zeta\dfrac{\omega}{\omega_n}}{1-\left(\dfrac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \tag{3.31}$$

As stated in Section 3.3, the quantity $M = X/\delta_{st}$ is called the *magnification factor, amplification factor*, or *amplitude ratio*. The variations of X/δ_{st} and ϕ with the frequency ratio r and the damping ratio ζ are shown in Fig. 3.11.

The following characteristics of the magnification factor (M) can be noted from Eq. (3.30) and Fig. 3.11(a):

1. For an undamped system ($\zeta = 0$), Eq. (3.30) reduces to Eq. (3.10), and $M \to \infty$ as $r \to 1$.
2. Any amount of damping ($\zeta > 0$) reduces the magnification factor (M) for all values of the forcing frequency.
3. For any specified value of r, a higher value of damping reduces the value of M.
4. In the degenerate case of a constant force (when $r = 0$), the value of $M = 1$.
5. The reduction in M in the presence of damping is very significant at or near resonance.
6. The amplitude of forced vibration becomes smaller with increasing values of the forcing frequency (that is, $M \to 0$ as $r \to \infty$).

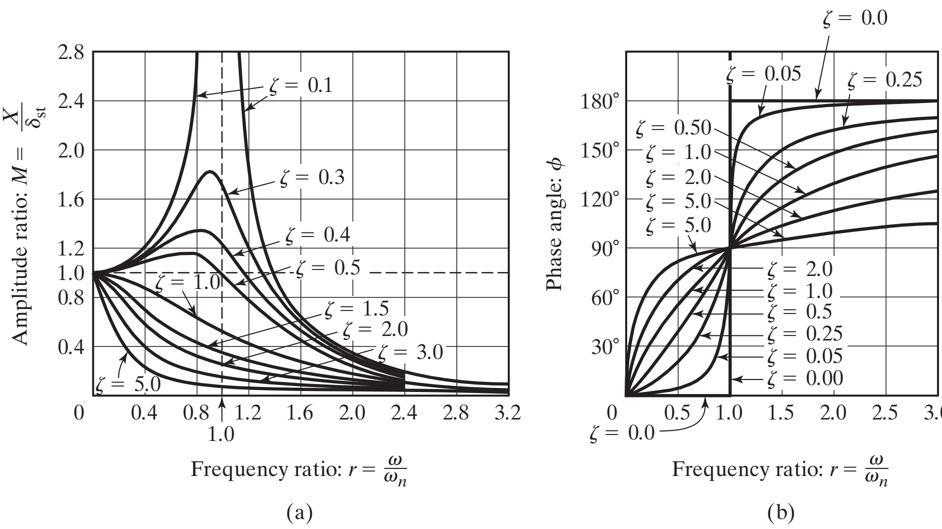


FIGURE 3.11 Variation of X and ϕ with frequency ratio r.

7. For $0 < \zeta < \frac{1}{\sqrt{2}}$, the maximum value of M occurs when (see Problem 3.32)

$$r = \sqrt{1 - 2\zeta^2} \qquad \text{or} \qquad \omega = \omega_n \sqrt{1 - 2\zeta^2} \tag{3.32}$$

which can be seen to be lower than the undamped natural frequency ω_n and the damped natural frequency $\omega_d = \omega_n \sqrt{1 - \zeta^2}$.

8. The maximum value of X (when $r = \sqrt{1 - 2\zeta^2}$) is given by

$$\left(\frac{X}{\delta_{st}}\right)_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{3.33}$$

and the value of X at $\omega = \omega_n$ by

$$\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} \tag{3.34}$$

Equation (3.33) can be used for the experimental determination of the measure of damping present in the system. In a vibration test, if the maximum amplitude of the response $(X)_{\max}$ is measured, the damping ratio of the system can be found using Eq. (3.33). Conversely, if the amount of damping is known, one can make an estimate of the maximum amplitude of vibration.

9. For $\zeta = \frac{1}{\sqrt{2}}$, $\frac{dM}{dr} = 0$ when $r = 0$. For $\zeta > \frac{1}{\sqrt{2}}$, the graph of M monotonically decreases with increasing values of r.

The following characteristics of the phase angle can be observed from Eq. (3.31) and Fig. 3.11(b):

1. For an undamped system $(\zeta = 0)$, Eq. (3.31) shows that the phase angle is 0 for $0 < r < 1$ and 180° for $r > 1$. This implies that the excitation and response are in phase for $0 < r < 1$ and out of phase for $r > 1$ when $\zeta = 0$.
2. For $\zeta > 0$ and $0 < r < 1$, the phase angle is given by $0 < \phi < 90°$, implying that the response lags the excitation.
3. For $\zeta > 0$ and $r > 1$, the phase angle is given by $90° < \phi < 180°$, implying that the response leads the excitation.
4. For $\zeta > 0$ and $r = 1$, the phase angle is given by $\phi = 90°$, implying that the phase difference between the excitation and the response is 90°.
5. For $\zeta > 0$ and large values of r, the phase angle approaches 180°, implying that the response and the excitation are out of phase.

3.4.1 Total Response

The complete solution is given by $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ is given by Eq. (2.70). Thus, for an underdamped system, we have

$$x(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi_0) + X\cos(\omega t - \phi) \tag{3.35}$$

$$\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$$

X and ϕ are given by Eqs. (3.30) and (3.31), respectively, and X_0 and ϕ_0 [different from those of Eq. (2.70)] can be determined from the initial conditions. For the initial conditions, $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, Eq. (3.35) yields

$$\begin{aligned} x_0 &= X_0 \cos \phi_0 + X \cos \phi \\ \dot{x}_0 &= -\zeta\omega_n X_0 \cos \phi_0 + \omega_d X_0 \sin \phi_0 + \omega X \sin \phi \end{aligned} \tag{3.36}$$

The solution of Eq. (3.36) gives X_0 and ϕ_0 as

$$\left.\begin{aligned} X_0 &= \left[(x_0 - X\cos\phi)^2 + \frac{1}{\omega_d^2}(\zeta\omega_n x_0 + \dot{x}_0 - \zeta\omega_n X \cos\phi - \omega X \sin\phi)^2\right]^{\frac{1}{2}} \\ \tan\phi_0 &= \frac{\zeta\omega_n x_0 + \dot{x}_0 - \zeta\omega_n X\cos\phi - \omega X\sin\phi}{\omega_d(x_0 - X\cos\phi)} \end{aligned}\right\} \tag{3.37}$$

EXAMPLE 3.3 Total Response of a System

Find the total response of a single-degree-of-freedom system with $m = 10$ kg, $c = 20$ N-s/m, $k = 4000$ N/m, $x_0 = 0.01$ m, and $\dot{x}_0 = 0$ under the following conditions:

a. An external force $F(t) = F_0 \cos \omega t$ acts on the system with $F_0 = 100$ N and $\omega = 10$ rad/s.
b. Free vibration with $F(t) = 0$.

Solution:

a. From the given data, we obtain

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$\delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025 \text{ m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{(4000)(10)}} = 0.05$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.05)^2}(20) = 19.974984 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.025}{\left[(1-0.05^2)^2 + (2\cdot 0.5\cdot 0.5)^2\right]^{1/2}} = 0.03326 \text{ m} \tag{E.1}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2\cdot 0.05\cdot 0.5}{1-0.5^2}\right) = 3.814075^\circ \tag{E.2}$$

Using the initial conditions, $x_0 = 0.01$ and $\dot{x}_0 = 0$, Eq. (3.36) yields:

$$0.01 = X_0\cos\phi_0 + (0.03326)(0.997785)$$

or

$$X_0 \cos \phi_0 = -0.023186 \tag{E.3}$$

$$0 = -(0.05)(20)\, X_0 \cos \phi_0 + X_0(19.974984) \sin \phi_0 + (0.03326)(10) \sin(3.814075°) \tag{E.4}$$

Substituting the value of $X_0 \cos \phi_0$ from Eq. (E.3) into (E.4), we obtain

$$X_0 \sin \phi_0 = -0.002268 \tag{E.5}$$

Solution of Eqs. (E.3) and (E.5) yields

$$X_0 = \left[(X_0 \cos \phi_0)^2 + (X_0 \sin \phi_0)^2\right]^{1/2} = 0.023297 \tag{E.6}$$

and

$$\tan \phi_0 = \frac{X_0 \sin \phi_0}{X_0 \cos \phi_0} = 0.0978176$$

or

$$\phi_0 = 5.586765° \tag{E.7}$$

b. For free vibration, the total response is given by

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) \tag{E.8}$$

Using the initial conditions $x(0) = x_0 = 0.01$ and $\dot{x}(0) = \dot{x}_0 = 0$, X_0 and ϕ_0 of Eq. (E.8) can be determined as (see Eqs. 2.73 and 2.75):

$$X_0 = \left[x_0^2 + \left(\frac{\zeta \omega_n x_0}{\omega_d}\right)^2\right]^{1/2} = \left[0.01^2 + \left(\frac{0.05 \cdot 20 \cdot 0.01}{19.974984}\right)^2\right]^{1/2} = 0.010012 \tag{E.9}$$

$$\phi_0 = \tan^{-1}\left(-\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d\, x_0}\right) = \tan^{-1}\left(-\frac{0.05 \cdot 20}{19.974984}\right) = -2.865984° \tag{E.10}$$

Note that the constants X_0 and ϕ_0 in cases (a) and (b) are very different.

■

3.4.2 Quality Factor and Bandwidth

For small values of damping ($\zeta < 0.05$), we can take

$$\left(\frac{X}{\delta_{st}}\right)_{max} \simeq \left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} = Q \tag{3.38}$$

The value of the amplitude ratio at resonance is also called *Q factor* or *quality factor* of the system, in analogy with some electrical-engineering applications, such as the tuning circuit of a radio, where the interest lies in an amplitude at resonance that is as large as possible [3.2]. The points R_1 and R_2, where the amplification factor falls to $Q/\sqrt{2}$, are called *half power points* because the power absorbed (ΔW) by the damper (or by the resistor in an electrical circuit), responding harmonically at a given frequency, is proportional to the square of the amplitude (see Eq. (2.94)):

$$\Delta W = \pi c \omega X^2 \tag{3.39}$$

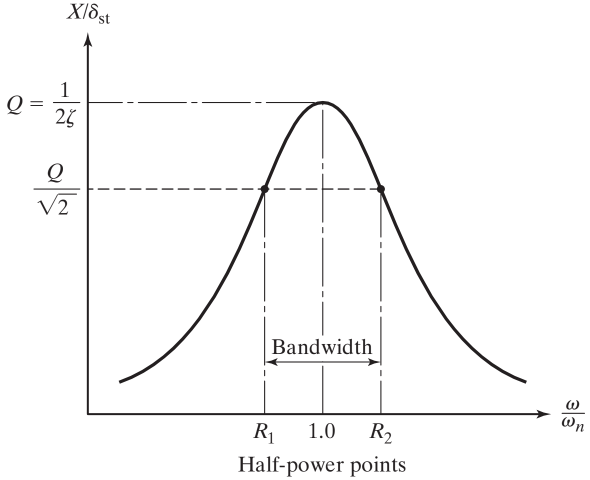


FIGURE 3.12 Harmonic-response curve showing half-power points and bandwidth.

The difference between the frequencies associated with the half-power points R_1 and R_2 is called the *bandwidth* of the system (see Fig. 3.12). To find the values of R_1 and R_2, we set $X/\delta_{st} = Q/\sqrt{2}$ in Eq. (3.30) so that

$$\frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{Q}{\sqrt{2}} = \frac{1}{2\sqrt{2}\zeta}$$

or

$$r^4 - r^2(2 - 4\zeta^2) + (1 - 8\zeta^2) = 0 \tag{3.40}$$

The solution of Eq. (3.40) gives

$$r_1^2 = 1 - 2\zeta^2 - 2\zeta\sqrt{1 + \zeta^2}, \qquad r_2^2 = 1 - 2\zeta^2 + 2\zeta\sqrt{1 + \zeta^2} \tag{3.41}$$

For small values of ζ, Eq. (3.41) can be approximated as

$$r_1^2 = R_1^2 = \left(\frac{\omega_1}{\omega_n}\right)^2 \simeq 1 - 2\zeta, \qquad r_2^2 = R_2^2 = \left(\frac{\omega_2}{\omega_n}\right)^2 \simeq 1 + 2\zeta \tag{3.42}$$

where $\omega_1 = \omega|_{R_1}$ and $\omega_2 = \omega|_{R_2}$. From Eq. (3.42),

$$\omega_2^2 - \omega_1^2 = (\omega_2 + \omega_1)(\omega_2 - \omega_1) = (R_2^2 - R_1^2)\omega_n^2 \simeq 4\zeta\omega_n^2 \tag{3.43}$$

Using the relation

$$\omega_2 + \omega_1 = 2\omega_n \tag{3.44}$$

in Eq. (3.43), we find that the bandwidth $\Delta\omega$ is given by

$$\Delta\omega = \omega_2 - \omega_1 \simeq 2\zeta\omega_n \tag{3.45}$$

Combining Eqs. (3.38) and (3.45), we obtain

$$Q \simeq \frac{1}{2\zeta} \simeq \frac{\omega_n}{\omega_2 - \omega_1} \tag{3.46}$$

It can be seen that the quality factor Q can be used for estimating the equivalent viscous damping in a mechanical system.[2]

3.5 Response of a Damped System Under $F(t) = F_0 e^{i\omega t}$

Let the harmonic forcing function be represented in complex form as $F(t) = F_0 e^{i\omega t}$ so that the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} \tag{3.47}$$

Since the actual excitation is given only by the real part of $F(t)$, the response will also be given only by the real part of $x(t)$, where $x(t)$ is a complex quantity satisfying the differential equation (3.47). F_0 in Eq. (3.47) is, in general, a complex number. By assuming the particular solution $x_p(t)$

$$x_p(t) = X e^{i\omega t} \tag{3.48}$$

we obtain, by substituting Eq. (3.48) into Eq. (3.47),[3]

$$X = \frac{F_0}{(k - m\omega^2) + ic\omega} \tag{3.49}$$

Multiplying the numerator and denominator on the right side of Eq. (3.49) by $[(k - m\omega^2) - ic\omega]$ and separating the real and imaginary parts, we obtain

$$X = F_0 \left[\frac{k - m\omega^2}{(k - m\omega^2)^2 + c^2\omega^2} - i \frac{c\omega}{(k - m\omega^2)^2 + c^2\omega^2} \right] \tag{3.50}$$

Using the relation $x + iy = Ae^{i\phi}$, where $A = \sqrt{x^2 + y^2}$ and $\tan\phi = y/x$, Eq. (3.50) can be expressed as

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}} e^{-i\phi} \tag{3.51}$$

---

[2]The determination of the system parameters (m, c, and k) based on half-power points and other response characteristics of the system is considered in Section 10.8.

[3]Equation (3.49) can be written as $Z(i\omega)X = F_0$, where $Z(i\omega) = -m\omega^2 + i\omega c + k$ is called the *mechanical impedance* of the system [3.8].

where

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \tag{3.52}$$

Thus the steady-state solution, Eq. (3.48), becomes

$$x_p(t) = \frac{F_0}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}} e^{i(\omega t - \phi)} \tag{3.53}$$

Frequency Response. Equation (3.49) can be rewritten in the form

$$\frac{kX}{F_0} = \frac{1}{1 - r^2 + i2\zeta r} \equiv H(i\omega) \tag{3.54}$$

where $H(i\omega)$ is known as the *complex frequency response* of the system. The absolute value of $H(i\omega)$ given by

$$|H(i\omega)| = \left|\frac{kX}{F_0}\right| = \frac{1}{[(1 - r^2)^2 + (2\zeta r)^2]^{1/2}} \tag{3.55}$$

denotes the magnification factor defined in Eq. (3.30). Recalling that $e^{i\phi} = \cos\phi + i\sin\phi$, we can show that Eqs. (3.54) and (3.55) are related:

$$H(i\omega) = |H(i\omega)| e^{-i\phi} \tag{3.56}$$

where ϕ is given by Eq. (3.52), which can also be expressed as

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{3.57}$$

Thus Eq. (3.53) can be expressed as

$$x_p(t) = \frac{F_0}{k} |H(i\omega)| e^{i(\omega t - \phi)} \tag{3.58}$$

It can be seen that the complex frequency-response function, $H(i\omega)$, contains both the magnitude and phase of the steady-state response. The use of this function in the experimental determination of the system parameters (m, c, and k) is discussed in Section 10.8. If $F(t) = F_0 \cos\omega t$, the corresponding steady-state solution is given by the real part of Eq. (3.53):

$$\begin{aligned} x_p(t) &= \frac{F_0}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}} \cos(\omega t - \phi) \\ &= \mathrm{Re}\left[\frac{F_0}{k} H(i\omega) e^{i\omega t}\right] = \mathrm{Re}\left[\frac{F_0}{k} |H(i\omega)| e^{i(\omega t - \phi)}\right] \end{aligned} \tag{3.59}$$

which can be seen to be the same as Eq. (3.25). Similarly, if $F(t) = F_0 \sin \omega t$, the corresponding steady-state solution is given by the imaginary part of Eq. (3.53):

$$\begin{aligned} x_p(t) &= \frac{F_0}{\left[(k - m\omega^2)^2 + (c\omega)^2\right]^{1/2}} \sin(\omega t - \phi) \\ &= \operatorname{Im}\left[\frac{F_0}{k} |H(i\omega)| e^{i(\omega t - \phi)}\right] \end{aligned} \tag{3.60}$$

Complex Vector Representation of Harmonic Motion. The harmonic excitation and the response of the damped system to that excitation can be represented graphically in the complex plane, and an interesting interpretation can be given to the resulting diagram. We first differentiate Eq. (3.58) with respect to time and obtain

$$\begin{aligned} \text{Velocity} &= \dot{x}_p(t) = i\omega \frac{F_0}{k} |H(i\omega)| e^{i(\omega t - \phi)} = i\omega x_p(t) \\ \text{Acceleration} &= \ddot{x}_p(t) = (i\omega)^2 \frac{F_0}{k} |H(i\omega)| e^{i(\omega t - \phi)} = -\omega^2 x_p(t) \end{aligned} \tag{3.61}$$

Because i can be expressed as

$$i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = e^{i\frac{\pi}{2}} \tag{3.62}$$

we can conclude that the velocity leads the displacement by the phase angle $\pi/2$ and that it is multiplied by ω. Similarly, -1 can be written as

$$-1 = \cos \pi + i \sin \pi = e^{i\pi} \tag{3.63}$$

Hence the acceleration leads the displacement by the phase angle π, and it is multiplied by ω^2.

Thus the various terms of the equation of motion (3.47) can be represented in the complex plane, as shown in Fig. 3.13. The interpretation of this figure is that the sum of the complex vectors $m\ddot{x}(t)$, $c\dot{x}(t)$, and $kx(t)$ balances $F(t)$, which is precisely what is required to satisfy Eq. (3.47). It is to also be noted that the entire diagram rotates with angular velocity ω in the complex plane. If only the real part of the response is to be considered, then the entire diagram must be projected onto the real axis. Similarly, if only the imaginary part of the response is to be considered, then the diagram must be projected onto the imaginary axis. In Fig. 3.13, notice that the force $F(t) = F_0 e^{i\omega t}$ is represented as a vector located at an angle ωt to the real axis. This implies that F_0 is real. If F_0 is also complex, then the force vector $F(t)$ will be located at an angle of $(\omega + \psi)$, where ψ is some phase angle introduced by F_0. In such a case, all the other vectors—namely, $m\ddot{x}$, $c\dot{x}$, and kx—will be shifted by the same angle ψ. This is equivalent to multiplying both sides of Eq. (3.47) by $e^{i\psi}$.

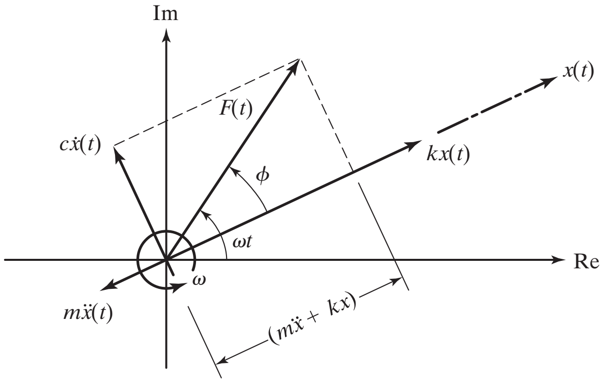


FIGURE 3.13 Representation of Eq. (3.47) in a complex plane.

3.6 Response of a Damped System Under the Harmonic Motion of the Base

Sometimes the base or support of a spring-mass-damper system undergoes harmonic motion, as shown in Fig. 3.14(a). Let $y(t)$ denote the displacement of the base and $x(t)$ the displacement of the mass from its static equilibrium position at time t. Then the net elongation of the spring is $x - y$ and the relative velocity between the two ends of the damper is $\dot{x} - \dot{y}$. From the free-body diagram shown in Fig. 3.14(b), we obtain the equation of motion:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \tag{3.64}$$

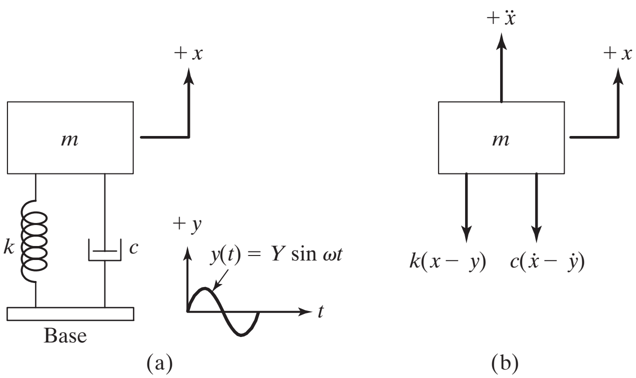


FIGURE 3.14 Base excitation.

If $y(t) = Y \sin \omega t$, Eq. (3.64) becomes

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = kY \sin \omega t + c\omega Y \cos \omega t$$
$$= A \sin(\omega t - \alpha) \tag{3.65}$$

where $A = Y\sqrt{k^2 + (c\omega)^2}$ and $\alpha = \tan^{-1}\left[-\frac{c\omega}{k}\right]$. This shows that giving excitation to the base is equivalent to applying a harmonic force of magnitude A to the mass. By using the solution indicated by Eq. (3.60), the steady-state response of the mass, $x_p(t)$, can be expressed as

$$x_p(t) = \frac{Y\sqrt{k^2 + (c\omega)^2}}{\left[(k - m\omega^2)^2 + (c\omega)^2\right]^{1/2}} \sin(\omega t - \phi_1 - \alpha) \tag{3.66}$$

where

$$\phi_1 = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

Using trigonometric identities, Eq. (3.66) can be rewritten in a more convenient form as

$$x_p(t) = X \sin(\omega t - \phi) \tag{3.67}$$

where X and ϕ are given by

$$\frac{X}{Y} = \left[\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}\right]^{1/2} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2} \tag{3.68}$$

and

$$\phi = \tan^{-1}\left[\frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2}\right] = \tan^{-1}\left[\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right] \tag{3.69}$$

The ratio of the amplitude of the response $x_p(t)$ to that of the base motion $y(t)$, $\frac{X}{Y}$, is called the *displacement transmissibility*.[4] The variations of $\frac{X}{Y} \equiv T_d$ and ϕ given by Eqs. (3.68) and (3.69) are shown in Figs. 3.15(a) and (b), respectively, for different values of r and ζ.

Note that if the harmonic excitation of the base is expressed in complex form as $y(t) = \text{Re}(Ye^{i\omega t})$, the response of the system can be expressed, using the analysis of Section 3.5, as

$$x_p(t) = \text{Re}\left\{\left(\frac{1 + i2\zeta r}{1 - r^2 + i2\zeta r}\right)Ye^{i\omega t}\right\} \tag{3.70}$$

and the displacement transmissibility as

$$\frac{X}{Y} = T_d = \left[1 + (2\zeta r)^2\right]^{1/2} |H(i\omega)| \tag{3.71}$$

where $|H(i\omega)|$ is given by Eq. (3.55).

[4]The expression for the displacement transmissibility can also be derived using the transfer-function approach described in Section 3.14.

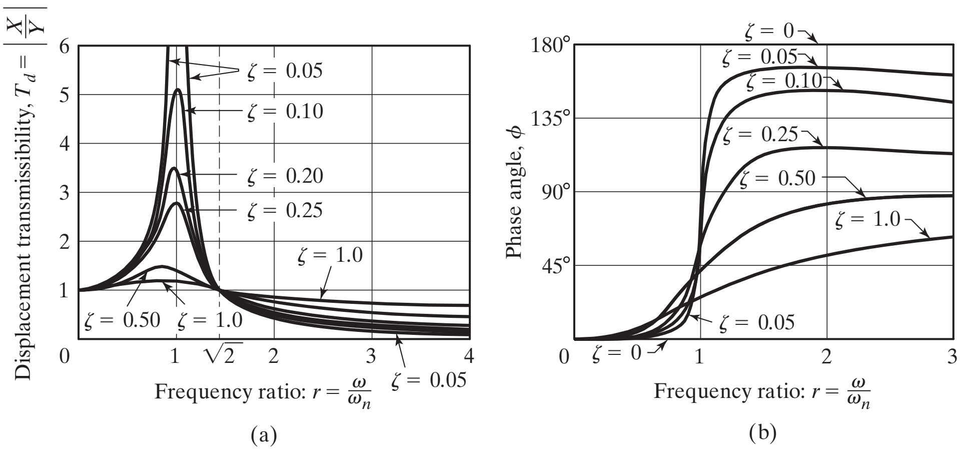


FIGURE 3.15 Variations of T_d and ϕ with r.

The following aspects of *displacement transmissibility,* $T_d = \frac{X}{Y}$, can be noted from Fig. 3.15(a):

1. The value of T_d is unity at $r = 0$ and close to unity for small values of r.
2. For an undamped system $(\zeta = 0)$, $T_d \rightarrow \infty$ at resonance $(r = 1)$.
3. The value of T_d is less than unity $(T_d < 1)$ for values of $r > \sqrt{2}$ (for any amount of damping ζ).
4. The value of T_d is unity for all values of ζ at $r = \sqrt{2}$.
5. For $r < \sqrt{2}$, smaller damping ratios lead to larger values of T_d. On the other hand, for $r > \sqrt{2}$, smaller values of damping ratio lead to smaller values of T_d.
6. The displacement transmissibility, T_d, attains a maximum for $0 < \zeta < 1$ at the frequency ratio $r = r_m < 1$ given by (see Problem 3.60):

$$r_m = \frac{1}{2\zeta}\left[\sqrt{1 + 8\zeta^2} - 1\right]^{1/2}$$

3.6.1 Force Transmitted

In Fig. 3.14, a force, F, is transmitted to the base or support due to the reactions from the spring and the dashpot. This force can be determined as

$$F = k(x - y) + c(\dot{x} - \dot{y}) = -m\ddot{x} \tag{3.72}$$

From Eq. (3.67), Eq. (3.72) can be written as

$$F = m\omega^2 X \sin(\omega t - \phi) = F_T \sin(\omega t - \phi) \tag{3.73}$$

where F_T is the amplitude or maximum value of the force transmitted to the base given by

$$\frac{F_T}{kY} = r^2\left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2} \tag{3.74}$$

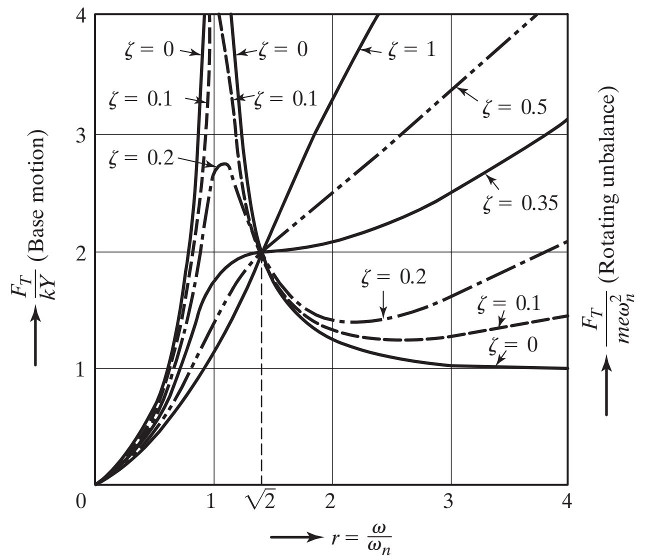


FIGURE 3.16 Force transmissibility.

The ratio (F_T/kY) is known as the *force transmissibility*.[5] Note that the transmitted force is in phase with the motion of the mass $x(t)$. The variation of the force transmitted to the base with the frequency ratio r is shown in Fig. 3.16 for different values of ζ.

3.6.2 Relative Motion

If $z = x - y$ denotes the motion of the mass relative to the base, the equation of motion, Eq. (3.64), can be rewritten as

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y \sin \omega t \tag{3.75}$$

The steady-state solution of Eq. (3.75) is given by

$$z(t) = \frac{m\omega^2 Y \sin(\omega t - \phi_1)}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}} = Z \sin(\omega t - \phi_1) \tag{3.76}$$

where Z, the amplitude of $z(t)$, can be expressed as

$$Z = \frac{m\omega^2 Y}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = Y \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{3.77}$$

[5]The use of the concept of transmissibility in the design of vibration isolation systems is discussed in Chapter 9. The expression for the force transmissibility can also be derived using the transfer-function approach described in Section 3.14.

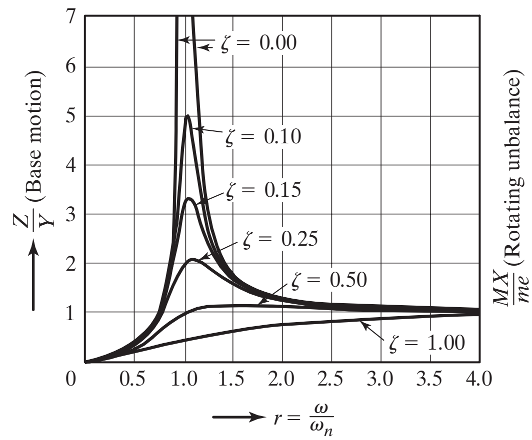


FIGURE 3.17 Variation of (Z/Y) or (MX/me) with frequency ratio $r = (\omega/\omega_n)$.

and ϕ_1 by

$$\phi_1 = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

The ratio Z/X is shown graphically in Fig. 3.17. The variation of ϕ_1 is same as that of ϕ shown in Fig. 3.11(b).

EXAMPLE 3.4 Vehicle Moving on a Rough Road

Figure 3.18 shows a simple model of a motor vehicle that can vibrate in the vertical direction while traveling over a rough road. The vehicle has a mass of 1200 kg. The suspension system has a spring constant of 400 kN/m and a damping ratio of $\zeta = 0.5$. If the vehicle speed is 20 km/hr, determine the displacement amplitude of the vehicle. The road surface varies sinusoidally with an amplitude of $Y = 0.05$ m and a wavelength of 6 m.

Solution: The frequency ω of the base excitation can be found by dividing the vehicle speed v km/hr by the length of one cycle of road roughness:

$$\omega = 2\pi f = 2\pi\left(\frac{v \times 1000}{3600}\right)\frac{1}{6} = 0.290889v \text{ rad/s}$$

For $v = 20$ km/hr, $\omega = 5.81778$ rad/s. The natural frequency of the vehicle is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{400 \times 10^3}{1200}\right)^{1/2} = 18.2574 \text{ rad/s}$$

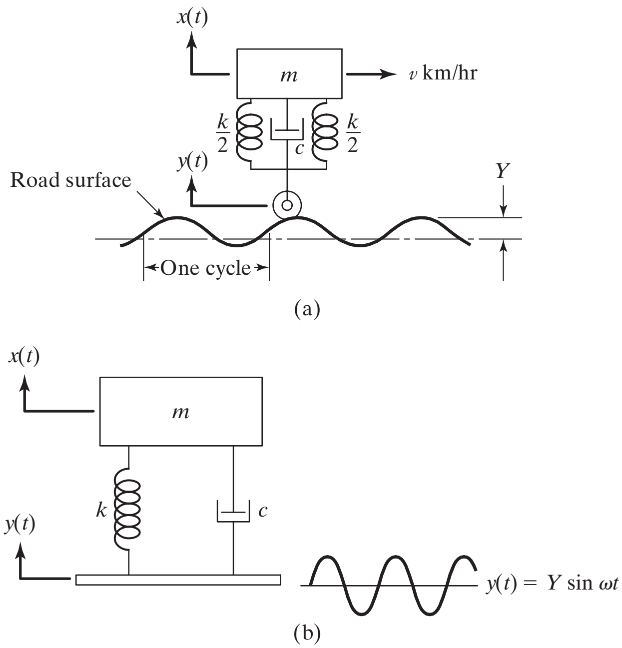


FIGURE 3.18 Vehicle moving over a rough road.

and hence the frequency ratio r is

$$r = \frac{\omega}{\omega_n} = \frac{5.81778}{18.2574} = 0.318653$$

The amplitude ratio can be found from Eq. (3.68):

$$\frac{X}{Y} = \left\{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right\}^{1/2} = \left\{\frac{1 + (2 \times 0.5 \times 0.318653)^2}{(1 - 0.318653)^2 + (2 \times 0.5 \times 0.318653)^2}\right\}^{1/2}$$

$$= 1.100964$$

Thus the displacement amplitude of the vehicle is given by

$$X = 1.100964Y = 1.100964(0.05) = 0.055048 \text{ m}$$

This indicates that a 5-cm bump in the road is transmitted as a 5.5-cm bump to the chassis and the passengers of the car. Thus in the present case the passengers feel an amplified motion (see Problem 3.107 for other situations).

■

EXAMPLE 3.5

Machine on Resilient Foundation

A heavy machine, weighing 3000 N, is supported on a resilient foundation. The static deflection of the foundation due to the weight of the machine is found to be 7.5 cm. It is observed that the machine vibrates with an amplitude of 1 cm when the base of the foundation is subjected to harmonic oscillation at the undamped natural frequency of the system with an amplitude of 0.25 cm. Find

a. the damping constant of the foundation,
b. the dynamic force amplitude on the base, and
c. the amplitude of the displacement of the machine relative to the base.

Solution:

a. The stiffness of the foundation can be found from its static deflection: k = weight of machine$/\delta_{st} = 3000/0.075 = 40{,}000$ N/m.

At resonance $(\omega = \omega_n$ or $r = 1)$, Eq. (3.68) gives

$$\frac{X}{Y} = \frac{0.010}{0.0025} = 4 = \left[\frac{1 + (2\zeta)^2}{(2\zeta)^2}\right]^{1/2} \tag{E.1}$$

The solution of Eq. (E.1) gives $\zeta = 0.1291$. The damping constant is given by

$$\begin{aligned} c = \zeta \cdot c_c = \zeta 2\sqrt{km} &= 0.1291 \times 2 \times \sqrt{40{,}000 \times (3000/9.81)} \\ &= 903.0512 \text{ N-s/m} \end{aligned} \tag{E.2}$$

b. The dynamic force amplitude on the base at $r = 1$ can be found from Eq. (3.74):

$$F_T = Yk\left[\frac{1 + 4\zeta^2}{4\zeta^2}\right]^{1/2} = kX = 40{,}000 \times 0.01 = 400 \text{ N} \tag{E.3}$$

c. The amplitude of the relative displacement of the machine at $r = 1$ can be obtained from Eq. (3.77):

$$Z = \frac{Y}{2\zeta} = \frac{0.0025}{2 \times 0.1291} = 0.00968 \text{ m} \tag{E.4}$$

It can be noticed that $X = 0.01$ m, $Y = 0.0025$ m, and $Z = 0.00968$ m; therefore, $Z \neq X - Y$. This is due to the phase differences between x, y, and z.

■

3.7 Response of a Damped System Under Rotating Unbalance

Unbalance in rotating machinery is one of the main causes of vibration. A simplified model of such a machine is shown in Fig. 3.19. The total mass of the machine is M, and there are two eccentric masses $m/2$ rotating in opposite directions with a constant angular velocity ω. The centrifugal force $(me\omega^2)/2$ due to each mass will cause excitation of the mass M. We consider two equal masses $m/2$ rotating in opposite directions in order to have the horizontal components of excitation of the two masses cancel each other. However, the vertical components of excitation add together and act along the axis of symmetry A–A in

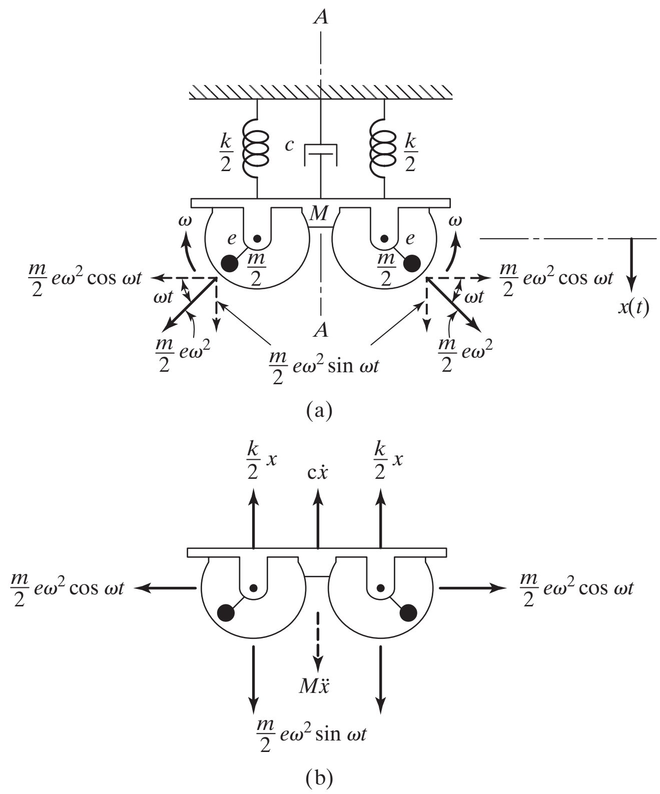


FIGURE 3.19 Rotating unbalanced masses.

Fig. 3.19. If the angular position of the masses is measured from a horizontal position, the total vertical component of the excitation is always given by $F(t) = me\omega^2 \sin \omega t$. The equation of motion can be derived by the usual procedure:

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t \tag{3.78}$$

The solution of this equation will be identical to Eq. (3.60) if we replace m and F_0 by M and $me\omega^2$, respectively. This solution can also be expressed as

$$x_p(t) = X \sin(\omega t - \phi) = \text{Im}\left[\frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2 |H(i\omega)| e^{i(\omega t - \phi)}\right] \tag{3.79}$$

where $\omega_n = \sqrt{k/M}$ and X and ϕ denote the amplitude and the phase angle of vibration given by

$$X = \frac{me\omega^2}{\left[(k - M\omega^2)^2 + (c\omega)^2\right]^{1/2}} = \frac{me}{M}\left(\frac{\omega}{\omega_n}\right)^2 |H(i\omega)|$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - M\omega^2}\right) \tag{3.80}$$

By defining $\zeta = c/c_c$ and $c_c = 2M\omega_n$, Eqs. (3.80) can be rewritten as

$$\frac{MX}{me} = \frac{r^2}{\left[(1 - r^2)^2 + (2\zeta r)^2\right]^{1/2}} = r^2 |H(i\omega)|$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{3.81}$$

The variation of MX/me with r for different values of ζ is shown in Fig. 3.17. On the other hand, the graph of ϕ versus r remains as in Fig. 3.11(b). The following observations can be made from Eq. (3.81) and Fig. 3.17:

1. All the curves begin at zero amplitude. The amplitude near resonance ($\omega = \omega_n$) is markedly affected by damping. Thus if the machine is to be run near resonance, damping should be introduced purposefully to avoid dangerous amplitudes.
2. At very high speeds (ω large), MX/me is almost unity, and the effect of damping is negligible.
3. For $0 < \zeta < \frac{1}{\sqrt{2}}$, the maximum of $\frac{MX}{me}$ occurs when

$$\frac{d}{dr}\left(\frac{MX}{me}\right) = 0 \tag{3.82}$$

The solution of Eq. (3.82) gives

$$r = \frac{1}{\sqrt{1 - 2\zeta^2}} > 1$$

with the corresponding maximum value of $\frac{MX}{me}$ given by

$$\left(\frac{MX}{me}\right)_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{3.83}$$

Thus the peaks occur to the right of the resonance value of $r = 1$.

4. For $\zeta > \frac{1}{\sqrt{2}}$, $\left[\frac{MX}{me}\right]$ does not attain a maximum. Its value grows from 0 at $r = 0$ to 1 at $r \to \infty$.

5. The force transmitted to the foundation due to rotating unbalanced force (F) can be found as $F(t) = kx(t) + c\dot{x}(t)$. The magnitude (or maximum value) of F can be derived as (see Problem 3.73):

$$|F| = me\omega^2 \left[\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2} \right]^{\frac{1}{2}} \tag{3.84}$$

EXAMPLE 3.6 Deflection of an Electric Motor due to Rotating Unbalance

An electric motor of mass M, mounted on an elastic foundation, is found to vibrate with a deflection of 0.15 m at resonance (Fig. 3.20). It is known that the unbalanced mass of the motor is 8% of the mass of the rotor due to manufacturing tolerances used, and the damping ratio of the foundation is $\zeta = 0.025$. Determine the following:

a. the eccentricity or radial location of the unbalanced mass (e),
b. the peak deflection of the motor when the frequency ratio varies from resonance, and
c. the additional mass to be added uniformly to the motor if the deflection of the motor at resonance is to be reduced to 0.1 m.

Assume that the eccentric mass remains unaltered when the additional mass is added to the motor.

a. From Eq. (3.81), the deflection at resonance ($r = 1$) is given by

$$\frac{MX}{me} = \frac{1}{2\zeta} = \frac{1}{2(0.025)} = 20$$

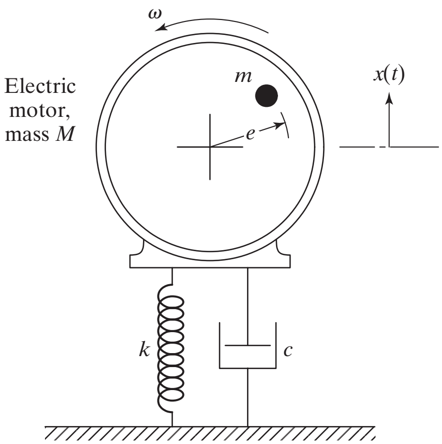


FIGURE 3.20

from which the eccentricity can be found as

$$e = \frac{MX}{20m} = \frac{M(0.15)}{20(0.08\ M)} = 0.09375 \text{ m}$$

b. The peak deflection of the motor is given by Eq. (3.83):

$$\left(\frac{MX}{me}\right)_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{2(0.025)\sqrt{1-0.025^2}} = 20.0063$$

from which the peak deflection can be determined as

$$X_{\max} = \frac{20.0063me}{M} = \frac{20.0063(0.08M)(0.09375)}{M} = 0.150047 \text{ m}$$

c. If the additional mass added to the motor is denoted as M_a, the corresponding deflection is given by Eq. (3.81):

$$\frac{(M + M_a)(0.1)}{(0.08M)(0.09375)} = 20$$

which yields $M_a = 0.5M$. Thus the mass of the motor is to be increased by 50% in order to reduce the deflection at resonance from 0.15 m to 0.10 m.

■

EXAMPLE 3.7

Francis Water Turbine

Figure 3.21 is a schematic diagram of a Francis water turbine, in which water flows from *A* into the blades *B* and down into the tail race *C*. The rotor has a mass of 250 kg and an unbalance (me) of 5 kg-mm. The radial clearance between the rotor and the stator is 5 mm. The turbine operates in the speed range 600 to 6000 rpm. The steel shaft carrying the rotor can be assumed to be clamped at the bearings. Determine the diameter of the shaft so that the rotor is always clear of the stator at all the operating speeds of the turbine. Assume damping to be negligible.

Solution: The maximum amplitude of the shaft (rotor) due to rotating unbalance can be obtained from Eq. (3.80) by setting $c = 0$ as

$$X = \frac{me\omega^2}{(k - M\omega^2)} = \frac{me\omega^2}{k(1 - r^2)} \tag{E.1}$$

where $me = 5$ kg-mm, $M = 250$ kg, and the limiting value of $X = 5$ mm. The value of ω ranges from

$$600 \text{ rpm} = 600 \times \frac{2\pi}{60} = 20\pi \text{ rad/s}$$

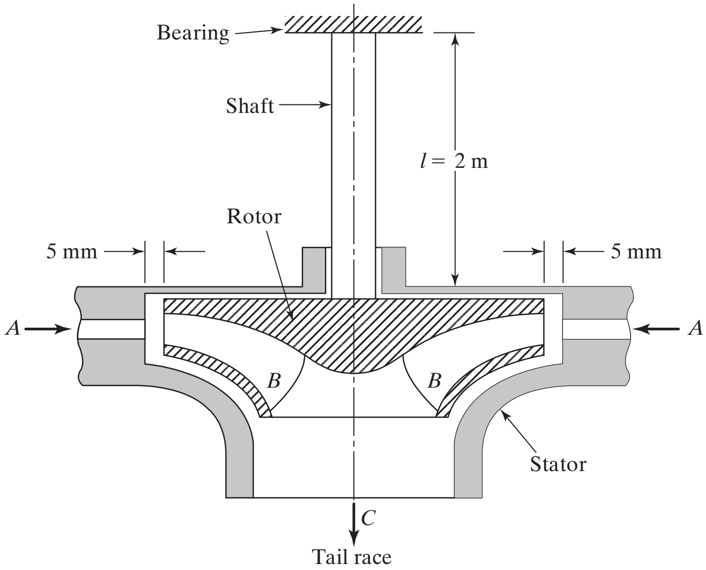


FIGURE 3.21 Francis water turbine.

to

$$6000 \text{ rpm} = 6000 \times \frac{2\pi}{60} = 200\pi \text{ rad/s}$$

while the natural frequency of the system is given by

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{250}} = 0.063245\sqrt{k} \text{ rad/s} \tag{E.2}$$

if k is in N/m. For $\omega = 20\pi$ rad/s, Eq. (E.1) gives

$$0.005 = \frac{(5.0 \times 10^{-3}) \times (20\pi)^2}{k\left[1 - \dfrac{(20\pi)^2}{0.004\,k}\right]} = \frac{2\pi^2}{k - 10^5\pi^2}$$

$$k = 10.04 \times 10^4 \pi^2 \text{ N/m} \tag{E.3}$$

For $\omega = 200\pi$ rad/s, Eq. (E.1) gives

$$0.005 = \frac{(5.0 \times 10^{-3}) \times (200\pi)^2}{k\left[1 - \dfrac{(200\pi)^2}{0.004k}\right]} = \frac{200\pi^2}{k - 10^7\pi^2}$$

$$k = 10.04 \times 10^6 \pi^2 \text{ N/m} \tag{E.4}$$

From Fig. 3.17, we find that the amplitude of vibration of the rotating shaft can be minimized by making $r = \omega/\omega_n$ very large. This means that ω_n must be made small compared to ω—that is, k must be made small. This can be achieved by selecting the value of k as $10.04 \times 10^4 \pi^2$ N/m. Since the stiffness of a cantilever beam (shaft) supporting a load (rotor) at the end is given by

$$k = \frac{3EI}{l^3} = \frac{3E}{l^3}\left(\frac{\pi d^4}{64}\right) \tag{E.5}$$

the diameter of the beam (shaft) can be found:

$$d^4 = \frac{64kl^3}{3\pi E} = \frac{(64)(10.04 \times 10^4 \pi^2)(2^3)}{3\pi(2.07 \times 10^{11})} = 2.6005 \times 10^{-4} \text{ m}^4$$

or

$$d = 0.1270 \text{ m} = 127 \text{ mm} \tag{E.6}$$

■

3.8 Forced Vibration with Coulomb Damping

For a single-degree-of-freedom system with Coulomb or dry-friction damping, subjected to a harmonic force $F(t) = F_0 \sin \omega t$ as in Fig. 3.22, the equation of motion is given by

$$m\ddot{x} + kx \pm \mu N = F(t) = F_0 \sin \omega t \tag{3.85}$$

where the sign of the friction force $(\mu N = \mu mg)$ is positive (negative) when the mass moves from left to right (right to left). The exact solution of Eq. (3.85) is quite involved. However, we can expect that if the dry-friction damping force is large, the motion of the mass will be discontinuous. On the other hand, if the dry-friction force is small compared to the amplitude of the applied force F_0, the steady-state solution is expected to be nearly harmonic. In this case, we can find an approximate solution of Eq. (3.85) by finding an equivalent viscous-damping ratio. To find such a ratio, we equate the energy dissipated due to dry friction to the energy dissipated by an equivalent viscous damper during a full cycle of motion. If the amplitude of motion is denoted as X, the energy dissipated by the friction force μN in a quarter cycle is μNX. Hence in a full cycle, the energy dissipated by dry-friction damping is given by

$$\Delta W = 4\mu NX \tag{3.86}$$

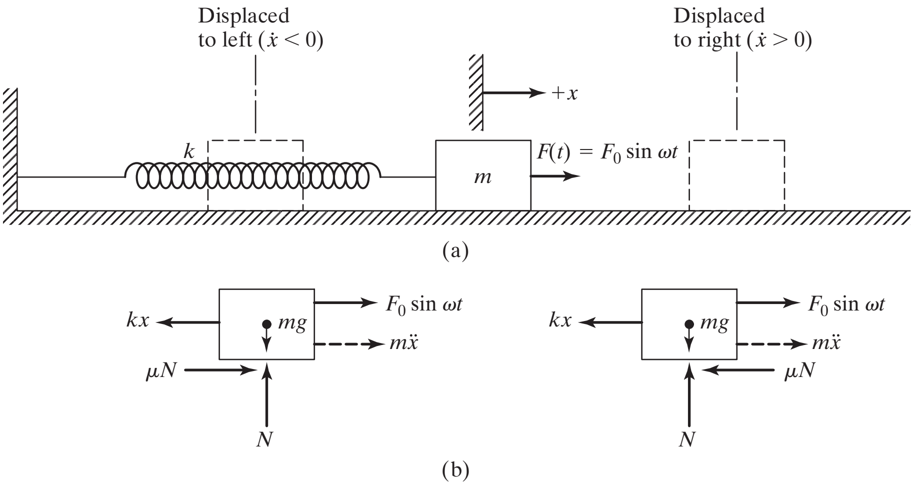


FIGURE 3.22 Single-degree-of-freedom system with Coulomb damping.

If the equivalent viscous-damping constant is denoted as c_{eq}, the energy dissipated during a full cycle (see Eq. (2.94)) will be

$$\Delta W = \pi c_{eq} \omega X^2 \tag{3.87}$$

By equating Eqs. (3.86) and (3.87), we obtain

$$c_{eq} = \frac{4\mu N}{\pi \omega X} \tag{3.88}$$

Thus the steady-state response is given by

$$x_p(t) = X \sin(\omega t - \phi) \tag{3.89}$$

where the amplitude X can be found from Eq. (3.60):

$$X = \frac{F_0}{\left[(k - m\omega^2)^2 + (c_{eq}\omega)^2\right]^{1/2}} = \frac{(F_0/k)}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta_{eq}\frac{\omega}{\omega_n}\right)^2\right]^{1/2}} \tag{3.90}$$

with

$$\zeta_{eq} = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{2m\omega_n} = \frac{4\mu N}{2m\omega_n \pi \omega X} = \frac{2\mu N}{\pi m \omega \omega_n X} \tag{3.91}$$

Substitution of Eq. (3.91) into Eq. (3.90) gives

$$X = \frac{(F_0/k)}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{4\mu N}{\pi k X}\right)^2\right]^{1/2}} \tag{3.92}$$

The solution of this equation gives the amplitude X as

$$X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}\right]^{1/2} \tag{3.93}$$

As stated earlier, Eq. (3.93) can be used only if the friction force is small compared to F_0. In fact, the limiting value of the friction force μN can be found from Eq. (3.93). To avoid imaginary values of X, we need to have

$$1 - \left(\frac{4\mu N}{\pi F_0}\right)^2 > 0 \qquad \text{or} \qquad \frac{F_0}{\mu N} > \frac{4}{\pi}$$

If this condition is not satisfied, the exact analysis, given in reference [3.3], is to be used. The phase angle ϕ appearing in Eq. (3.89) can be found using Eq. (3.52):

$$\phi = \tan^{-1}\left(\frac{c_{\text{eq}}\omega}{k - m\omega^2}\right) = \tan^{-1}\left[\frac{2\zeta_{\text{eq}}\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right] = \tan^{-1}\left\{\frac{\frac{4\mu N}{\pi k X}}{1 - \frac{\omega^2}{\omega_n^2}}\right\} \tag{3.94}$$

Substituting Eq. (3.93) into Eq. (3.94) for X, we obtain

$$\phi = \tan^{-1}\left[\frac{\frac{4\mu N}{\pi F_0}}{\left\{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2\right\}^{1/2}}\right] \tag{3.95}$$

Equation (3.94) shows that $\tan\phi$ is a constant for a given value of $F_0/\mu N$. ϕ is discontinuous at $\omega/\omega_n = 1$ (resonance), since it takes a positive value for $\omega/\omega_n < 1$ and a negative value for $\omega/\omega_n > 1$. Thus Eq. (3.95) can also be expressed as

$$\phi = \tan^{-1}\left[\frac{\pm\dfrac{4\mu N}{\pi F_0}}{\left\{1 - \left(\dfrac{4\mu N}{\pi F_0}\right)^2\right\}^{1/2}}\right] \tag{3.96}$$

Equation (3.93) shows that friction serves to limit the amplitude of forced vibration for $\omega/\omega_n \neq 1$. However, at resonance $(\omega/\omega_n = 1)$, the amplitude becomes infinite. This can be explained as follows. The energy directed into the system over one cycle when it is excited harmonically at resonance is

$$\Delta W' = \int_{\text{cycle}} F \cdot dx = \int_0^{\tau} F \frac{dx}{dt} dt$$

$$= \int_0^{\tau = 2\pi/\omega} F_0 \sin \omega t \cdot [\omega X \cos(\omega t - \phi)]\, dt \tag{3.97}$$

Since Eq. (3.94) gives $\phi = 90°$ at resonance, Eq. (3.97) becomes

$$\Delta W' = F_0 X \omega \int_0^{2\pi/\omega} \sin^2 \omega t\, dt = \pi F_0 X \tag{3.98}$$

The energy dissipated from the system is given by Eq. (3.86). Since $\pi F_0 X > 4\mu N X$ for X to be real-valued, $\Delta W' > \Delta W$ at resonance (see Fig. 3.23). Thus more energy is directed into the system per cycle than is dissipated per cycle. This extra energy is used to build up the amplitude of vibration. For the nonresonant condition $(\omega/\omega_n \neq 1)$, the energy input can be found from Eq. (3.97):

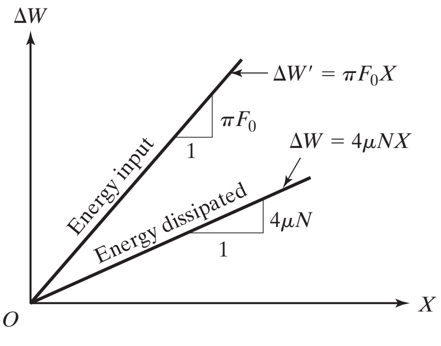


FIGURE 3.23 Energy input and energy dissipated with Coulomb damping.

$$\Delta W' = \omega F_0 X \int_0^{2\pi/\omega} \sin \omega t \cos(\omega t - \phi)\, dt = \pi F_0 X \sin \phi \tag{3.99}$$

Due to the presence of $\sin \phi$ in Eq. (3.99), the input energy curve in Fig. 3.23 is made to coincide with the dissipated energy curve, so the amplitude is limited. Thus the phase of the motion ϕ can be seen to limit the amplitude of the motion.

The periodic response of a spring-mass system with Coulomb damping subjected to base excitation is given in references [3.10, 3.11].

EXAMPLE 3.8 Spring-Mass System with Coulomb Damping

A spring-mass system, having a mass of 10 kg and a spring of stiffness of 4000 N/m, vibrates on a horizontal surface. The coefficient of friction is 0.12. When subjected to a harmonic force of frequency 2 Hz, the mass is found to vibrate with an amplitude of 40 mm. Find the amplitude of the harmonic force applied to the mass.

Solution: The vertical force (weight) of the mass is $N = mg = 10 \times 9.81 = 98.1$ N. The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

and the frequency ratio is

$$\frac{\omega}{\omega_n} = \frac{2 \times 2\pi}{20} = 0.6283$$

The amplitude of vibration X is given by Eq. (3.93):

$$X = \frac{F_0}{k}\left[\frac{1 - \left(\dfrac{4\mu N}{\pi F_0}\right)^2}{\left\{1 - \left(\dfrac{\omega}{\omega_n}\right)^2\right\}^2}\right]^{1/2}$$

$$0.04 = \frac{F_0}{4000}\left[\frac{1 - \left\{\dfrac{4(0.12)(98.1)}{\pi F_0}\right\}^2}{(1 - 0.6283^2)^2}\right]^{1/2}$$

The solution of this equation gives $F_0 = 97.9874$ N.

■

3.9 Forced Vibration with Hysteresis Damping

Consider a single-degree-of-freedom system with hysteresis damping and subjected to a harmonic force $F(t) = F_0 \sin \omega t$, as indicated in Fig. 3.24. The equation of motion of the mass can be derived, using Eq. (2.157), as

$$m\ddot{x} + \frac{\beta k}{\omega}\dot{x} + kx = F_0 \sin \omega t \tag{3.100}$$

where $(\beta k/\omega)\dot{x} = (h/\omega)\dot{x}$ denotes the damping force.[6] Although the solution of Eq. (3.100) is quite involved for a general forcing function $F(t)$, our interest is to find the response under a harmonic force.

The steady-state solution of Eq. (3.100) can be assumed:

$$x_p(t) = X \sin(\omega t - \phi) \tag{3.101}$$

By substituting Eq. (3.101) into Eq. (3.100), we obtain

$$X = \frac{F_0}{k\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \beta^2\right]^{1/2}} \tag{3.102}$$

and

$$\phi = \tan^{-1}\left[\frac{\beta}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}\right] \tag{3.103}$$

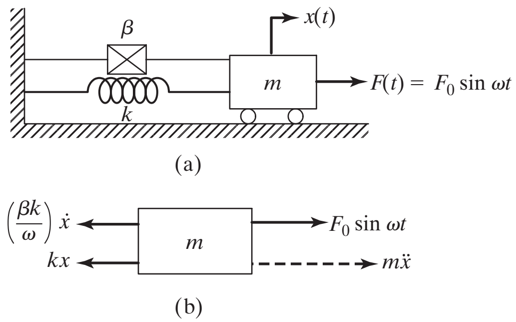


FIGURE 3.24 System with hysteresis damping.

---

[6]In contrast to viscous damping, the damping force here can be seen to be a function of the forcing frequency ω (see Section 2.10).

Equations (3.102) and (3.103) are shown plotted in Fig. 3.25 for several values of β. A comparison of Fig. 3.25 with Fig. 3.11 for viscous damping reveals the following:

1. The amplitude ratio

$$\frac{X}{(F_0/k)}$$

attains its maximum value of $F_0/k\beta$ at the resonant frequency ($\omega = \omega_n$) in the case of hysteresis damping, while it occurs at a frequency below resonance ($\omega < \omega_n$) in the case of viscous damping.

2. The phase angle ϕ has a value of $\tan^{-1}(\beta)$ at $\omega = 0$ in the case of hysteresis damping, while it has a value of zero at $\omega = 0$ in the case of viscous damping. This indicates that the response can never be in phase with the forcing function in the case of hysteresis damping.

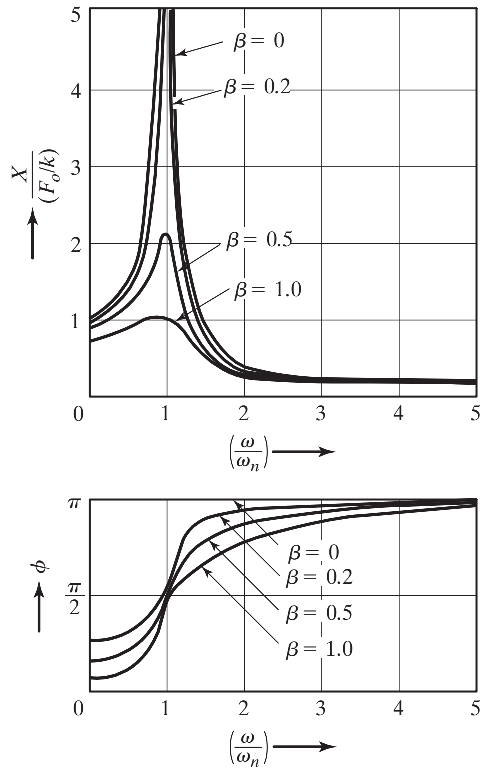


FIGURE 3.25 Steady-state response.

Note that if the harmonic excitation is assumed to be $F(t) = F_0 e^{i\omega t}$ in Fig. 3.24, the equation of motion becomes

$$m\ddot{x} + \frac{\beta k}{\omega}\dot{x} + kx = F_0 e^{i\omega t} \tag{3.104}$$

In this case, the response $x(t)$ is also a harmonic function involving the factor $e^{i\omega t}$. Hence $\dot{x}(t)$ is given by $i\omega x(t)$, and Eq. (3.104) becomes

$$m\ddot{x} + k(1 + i\beta)x = F_0 e^{i\omega t} \tag{3.105}$$

where the quantity $k(1 + i\beta)$ is called the *complex stiffness* or *complex damping* [3.7]. The steady-state solution of Eq. (3.105) is given by the real part of

$$x(t) = \frac{F_0 e^{i\omega t}}{k\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + i\beta\right]} \tag{3.106}$$

3.10 Forced Motion with Other Types of Damping

Viscous damping is the simplest form of damping to use in practice, since it leads to linear equations of motion. In the cases of Coulomb and hysteretic damping, we defined equivalent viscous-damping coefficients to simplify the analysis. Even for a more complex form of damping, we define an equivalent viscous-damping coefficient, as illustrated in the following examples. The practical use of equivalent damping is discussed in reference [3.12].

EXAMPLE 3.9 Quadratic Damping

Find the equivalent viscous-damping coefficient corresponding to *quadratic* or *velocity-squared damping* that is present when a body moves in a turbulent fluid flow.

Solution: The damping force is assumed to be

$$F_d = \pm a(\dot{x})^2 \tag{E.1}$$

where a is a constant, $\dot{x}$ is the relative velocity across the damper, and the negative (positive) sign must be used in Eq. (E.1) when $\dot{x}$ is positive (negative). The energy dissipated per cycle during harmonic motion $x(t) = X \sin \omega t$ is given by

$$\Delta W = 2\int_{-X}^{X} a(\dot{x})^2\, dx = 2X^3 \int_{-\pi/2}^{\pi/2} a\omega^2 \cos^3 \omega t\, d(\omega t) = \frac{8}{3}\omega^2 a X^3 \tag{E.2}$$

Jean Baptiste Joseph Fourier (1768–1830) was a French mathematician and a professor at the Ecole Polytechnique in Paris. His works on heat flow, published in 1822, and on trigonometric series are well known. The expansion of a periodic function in terms of harmonic functions has been named after him as the "Fourier series."
(Reproduced with permission from *Applied Mechanics Reviews.*)

CHAPTER 4

Vibration Under General Forcing Conditions

Chapter Outline

This chapter is devoted to the vibration response of a single-degree-of-freedom system under arbitrary forcing conditions. The response of the system under a general periodic force is presented by first expanding the periodic force into a series of harmonic forces using Fourier series and then superposing the responses due to the individual harmonic forces. The response of the system under a nonperiodic force is presented using two methods—those of convolution integral and of Laplace transform. The method of convolution or Duhamel integral makes use of the impulse response function of the system. The method is also used to find the response to base excitation. Several examples are presented to illustrate its use. The concept of response spectra corresponding to specific forcing functions and their use in finding the maximum response of the system is also outlined. The response spectrum corresponding to the base excitation, such as the one caused by an earthquake, is also considered. Typical earthquake response spectra and their use in finding the responses of building frames are illustrated. The concept of pseudo velocity and the associated pseudo spectrum are also defined. The design of mechanical systems under a shock environment is presented with an illustrative example. The Laplace transform method and its use in finding the response of both first- and second-order systems are presented. The responses under impulse, step, and ramp forcing functions are considered. Inelastic and elastic collision problems are considered as applications of impulse response computations. The analysis of the step response and the description of transient response in terms of peak time, rise time, maximum overshoot, settling time, and delay time are presented. The response of systems under irregular forcing conditions using numerical methods, including the fourth-order Runge-Kutta method, is presented with illustrative examples. Finally the use of MATLAB programs in finding the response of a system under arbitrary forcing functions is illustrated with examples.

Learning Objectives

After completing this chapter, you should be able to do the following:

- Find the responses of single-degree-of-freedom systems subjected to general periodic forces using Fourier series.
- Use the method of convolution or Duhamel integral to solve vibration problems of systems subjected to arbitrary forces.
- Find the response of systems subjected to earthquakes using response spectra.
- Solve undamped and damped systems subjected to arbitrary forces, including impulse, step, and ramp forces, using Laplace transform.
- Understand the characteristics of transient response, such as peak time, overshoot, settling time, rise time, and decay time, and procedures for their estimation.
- Apply numerical methods to solve vibration problems of systems subjected to forces that are described numerically.
- Solve forced-vibration problems using MATLAB.

4.1 Introduction

In Chapter 3, we considered the response of single-degree-of-freedom systems subjected to harmonic excitation. However, many practical systems are subjected to several types of forcing functions that are not harmonic. The general forcing functions may be periodic (nonharmonic) or nonperiodic. The nonperiodic forces include forces such as a suddenly applied constant force (called a *step force*), a linearly increasing force (called a ramp force), and an exponentially varying force. A nonperiodic forcing function may be acting for a short, long, or infinite duration. A forcing function or excitation of short duration compared to the natural time period of the system is called a shock. Examples of general forcing functions are the motion imparted by a cam to the follower, the vibration felt by an instrument when its package is dropped from a height, the force applied to the foundation of a forging press, the motion of an automobile when it hits a pothole, and the ground vibration of a building frame during an earthquake.

If the forcing function is periodic but not harmonic, it can be replaced by a sum of harmonic functions using the harmonic analysis procedure discussed in Section 1.11. Using the principle of superposition, the response of the system can then be determined by superposing the responses due to the individual harmonic forcing functions.

The response of a system subjected to any type of nonperiodic force is commonly found using the following methods:

1. Convolution integral.
2. Laplace transform.
3. Numerical methods.

The first two methods are analytical ones, in which the response or solution is expressed in a way that helps in studying the behavior of the system under the applied force with respect to various parameters and in designing the system. The third method, on the other hand, can be used to find the response of a system under any arbitrary force for which an analytical solution is difficult or impossible to find. However, the solution found is applicable only for the particular set of parameter values used in finding the solution. This makes it difficult to study the behavior of the system when the parameters are varied. This chapter presents all three methods of solution.

4.2 Response Under a General Periodic Force

When the external force $F(t)$ is periodic with period $\tau = 2\pi/\omega$, it can be expanded in a Fourier series (see Section 1.11):

$$F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \tag{4.1}$$

where

$$a_j = \frac{2}{\tau}\int_0^{\tau} F(t) \cos j\omega t\, dt, \qquad j = 0, 1, 2, \ldots \tag{4.2}$$

and

$$b_j = \frac{2}{\tau}\int_0^{\tau} F(t) \sin j\omega t\, dt, \qquad j = 1, 2, \ldots \tag{4.3}$$

The response of systems under general periodic forces is considered in this section for both first- and second-order systems. First-order systems are those for which the equation of

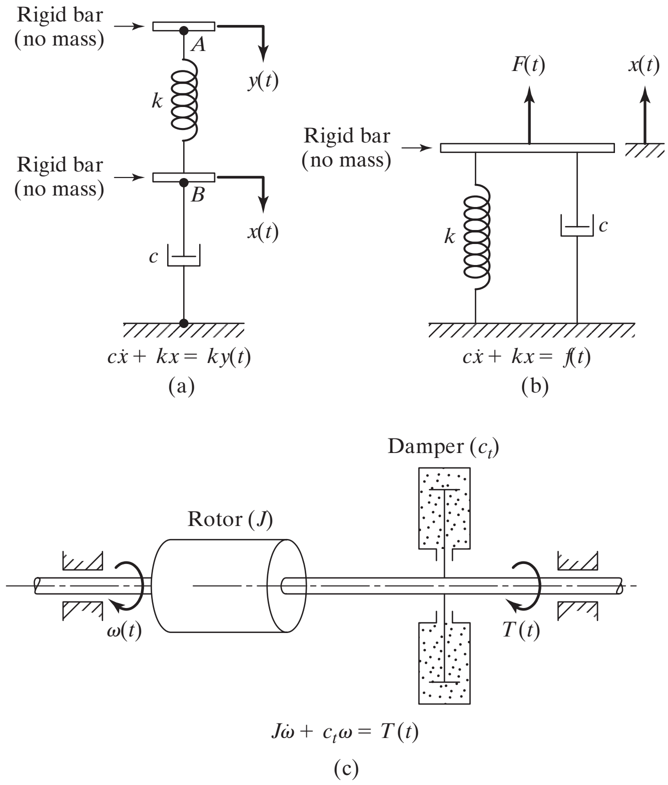


FIGURE 4.1 Examples of first-order systems.

motion is a first-order differential equation. Similarly, second-order systems are those for which the equation of motion is a second-order differential equation. Typical examples of first- and second-order systems are shown in Figs. 4.1 and 4.2, respectively.

4.2.1 First-Order Systems

Consider a spring-damper system subjected to a periodic excitation as shown in Fig. 4.1(a). The equation of motion of the system is given by

$$c\dot{x} + k(x - y) = 0 \tag{4.4}$$

where $y(t)$ is the periodic motion (or excitation) imparted to the system at point A (for example, by a cam). If the periodic displacement of point A, $y(t)$, is expressed in Fourier series as indicated by the right-hand side of Eq. (4.1), the equation of motion of the system can be expressed as

$$\dot{x} + ax = ay = A_0 + \sum_{j=1}^{\infty} A_j \sin \omega_j t + \sum_{j=1}^{\infty} B_j \cos \omega_j t \tag{4.5}$$

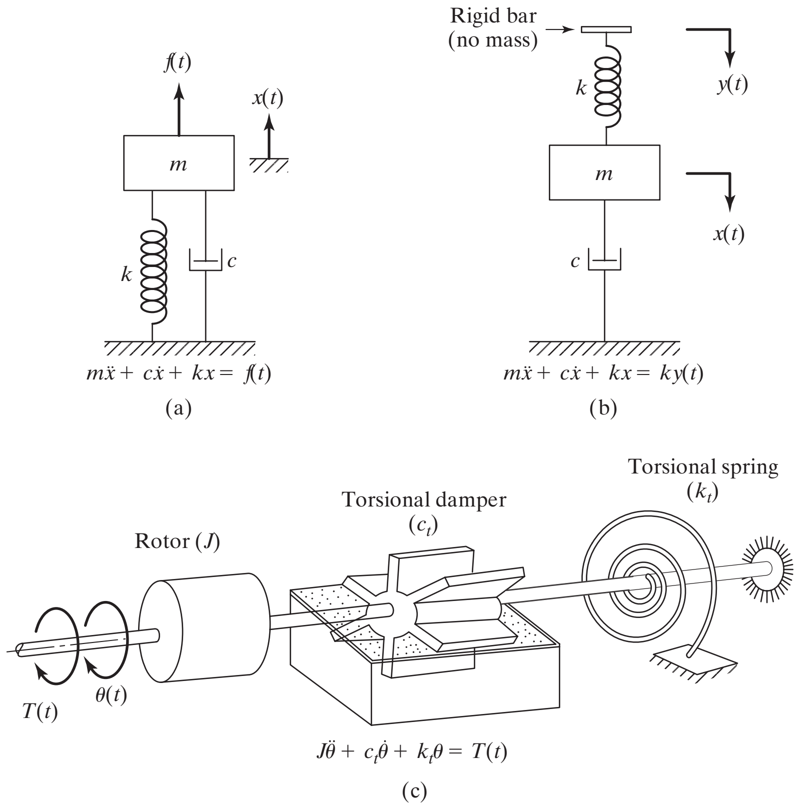


FIGURE 4.2 Examples of second-order systems.

where

$$a = \frac{k}{c}, \quad A_0 = \frac{a a_0}{2}, \quad A_j = a a_j, \quad B_j = a b_j, \quad \omega_j = j\omega, \quad j = 1, 2, 3, \ldots \tag{4.6}$$

The solution of Eq. (4.5) is presented in Example 4.1.

EXAMPLE 4.1 Response of a First-Order System under Periodic Force

Find the response of the spring-damper system shown in Fig. 4.1(a) subjected to a periodic force with the equation of motion given by Eq. (4.5).

Solution: It can be seen that the right-hand side of the equation of motion, Eq. (4.5), is a constant plus a linear sum of harmonic (sine and cosine) functions. Using the principle of superposition, the steady-state solution of Eq. (4.5) can be found by summing the steady-state solutions corresponding to the individual forcing terms on the right-hand side of Eq. (4.5).

The equation of motion corresponding to the constant force A_0 can be expressed, using x_0 for x, as

$$\dot{x}_0 + ax_0 = A_0 \tag{E.1}$$

The solution of Eq. (E.1) is given by (can be verified by substituting in Eq. (E.1)):

$$x_0(t) = \frac{A_0}{a} \tag{E.2}$$

The equation of motion under the force $A_j \sin \omega_j t$ can be expressed as

$$\dot{x}_j + ax_j = A_j \sin \omega_j t \tag{E.3}$$

in which the steady-state solution of Eq. (E.3) can be assumed in the form

$$x_j(t) = X_j \sin(\omega_j t - \phi_j) \tag{E.4}$$

where the magnitude X_j and the phase angle ϕ_j denote the unknown constants to be determined. The solution in Eq. (E.4) can be expressed as the imaginary part of the following solution in complex form:

$$x_j(t) = \mathrm{Im}\lfloor X_j e^{i(\omega_j t - \phi_j)} \rfloor = X_j e^{i\omega_j t} e^{-i\phi_j} = U_j e^{i\omega_j t} \tag{E.5}$$

where U_j denotes the complex number:

$$U_j = X_j e^{-i\phi_j} \tag{E.6}$$

Noting that the time derivative of $x_j(t)$ is given by

$$\dot{x}_j(t) = i\omega_j U_j e^{i\omega_j t} \tag{E.7}$$

Eq. (E.3) can be expressed with the forcing term in complex form (with the understanding that we are interested only in the imaginary part of the solution):

$$\dot{x}_j + ax_j = A_j e^{i\omega_j t} = A_j(\cos \omega_j t + i \sin \omega_j t) \tag{E.8}$$

By inserting Eqs. (E.5) and (E.7) into Eq. (E.8), we obtain

$$i\omega_j U_j e^{i\omega_j t} + aU_j e^{i\omega_j t} = A_j e^{i\omega_j t} \tag{E.9}$$

Since $e^{i\omega_j t} \neq 0$, Eq. (E.9) can be reduced to

$$i\omega_j U_j + aU_j = A_j \tag{E.10}$$

or

$$U_j = \frac{A_j}{a + i\omega_j} \tag{E.11}$$

Equations (E.6) and (E.11) yield

$$U_j = X_j e^{-i\phi_j} = \frac{A_j}{a + i\omega_j} \tag{E.12}$$

By expressing $\dfrac{1}{a + i\omega_j}$ as

$$\frac{1}{a + i\omega_j} = \frac{a - i\omega_j}{(a + i\omega_j)(a - i\omega_j)} = \frac{1}{\sqrt{a^2 + \omega_j^2}}\left[\frac{a}{\sqrt{a^2 + \omega_j^2}} - i\frac{\omega_j}{\sqrt{a^2 + \omega_j^2}}\right] \tag{E.13}$$

Eq. (E.13) can be rewritten as

$$\frac{1}{a + i\omega_j} = \frac{1}{\sqrt{a^2 + \omega_j^2}}[\cos\phi_j - i\sin\phi_j] = \frac{1}{\sqrt{a^2 + \omega_j^2}}e^{-i\phi_j} \tag{E.14}$$

where

$$\phi_j = \tan^{-1}\left(\frac{\omega_j}{a}\right) \tag{E.15}$$

By using Eq. (E.14) in Eq. (E.12), we find that

$$X_j = \frac{A_j}{\sqrt{a^2 + \omega_j^2}}, \qquad \phi_j = \tan^{-1}\left(\frac{\omega_j}{a}\right) \tag{E.16}$$

The solution of Eq. (E.3) is thus given by Eq. (E.4) with X_j and ϕ_j given by Eq. (E.16). The equation of motion under the force $B_j \cos \omega_j t$ can be expressed as

$$\dot{x}_j + a x_j = B_j \cos \omega_j t \tag{E.17}$$

By assuming the steady-state solution of Eq. (E.17) in the form

$$x_j(t) = Y_j \cos(\omega_j t - \phi_j) \tag{E.18}$$

the constants Y_j and ϕ_j can be determined, by proceeding as in the case of the solution of Eq. (E.3), as

$$Y_j = \frac{B_j}{\sqrt{a^2 + \omega_j^2}}, \qquad \phi_j = \tan^{-1}\left(\frac{\omega_j}{a}\right) \tag{E.19}$$

The complete steady-state (or particular) solution of Eq. (4.5) can be expressed as

$$x_p(t) = \frac{A_0}{a} + \sum_{j=1}^{\infty}\frac{A_j}{\sqrt{a^2 + \omega_j^2}}\sin\left\{\omega_j t - \tan^{-1}\left(\frac{\omega_j}{a}\right)\right\}$$
$$+ \sum_{j=1}^{\infty}\frac{B_j}{\sqrt{a^2 + \omega_j^2}}\cos\left\{\omega_j t - \tan^{-1}\left(\frac{\omega_j}{a}\right)\right\} \tag{E.20}$$

where a, A_0, A_j, B_j, and ω_j are given by Eq. (4.6).

Note: The total solution of Eq. (4.5) is given by the sum of the homogeneous and particular (or steady-state) solutions:

$$x(t) = x_h(t) + x_p(t) \tag{E.21}$$

where the particular solution is given by Eq. (E.20) and the homogeneous solution of Eq. (4.5) can be expressed as

$$x_h(t) = Ce^{-at} \tag{E.22}$$

where C is an unknown constant to be determined using the initial condition of the system. The total solution can be expressed as

$$x(t) = Ce^{-at} + \frac{A_0}{a} + \sum_{j=1}^{\infty} X_j \sin(\omega_j t - \phi_j) + \sum_{j=1}^{\infty} Y_j \cos(\omega_j t - \phi_j) \tag{E.23}$$

When the initial condition $x(t = 0) = x_0$ is used in Eq. (E.23), we obtain

$$x_0 = C + \frac{A_0}{a} - \sum_{j=1}^{\infty} X_j \sin \phi_j + \sum_{j=1}^{\infty} Y_i \cos \phi_j \tag{E.24}$$

which yields

$$C = x_0 - \frac{A_0}{a} + \sum_{j=1}^{\infty} X_j \sin \phi_j - \sum_{j=1}^{\infty} Y_j \cos \phi_j \tag{E.25}$$

Thus the total solution of Eq. (4.5) becomes

$$x(t) = \left[x_0 - \frac{A_0}{a} + \sum_{j=1}^{\infty} X_j \sin \phi_j - \sum_{j=1}^{\infty} Y_j \cos \phi_j \right] e^{-at}$$

$$+ \frac{A_0}{a} + \sum_{j=1}^{\infty} X_j \sin(\omega_d t - \phi_j) + \sum_{j=1}^{\infty} Y_j \cos(\omega_d t - \phi_j) \tag{E.26}$$

The features of the response of the system can be studied by considering a simpler type of forcing function through the following example.

■

EXAMPLE 4.2 Response of a First-Order System

Determine the response of a spring-damper system, similar to the one shown in Fig. 4.1(a), with the equation of motion:

$$\dot{x} + 1.5x = 7.5 + 4.5 \cos t + 3 \sin 5t$$

Assume the initial condition as $x(t = 0) = 0$.

Solution: The equation of motion of the system is given by

$$\dot{x} + 1.5x = 7.5 + 4.5 \cos t + 3 \sin 5t \tag{E.1}$$

We first find the solution of the differential equation by considering one forcing term at a time given on the right-hand side of Eq. (E.1) and then adding the solutions to find the total solution of Eq. (E.1). For the constant term, the equation to be solved is

$$\dot{x} + 1.5x = 7.5 \tag{E.2}$$

The solution of Eq. (E.2) is $x(t) = 7.5/1.5 = 5$. For the cosine term, the equation to be solved is given by

$$\dot{x} + 1.5x = 4.5 \cos t \tag{E.3}$$

Using the steady-state solution indicated in Eq. (E.21) of Example 4.1, we can express the solution of Eq. (E.3) as

$$x(t) = Y \cos(t - \phi) \tag{E.4}$$

where

$$Y = \frac{4.5}{\sqrt{(1.5)^2 + (1)^2}} = \frac{4.5}{\sqrt{3.25}} = 2.4961 \tag{E.5}$$

and

$$\phi = \tan^{-1}\left(\frac{1}{1.5}\right) = 0.5880 \text{ rad} \tag{E.6}$$

Similarly, for the sine term, the equation to be solved is

$$\dot{x} + 1.5x = 3 \sin 5t \tag{E.7}$$

Using the steady-state solution indicated in Eq. (E.4) of Example 4.1, we can express the solution of Eq. (E.7) as

$$x(t) = X \sin(5t - \phi) \tag{E.8}$$

where

$$X = \frac{3}{\sqrt{(1.5)^2 + (5)^2}} = \frac{3}{\sqrt{27.25}} = 0.5747 \tag{E.9}$$

and

$$\phi = \tan^{-1}\left(\frac{5}{1.5}\right) = 1.2793 \text{ rad} \tag{E.10}$$

Thus the total particular solution of Eq. (E.1) is given by the sum of the solutions of Eqs. (E.2), (E.3) and (E.7):

$$x(t) = 5 + 2.4961 \cos(t - 0.5880) + 0.5747 \sin(5t - 1.2793) \tag{E.11}$$

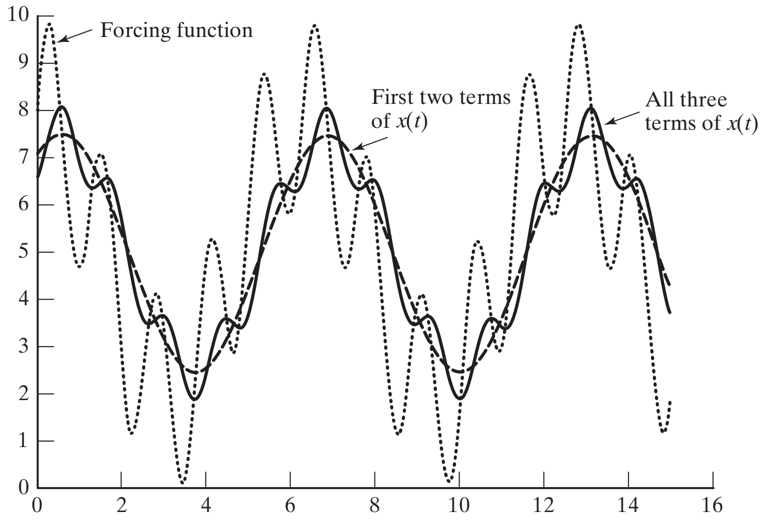


FIGURE 4.3

The forcing function given by the right-hand-side expression in Eq. (E.1) and the steady-state response of the system given by Eq. (E.11) are shown graphically in Fig. 4.3. The first two terms of the response (given by the first two terms on the right-hand side of Eq. (E.11)) are also shown in Fig. 4.3. It can be seen that system does not filter the constant term. However, it filters the lower-frequency (cosine term) to some extent and the higher-frequency (sine time) to a larger extent.

■

4.2.2 Second-Order Systems

Let a spring-mass-damper system, Fig. 4.2(a), be subjected to a periodic force. This is a second-order system because the governing equation is a second-order differential equation:

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{4.7}$$

If the forcing function $f(t)$ is periodic, it can be expressed in Fourier series so that the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \tag{4.8}$$

The determination of the solution of Eq. (4.8) is illustrated in Example 4.3.

EXAMPLE 4.3 Response of a Second-Order System Under Periodic Force

Determine the response of a spring-mass-damper system subjected to a periodic force with the equation of motion given by Eq. (4.8). Assume the initial conditions as zero.

Solution: The right-hand side of Eq. (4.8) is a constant plus a sum of harmonic functions. Using the principle of superposition, the steady-state solution of Eq. (4.4) is the sum of the steady-state solutions of the following equations:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} \tag{E.1}$$

$$m\ddot{x} + c\dot{x} + kx = a_j \cos j\omega t \tag{E.2}$$

$$m\ddot{x} + c\dot{x} + kx = b_j \sin j\omega t \tag{E.3}$$

Noting that the solution of Eq. (E.1) is given by

$$x_p(t) = \frac{a_0}{2k} \tag{E.4}$$

and, using the results of Section 3.4, we can express the solutions of Eqs. (E.2) and (E.3), respectively, as

$$x_p(t) = \frac{(a_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta jr)^2}} \cos(j\omega t - \phi_j) \tag{E.5}$$

$$x_p(t) = \frac{(b_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta jr)^2}} \sin(j\omega t - \phi_j) \tag{E.6}$$

where

$$\phi_j = \tan^{-1}\left(\frac{2\zeta jr}{1 - j^2r^2}\right) \tag{E.7}$$

and

$$r = \frac{\omega}{\omega_n} \tag{E.8}$$

Thus the complete steady-state solution of Eq. (4.8) is given by

$$x_p(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{(a_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta jr)^2}} \cos(j\omega t - \phi_j)$$
$$+ \sum_{j=1}^{\infty} \frac{(b_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta jr)^2}} \sin(j\omega t - \phi_j) \tag{E.9}$$

It can be seen from the solution, Eq. (E.9), that the amplitude and phase shift corresponding to the jth term depend on j. If $j\omega = \omega_n$, for any j, the amplitude of the corresponding harmonic will be comparatively large. This will be particularly true for small values of j and ζ. Further, as j becomes larger, the amplitude becomes smaller and the corresponding terms tend to zero. Thus the first few terms are usually sufficient to obtain the response with reasonable accuracy.

The solution given by Eq. (E.9) denotes the steady-state response of the system. The transient part of the solution arising from the initial conditions can also be included to find the complete solution. To find the complete solution, we need to evaluate the arbitrary constants by setting the value of the complete solution and its derivative to the specified values of initial displacement $x(0)$ and the initial velocity $\dot{x}(0)$. This results in a complicated expression for the transient part of the total solution.

■

EXAMPLE 4.4 Periodic Vibration of a Hydraulic Valve

In the study of vibrations of valves used in hydraulic control systems, the valve and its elastic stem are modeled as a damped spring-mass system, as shown in Fig. 4.4(a). In addition to the spring force and damping force, there is a fluid-pressure force on the valve that changes with the amount of opening or closing of the valve. Find the steady-state response of the valve when the pressure in the chamber varies as indicated in Fig. 4.4(b). Assume $k = 2500$ N/m, $c = 10$ N-s/m, and $m = 0.25$ kg.

Solution: The valve can be considered as a mass connected to a spring and a damper on one side and subjected to a forcing function $F(t)$ on the other side. The forcing function can be expressed as

$$F(t) = Ap(t) \tag{E.1}$$

where A is the cross-sectional area of the chamber, given by

$$A = \frac{\pi(50)^2}{4} = 625\pi \text{ mm}^2 = 0.000625\pi \text{ m}^2 \tag{E.2}$$

and $p(t)$ is the pressure acting on the valve at any instant t. Since $p(t)$ is periodic with period $\tau = 2$ seconds and A is a constant, $F(t)$ is also a periodic function of period $\tau = 2$ seconds. The

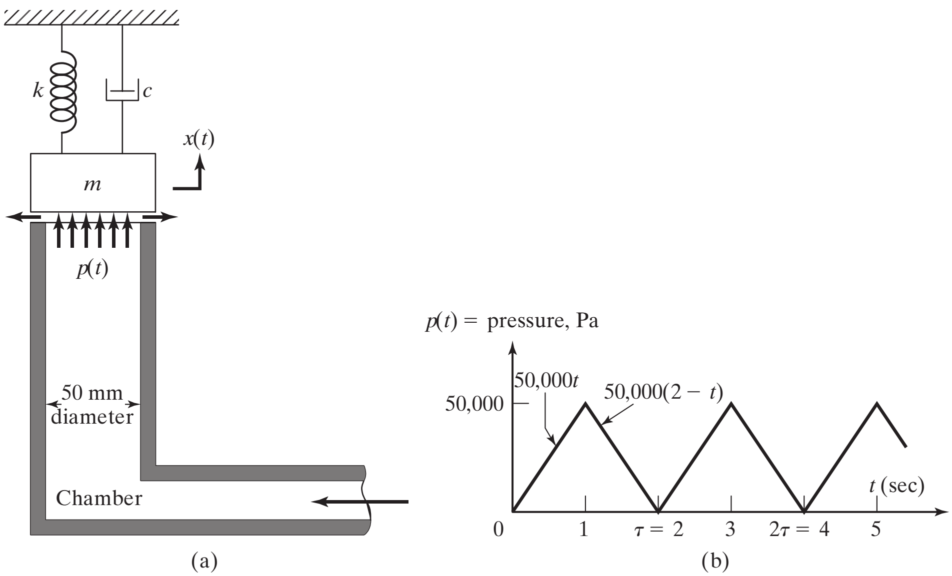


FIGURE 4.4 Periodic vibration of a hydraulic valve.

frequency of the forcing function is $\omega = (2\pi/\tau) = \pi$ rad/s. $F(t)$ can be expressed in a Fourier series as

$$F(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots + b_1 \sin \omega t + b_2 \sin 2\omega t + \cdots \tag{E.3}$$

where a_j and b_j are given by Eqs. (4.2) and (4.3). Since the function $F(t)$ is given by

$$F(t) = \begin{cases} 50{,}000At & \text{for } 0 \leqslant t \leqslant \dfrac{\tau}{2} \\ 50{,}000A(2-t) & \text{for } \dfrac{\tau}{2} \leqslant t \leqslant \tau \end{cases} \tag{E.4}$$

the Fourier coefficients a_j and b_j can be computed with the help of Eqs. (4.2) and (4.3):

$$a_0 = \frac{2}{2}\left[\int_0^1 50{,}000At\,dt + \int_1^2 50{,}000A(2-t)\,dt\right] = 50{,}000A \tag{E.5}$$

$$a_1 = \frac{2}{2}\left[\int_0^1 50{,}000At \cos \pi t\,dt + \int_1^2 50{,}000A(2-t) \cos \pi t\,dt\right] = -\frac{2 \times 10^5 A}{\pi^2} \tag{E.6}$$

$$b_1 = \frac{2}{2}\left[\int_0^1 50{,}000At \sin \pi t\,dt + \int_1^2 50{,}000A(2-t) \sin \pi t\,dt\right] = 0 \tag{E.7}$$

$$a_2 = \frac{2}{2}\left[\int_0^1 50{,}000At \cos 2\pi t\,dt + \int_1^2 50{,}000A(2-t) \cos 2\pi t\,dt\right] = 0 \tag{E.8}$$

$$b_2 = \frac{2}{2}\left[\int_0^1 50{,}000At \sin 2\pi t\,dt + \int_1^2 50{,}000A(2-t) \sin 2\pi t\,dt\right] = 0 \tag{E.9}$$

$$a_3 = \frac{2}{2}\left[\int_0^1 50{,}000At \cos 3\pi t\,dt + \int_1^2 50{,}000A(2-t) \cos 3\pi t\,dt\right] = -\frac{2 \times 10^5 A}{9\pi^2} \tag{E.10}$$

$$b_3 = \frac{2}{2}\left[\int_0^1 50{,}000At \sin 3\pi t\,dt + \int_1^2 50{,}000A(2-t) \sin 3\pi t\,dt\right] = 0 \tag{E.11}$$

Likewise, we can obtain $a_4 = a_6 = \cdots = b_4 = b_5 = b_6 = \cdots = 0$. By considering only the first three harmonics, the forcing function can be approximated:

$$F(t) \simeq 25{,}000A - \frac{2 \times 10^5 A}{\pi^2} \cos \omega t - \frac{2 \times 10^5 A}{9\pi^2} \cos 3\omega t \tag{E.12}$$

The steady-state response of the valve to the forcing function of Eq. (E.12) can be expressed as

$$x_p(t) = \frac{25{,}000A}{k} - \frac{(2 \times 10^5 A/(k\pi^2))}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \cos(\omega t - \phi_1)$$
$$- \frac{(2 \times 10^5 A/(9k\pi^2))}{\sqrt{(1 - 9r^2)^2 + (6\zeta r)^2}} \cos(3\omega t - \phi_3) \tag{E.13}$$

The natural frequency of the valve is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{0.25}} = 100 \text{ rad/s} \tag{E.14}$$

and the forcing frequency ω by

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{2} = \pi \text{ rad/s} \tag{E.15}$$

Thus the frequency ratio can be obtained:

$$r = \frac{\omega}{\omega_n} = \frac{\pi}{100} = 0.031416 \tag{E.16}$$

and the damping ratio:

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{10.0}{2(0.25)(100)} = 0.2 \tag{E.17}$$

The phase angles ϕ_1 and ϕ_3 can be computed as follows:

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$
$$= \tan^{-1}\left(\frac{2 \times 0.2 \times 0.031416}{1 - 0.031416^2}\right) = 0.0125664 \text{ rad} \tag{E.18}$$

and

$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1 - 9r^2}\right)$$
$$= \tan^{-1}\left(\frac{6 \times 0.2 \times 0.031416}{1 - 9(0.031416)^2}\right) = 0.0380483 \text{ rad} \tag{E.19}$$

In view of Eqs. (E.2) and (E.14) to (E.19), the solution can be written as

$$x_p(t) = 0.019635 - 0.015930 \cos(\pi t - 0.0125664)$$
$$- 0.0017828 \cos(3\pi t - 0.0380483) \text{ m} \tag{E.20}$$

■

EXAMPLE 4.5 Total Response Under Harmonic Base Excitation

Find the total response of a viscously damped single-degree-of-freedom system subjected to a harmonic base excitation for the following data: $m = 10$ kg, $c = 20$ N-s/m, $k = 4000$ N/m, $y(t) = 0.05 \sin 5t$ m, $x_0 = 0.02$ m, $\dot{x}_0 = 10$ m/s.

Solution: The equation of motion of the system is given by (see Eq. (3.65)):

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = kY \sin \omega t + c\omega Y \cos \omega t \tag{E.1}$$

Noting that Eq. (E.1) is similar to Eq. (4.8) with $a_0 = 0$, $a_1 = c\omega Y$, $b_1 = kY$, and $a_i = b_i = 0$; $i = 2, 3, \ldots$, the steady-state response of the system can be expressed, using Eq. (E.9) of Example 4.3, as

$$x_p(t) = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \left[\frac{a_1}{k} \cos(\omega t - \phi_1) + \frac{b_1}{k} \sin(\omega t - \phi_1) \right] \tag{E.2}$$

For the given data, we find

$$Y = 0.05 \text{ m}, \qquad \omega = 5 \text{ rad/s}, \qquad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{5}{20} = 0.25, \qquad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{(4000)(10)}} = 0.05$$

$$\omega_d = \sqrt{1 - \zeta^2}\, \omega_n = 19.975 \text{ rad/s}$$

$$a_1 = c\omega Y = (20)(5)(0.05) = 5, \qquad b_1 = kY = (4000)(0.05) = 200$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) = \tan^{-1}\left(\frac{2(0.05)(0.25)}{1 - (0.25)^2}\right) = 0.02666 \text{ rad}$$

$$\sqrt{(1 - r^2)^2 + (2\zeta r)^2} = \sqrt{(1 - 0.25^2)^2 + (2(0.05)(0.25))^2} = 0.937833.$$

The solution of the homogeneous equation is given by (see Eq. (2.70)):

$$x_h(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) = X_0 e^{-t} \cos(19.975t - \phi_0) \tag{E.3}$$

where X_0 and ϕ_0 are unknown constants. The total solution can be expressed as the superposition of $x_h(t)$ and $x_p(t)$ as

$$\begin{aligned} x(t) &= X_0 e^{-t} \cos(19.975t - \phi_0) + \frac{1}{0.937833}\left[\frac{5}{4000}\cos(5t - \phi_1) + \frac{200}{4000}\sin(5t - \phi_1)\right] \\ &= X_0 e^{-t} \cos(19.975t - \phi_0) + 0.001333 \cos(5t - 0.02666) \\ &\quad + 0.053314 \sin(5t - 0.02666) \end{aligned} \tag{E.4}$$

where the unknowns X_0 and ϕ_0 are to be found from the initial conditions. The velocity of the mass can be expressed from Eq. (E.4) as

$$\begin{aligned} \dot{x}(t) = \frac{dx}{dt}(t) &= -X_0 e^{-t} \cos(19.975t - \phi_0) - 19.975 X_0 e^{-t} \sin(19.975t - \phi_0) \\ &\quad - 0.006665 \sin(5t - 0.02666) + 0.266572 \cos(5t - 0.02666) \end{aligned} \tag{E.5}$$

Using Eqs. (E.4) and (E.5), we find

$$x_0 = x(t = 0) = 0.02 = X_0 \cos \phi_0 + 0.001333 \cos(0.02666) - 0.053314 \sin(0.02666)$$

or

$$X_0 \cos \phi_0 = 0.020088 \tag{E.6}$$

and

$$\dot{x}_0 = \dot{x}(t = 0) = 10 = -X_0 \cos \phi_0 + 19.975\, X_0 \sin \phi_0$$
$$+ 0.006665 \sin(0.02666) + 0.266572 \cos(0.02666)$$

or

$$- X_0 \cos \phi_0 + 19.975 \sin \phi_0 = 9.733345 \tag{E.7}$$

The solution of Eqs. (E.6) and (E.7) yields $X_0 = 0.488695$ and $\phi_0 = 1.529683$ rad. Thus the total response of the mass under base excitation, in meters, is given by

$$\begin{aligned} x(t) &= 0.488695 e^{-t} \cos(19.975 t - 1.529683) \\ &\quad + 0.001333 \cos(5t - 0.02666) + 0.053314 \sin(5t - 0.02666) \end{aligned} \tag{E.8}$$

Note: Equation (E.8) is plotted in Example 4.32.

■

4.3 Response Under a Periodic Force of Irregular Form

In some cases, the force acting on a system may be quite irregular and may be determined only experimentally. Examples of such forces include wind and earthquake-induced forces. In such cases, the forces will be available in graphical form and no analytical expression can be found to describe $F(t)$. Sometimes, the value of $F(t)$ may be available only at a number of discrete points $t_1, t_2, \ldots, t_N$. In all these cases, it is possible to find the Fourier coefficients by using a numerical integration procedure, as described in Section 1.11. If $F_1, F_2, \ldots, F_N$ denote the values of $F(t)$ at $t_1, t_2, \ldots, t_N$, respectively, where N denotes an even number of equidistant points in one time period $\tau (\tau = N \Delta t)$, as shown in Fig. 4.5, the application of trapezoidal rule [4.1] gives

$$a_0 = \frac{2}{N} \sum_{i=1}^{N} F_i \tag{4.9}$$

$$a_j = \frac{2}{N} \sum_{i=1}^{N} F_i \cos \frac{2 j \pi t_i}{\tau}, \qquad j = 1, 2, \ldots \tag{4.10}$$

$$b_j = \frac{2}{N} \sum_{i=1}^{N} F_i \sin \frac{2 j \pi t_i}{\tau}, \qquad j = 1, 2, \ldots \tag{4.11}$$

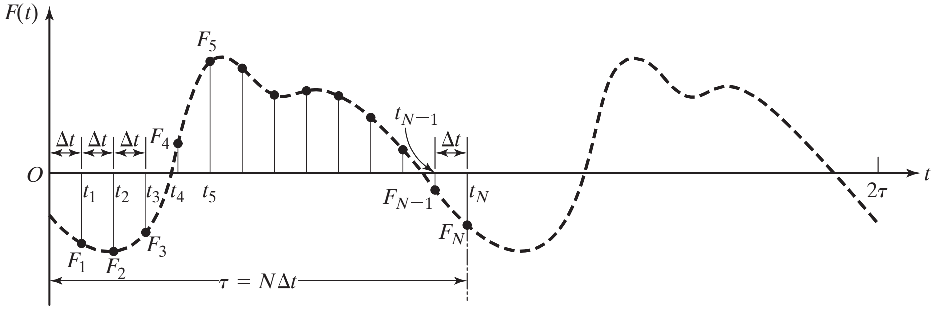


FIGURE 4.5 An irregular forcing function.

Once the Fourier coefficients a_0, a_j, and b_j are known, the steady-state response of the system can be found using Eq. (4.9) with

$$r = \left(\frac{2\pi}{\tau\omega_n}\right) \tag{4.12}$$

EXAMPLE 4.6

Steady-State Vibration of a Hydraulic Valve

Find the steady-state response of the valve in Example 4.4 if the pressure fluctuations in the chamber are found to be periodic. The values of pressure measured at 0.01-second intervals in one cycle are given below.

Time, t_i (seconds)	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12
$p_i = p(t_i)$ (kN/m^2)	0	20	34	42	49	53	70	60	36	22	16	7	0

Solution: Since the pressure fluctuations on the valve are periodic, the Fourier analysis of the given data of pressures in a cycle gives

$$\begin{aligned} p(t) &= 34083.3 - 26996.0 \cos 52.36t + 8307.7 \sin 52.36t \\ &\quad + 1416.7 \cos 104.72t + 3608.3 \sin 104.72t \\ &\quad - 5833.3 \cos 157.08t - 2333.3 \sin 157.08t + \ldots \text{ N/m}^2 \end{aligned} \tag{E.1}$$

(See Example 1.20.) Other quantities needed for the computation are

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.12} = 52.36 \text{ rad/s}$$

$$\omega_n = 100 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = 0.5236$$

$$\zeta = 0.2$$

$$A = 0.000625\pi \text{ m}^2$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) = \tan^{-1}\left(\frac{2 \times 0.2 \times 0.5236}{1 - 0.5236^2}\right) = 16.1°$$

$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1 - 4r^2}\right) = \tan^{-1}\left(\frac{4 \times 0.2 \times 0.5236}{1 - 4 \times 0.5236^2}\right) = -77.01°$$

$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1 - 9r^2}\right) = \tan^{-1}\left(\frac{6 \times 0.2 \times 0.5236}{1 - 9 \times 0.5236^2}\right) = -23.18°$$

The steady-state response of the valve can be expressed, using Eq. (E.9) of Example 4.3, as

$$\begin{aligned}
x_p(t) = {} & \frac{34083.3A}{k} - \frac{(26996.0A/k)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}\cos(52.36t - \phi_1) \\
& + \frac{(8309.7A/k)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}\sin(52.36t - \phi_1) \\
& + \frac{(1416.7A/k)}{\sqrt{(1 - 4r^2)^2 + (4\zeta r)^2}}\cos(104.72t - \phi_2) \\
& + \frac{(3608.3A/k)}{\sqrt{(1 - 4r^2)^2 + (4\zeta r)^2}}\sin(104.72t - \phi_2) \\
& - \frac{(5833.3A/k)}{\sqrt{(1 - 9r^2)^2 + (6\zeta r)^2}}\cos(157.08t - \phi_3) \\
& - \frac{(2333.3A/k)}{\sqrt{(1 - 9r^2)^2 + (6\zeta r)^2}}\sin(157.08t - \phi_3)
\end{aligned}$$

■

4.4 Response Under a Nonperiodic Force

We have seen that periodic forces of any general waveform can be represented by Fourier series as a superposition of harmonic components of various frequencies. The response of a linear system is then found by superposing the harmonic response to each of the exciting forces. When the exciting force $F(t)$ is nonperiodic, such as that due to the blast from an explosion, a different method of calculating the response is required. Various methods can be used to find the response of the system to an arbitrary excitation. Some of these methods are as follows:

1. Representing the excitation by a Fourier integral.
2. Using the method of convolution integral.

3. Using the method of Laplace transforms.
4. Numerically integrating the equations of motion (numerical solution of differential equations).

We shall discuss methods 2, 3 and 4 in the following sections. The numerical methods are also considered in Chapter 11.

4.5 Convolution Integral

A nonperiodic exciting force usually has a magnitude that varies with time; it acts for a specified period and then stops. The simplest form is the impulsive force—a force that has a large magnitude F and acts for a very short time Δt. From dynamics we know that impulse can be measured by finding the change it causes in momentum of the system [4.2]. If $\dot{x}_1$ and $\dot{x}_2$ denote the velocities of the mass m before and after the application of the impulse, we have

$$\text{Impulse} = F\Delta t = m\dot{x}_2 - m\dot{x}_1 \tag{4.12}$$

By designating the magnitude of the impulse $F\Delta t$ by $\boldsymbol{F}$, we can write, in general,

$$\boldsymbol{F} = \int_t^{t+\Delta t} F\,dt \tag{4.13}$$

A unit impulse acting at $t = 0$ ($\boldsymbol{f}$) is defined as

$$\boldsymbol{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F\,dt = F\,dt = 1 \tag{4.14}$$

It can be seen that in order for $F\,dt$ to have a finite value, F tends to infinity (since dt tends to zero).

The unit impulse, $\boldsymbol{f} = 1$, acting at $t = 0$, is also denoted by the Dirac delta function as

$$\boldsymbol{f} = \boldsymbol{f}\delta(t) = \delta(t) \tag{4.15}$$

and the impulse of magnitude $\boldsymbol{F}$, acting at $t = 0$, is denoted as[1]

$$\boldsymbol{F} = \boldsymbol{F}\,\delta(t) \tag{4.16}$$

---

[1]The unit impulse, $\boldsymbol{f}$ acting at $t = 0$, is also denoted by the Dirac delta function, $\delta(t)$. The Dirac delta function at time $t = \tau$, denoted as $\delta(t - \tau)$, has the properties

$$\delta(t - \tau) = 0 \quad \text{for } t \neq \tau;$$

$$\int_0^\infty \delta(t - \tau)\,dt = 1, \qquad \int_0^\infty \delta(t - \tau)F(t)\,dt = F(\tau)$$

where $0 < \tau < \infty$. Thus an impulse of magnitude $\boldsymbol{F}$, acting at $t = \tau$, can be denoted as $F(t) = \boldsymbol{F}\,\delta(t - \tau)$

4.5.1 Response to an Impulse

We first consider the response of a single-degree-of-freedom system to an impulse excitation; this case is important in studying the response under more general excitations. Consider a viscously damped spring-mass system subjected to a unit impulse at $t = 0$, as shown in Figs. 4.6(a) and (b). For an underdamped system, the solution of the equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{4.17}$$

is given by Eq. (2.72) as

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\} \tag{4.18}$$

where

$$\zeta = \frac{c}{2m\omega_n} \tag{4.19}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{4.20}$$

$$\omega_n = \sqrt{\frac{k}{m}} \tag{4.21}$$

If the mass is at rest before the unit impulse is applied ($x = \dot{x} = 0$ for $t < 0$ or at $t = 0^-$), we obtain, from the impulse-momentum relation,

$$\text{Impulse} = \boldsymbol{f} = 1 = m\dot{x}(t=0) - m\dot{x}(t=0^-) = m\dot{x}_0 \tag{4.22}$$

Thus the initial conditions are given by

$$x(t=0) = x_0 = 0 \tag{4.23}$$

$$\dot{x}(t=0) = \dot{x}_0 = \frac{1}{m} \tag{4.24}$$

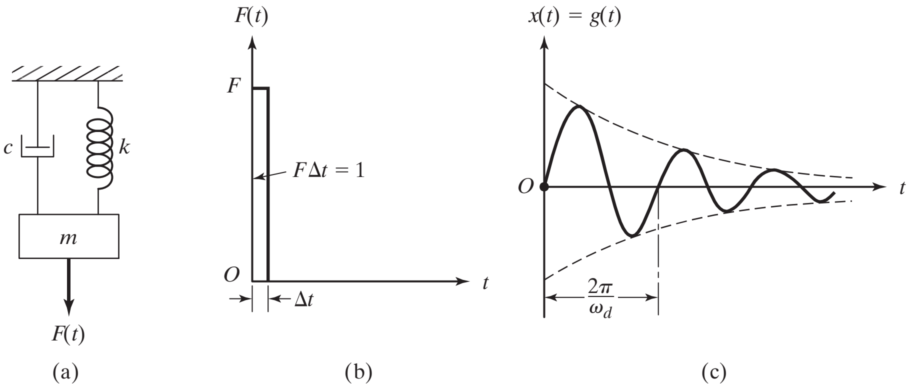


FIGURE 4.6 A single-degree-of-freedom system subjected to an impulse.

In view of Eqs. (4.23) and (4.24), Eq. (4.18) reduces to

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t \tag{4.25}$$

Equation (4.25) gives the response of a single-degree-of-freedom system to a unit impulse, which is also known as the *impulse response function*, denoted by $g(t)$. The function $g(t)$, Eq. (4.25), is shown in Fig. 4.6(c).

If the magnitude of the impulse is $\boldsymbol{F}$ instead of unity, the initial velocity $\dot{x}_0$ is $\boldsymbol{F}/m$ and the response of the system becomes

$$x(t) = \frac{\boldsymbol{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = \boldsymbol{F}g(t) \tag{4.26}$$

If the impulse $\boldsymbol{F}$ is applied at an arbitrary time $t = \tau$, as shown in Fig. 4.7(a), it will change the velocity at $t = \tau$ by an amount $\boldsymbol{F}/m$. Assuming that $x = 0$ until the impulse is applied, the displacement x at any subsequent time t, caused by a change in the velocity at time τ, is given by Eq. (4.26) with t replaced by the time elapsed after the application of the impulse—that is, $t - \tau$. Thus we obtain

$$x(t) = \boldsymbol{F}g(t - \tau) \tag{4.27}$$

This is shown in Fig. 4.7(b).

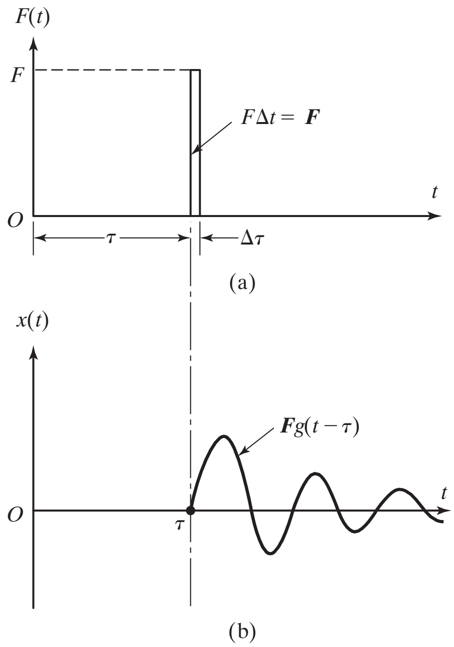


FIGURE 4.7 Impulse response.

EXAMPLE 4.7 Response of a Structure Under Impact

In the vibration testing of a structure, an impact hammer with a load cell to measure the impact force is used to cause excitation, as shown in Fig. 4.8(a). Assuming $m = 5$ kg, $k = 2000$ N/m, $c = 10$ N-s/m, and $\boldsymbol{F} = 20$ N-s, find the response of the system.

Solution: From the known data, we can compute

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{5}} = 20 \text{ rad/s}, \qquad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{10}{2\sqrt{2000(5)}} = 0.05$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 19.975 \text{ rad/s}$$

Assuming that the impact is given at $t = 0$, we find (from Eq. (4.26)) the response of the system as

$$x_1(t) = \boldsymbol{F}\,\frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

$$= \frac{20}{(5)(19.975)}e^{-0.05(20)t}\sin 19.975t = 0.20025e^{-t}\sin 19.975t \text{ m} \tag{E.1}$$

Note: The graph of Eq. (E.1) is shown in Example 4.33.

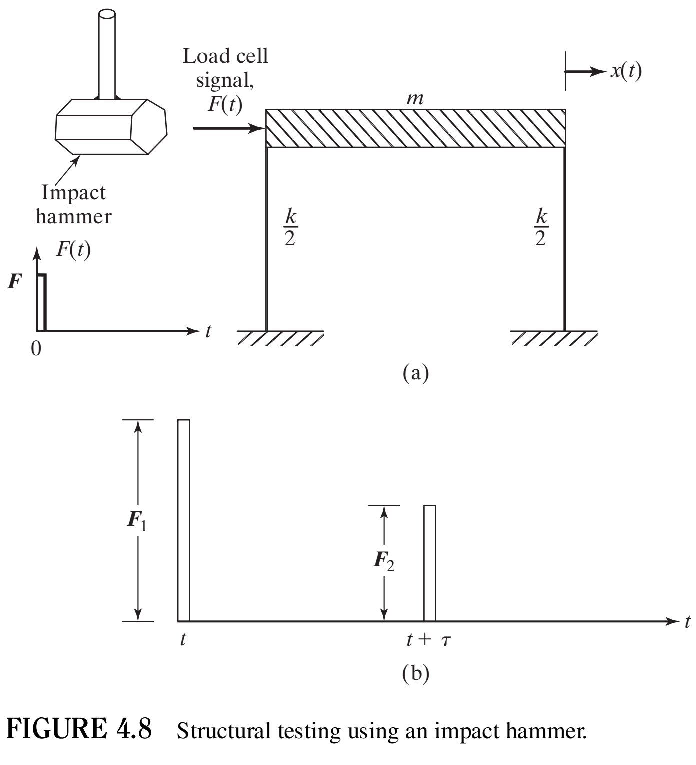


FIGURE 4.8 Structural testing using an impact hammer.

■

EXAMPLE 4.8 Response of a Structure Under Double Impact

In many cases, providing only one impact to the structure using an impact hammer is difficult. Sometimes a second impact takes place after the first, as shown in Fig. 4.8(b), and the applied force, $F(t)$, can be expressed as

$$F(t) = \boldsymbol{F}_1\,\delta(t) + \boldsymbol{F}_2\,\delta(t-\tau)$$

where $\delta(t)$ is the Dirac delta function and τ indicates the time between the two impacts of magnitudes $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. For a structure with $m = 5$ kg, $k = 2000$ N/m, $c = 10$ N-s/m and $F(t) = 20\,\delta(t) + 10\,\delta(t-0.2)$ N, find the response of the structure.

Solution: From the known data, we find $\omega_n = 20$ rad/s (see the solution for Example 4.7), $\zeta = 0.05$, and $\omega_d = 19.975$ rad/s. The response due to the impulse $\boldsymbol{F}_1\,\delta(t)$ is given by Eq. (E.1) of Example 4.7, while the response due to the impulse $\boldsymbol{F}_2\,\delta(t-0.2)$ can be determined from Eqs. (4.27) and (4.26) as

$$x_2(t) = \boldsymbol{F}_2\,\frac{e^{-\zeta\omega_n(t-\tau)}}{m\omega_d}\sin\omega_d(t-\tau) \tag{E.1}$$

For $\tau = 0.2$, Eq. (E.1) becomes

$$x_2(t) = \frac{10}{(5)(19.975)}\,e^{-0.05(20)(t-0.2)}\sin 19.975(t-0.2)$$

$$= 0.100125\mathrm{e}^{-(t-0.2)}\sin 19.975(t-0.2); \qquad t > 0.2 \tag{E.2}$$

Using the superposition of the two responses $x_1(t)$ and $x_2(t)$, the response due to two impacts, in meters, can be expressed as

$$x(t) = \begin{Bmatrix} 0.20025e^{-t}\sin 19.975t;\ 0 \le t \le 0.2 \\ 0.20025e^{-t}\sin 19.975t + 0.100125e^{-(t-0.2)}\sin 19.975(t-0.2);\ t > 0.2 \end{Bmatrix} \tag{E.3}$$

Note: The graph of Eq. (E.3) is shown in Example 4.33.

■

4.5.2 Response to a General Forcing Condition

Now we consider the response of the system under an arbitrary external force $F(t)$, shown in Fig. 4.9. This force may be assumed to be made up of a series of impulses of varying magnitude. Assuming that at time τ, the force $F(\tau)$ acts on the system for a short period of time $\Delta\tau$, the impulse acting at $t = \tau$ is given by $F(\tau)\,\Delta\tau$. At any time t, the elapsed time since the impulse is $t - \tau$, so the response of the system at t due to this impulse alone is given by Eq. (4.27) with $\boldsymbol{F} = F(\tau)\,\Delta\tau$:

$$\Delta x(t) = F(\tau)\,\Delta\tau\,g(t-\tau) \tag{4.28}$$

The total response at time t can be found by summing all the responses due to the elementary impulses acting at all times τ:

$$x(t) \simeq \sum F(\tau)g(t-\tau)\,\Delta\tau \tag{4.29}$$

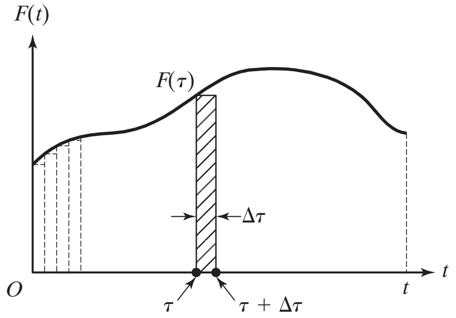


FIGURE 4.9 An arbitrary (nonperiodic) forcing function.

Letting $\Delta\tau \to 0$ and replacing the summation by integration, we obtain

$$x(t) = \int_0^t F(\tau) g(t - \tau)\, d\tau \tag{4.30}$$

By substituting Eq. (4.25) into Eq. (4.30), we obtain

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t - \tau)\, d\tau \tag{4.31}$$

which represents the response of an underdamped single-degree-of-freedom system to the arbitrary excitation $F(t)$. Note that Eq. (4.31) does not consider the effect of initial conditions of the system, because the mass is assumed to be at rest before the application of the impulse, as implied by Eqs. (4.25) and (4.28). The integral in Eq. (4.30) or Eq. (4.31) is called the *convolution* or *Duhamel integral*. In many cases the function $F(t)$ has a form that permits an explicit integration of Eq. (4.31). If such integration is not possible, we can evaluate numerically without much difficulty, as illustrated in Section 4.9 and in Chapter 11. An elementary discussion of the Duhamel integral in vibration analysis is given in reference [4.6].

4.5.3 Response to Base Excitation

If a spring-mass-damper system is subjected to an arbitrary base excitation described by its displacement, velocity, or acceleration, the equation of motion can be expressed in terms of the relative displacement of the mass $z = x - y$ as follows (see Section 3.6.2):

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4.32}$$

This is similar to the equation

$$m\ddot{x} + c\dot{x} + kx = F \tag{4.33}$$

with the variable z replacing x and the term $-m\ddot{y}$ replacing the forcing function F. Hence all of the results derived for the force-excited system are applicable to the base-excited system also for z when the term F is replaced by $-m\ddot{y}$. For an underdamped system subjected to base excitation, the relative displacement can be found from Eq. (4.31):

$$z(t) = -\frac{1}{\omega_d}\int_0^t \ddot{y}(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t - \tau)\, d\tau \tag{4.34}$$

EXAMPLE 4.9 Step Force on a Compacting Machine

A compacting machine, modeled as a single-degree-of-freedom system, is shown in Fig. 4.10(a). The force acting on the mass m (m includes the masses of the piston, the platform, and the material being compacted) due to a sudden application of the pressure can be idealized as a step force, as shown in Fig. 4.10(b). Determine the response of the system.

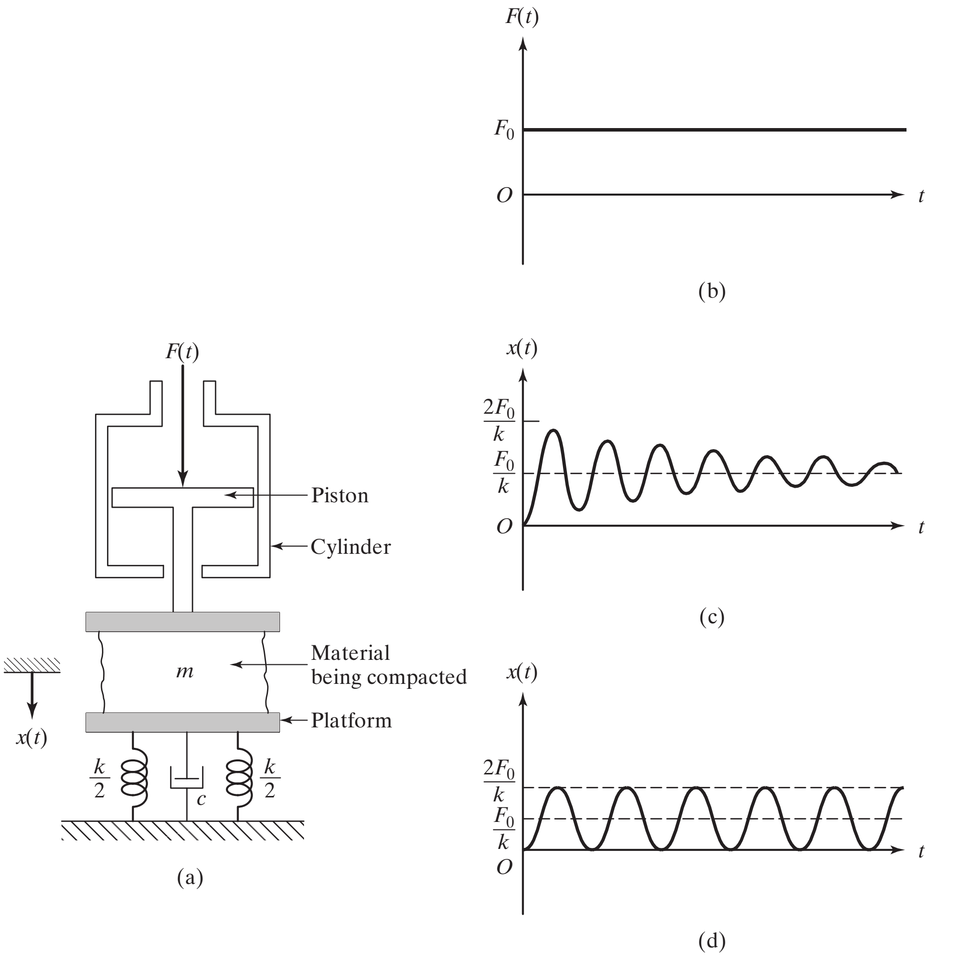


FIGURE 4.10 Step force applied to a compacting machine.

Solution: Since the compacting machine is modeled as a mass-spring-damper system, the problem is to find the response of a damped single-degree-of-freedom system subjected to a step force. By noting that $F(t) = F_0$, we can write Eq. (4.31) as

$$\begin{aligned} x(t) &= \frac{F_0}{m\omega_d} \int_0^t e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t - \tau)\, d\tau \\ &= \frac{F_0}{m\omega_d}\left[e^{-\zeta\omega_n(t-\tau)} \left\{ \frac{\zeta\omega_n \sin \omega_d(t - \tau) + \omega_d \cos \omega_d(t - \tau)}{(\zeta\omega_n)^2 + (\omega_d)^2} \right\} \right]_{\tau=0}^{t} \\ &= \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1 - \zeta^2}} \cdot e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) \right] \end{aligned} \tag{E.1}$$

where

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \tag{E.2}$$

This response is shown in Fig. 4.10(c). If the system is undamped ($\zeta = 0$ and $\omega_d = \omega_n$), Eq. (E.1) reduces to

$$x(t) = \frac{F_0}{k}[1 - \cos \omega_n t] \tag{E.3}$$

Equation (E.3) is shown graphically in Fig. 4.10(d). It can be seen that if the load is instantaneously applied to an undamped system, a maximum displacement of twice the static displacement will be attained—that is, $x_{max} = 2F_0/k$.

■

EXAMPLE 4.10 Time-Delayed Step Force

Find the response of the compacting machine shown in Fig. 4.10(a) when it is subjected to the force shown in Fig. 4.11.

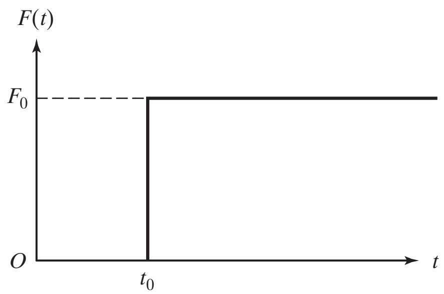


FIGURE 4.11 Step force applied with a time delay.

Solution: Since the forcing function starts at $t = t_0$ instead of at $t = 0$, the response can be obtained from Eq. (E.1) of Example 4.9 by replacing t by $t - t_0$. This gives

$$x(t) = \frac{F_0}{k\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2} - e^{-\zeta\omega_n(t-t_0)}\cos\{\omega_d(t-t_0) - \phi\}\right] \tag{E.1}$$

If the system is undamped, Eq. (E.1) reduces to

$$x(t) = \frac{F_0}{k}[1 - \cos\omega_n(t-t_0)] \tag{E.2}$$

■

EXAMPLE 4.11 Rectangular Pulse Load

If the compacting machine shown in Fig. 4.10(a) is subjected to a constant force only during the time $0 \le t \le t_0$ (Fig. 4.12a), determine the response of the machine.

Solution: The given forcing function, $F(t)$, can be considered as the sum of a step function $F_1(t)$ of magnitude $+F_0$ beginning at $t = 0$ and a second step function $F_2(t)$ of magnitude $-F_0$ starting at time $t = t_0$, as shown in Fig. 4.12(b).

Thus the response of the system can be obtained by subtracting Eq. (E.1) of Example 4.10 from Eq. (E.1) of Example 4.9. This gives

$$x(t) = \frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\left[-\cos(\omega_d t - \phi) + e^{\zeta\omega_n t_0}\cos\{\omega_d(t-t_0) - \phi\}\right] \tag{E.1}$$

with

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{E.2}$$

To see the vibration response graphically, we consider the system as undamped, so that Eq. (E.1) reduces to

$$x(t) = \frac{F_0}{k}\left[\cos\omega_n(t-t_0) - \cos\omega_n t\right] \tag{E.3}$$

The response is shown in Fig. 4.12(c) for two different pulse widths of t_0 for the following data (Problem 4.90): $m = 100$ kg, $c = 50$ N-s/m, $k = 1200$ N/m, and $F_0 = 100$ N. The responses will be different for the two cases $t_0 > \tau_n/2$ and $t_0 > \tau_n/2$, where τ_n is the undamped natural time period of the system. If $t_0 > \tau_n/2$, the peak will be larger and occur during the forced-vibration era (that is, during 0 to t_0) while the peak will be smaller and occur in the residual-vibration era (that is, after t_0) if $t_0 > \tau_n/2$. In Fig. 4.12(c), $\tau_n = 1.8138$ s and the peak corresponding to $t_0 = 1.5$ s is about six times larger than the one with $t_0 = 0.1$ s.

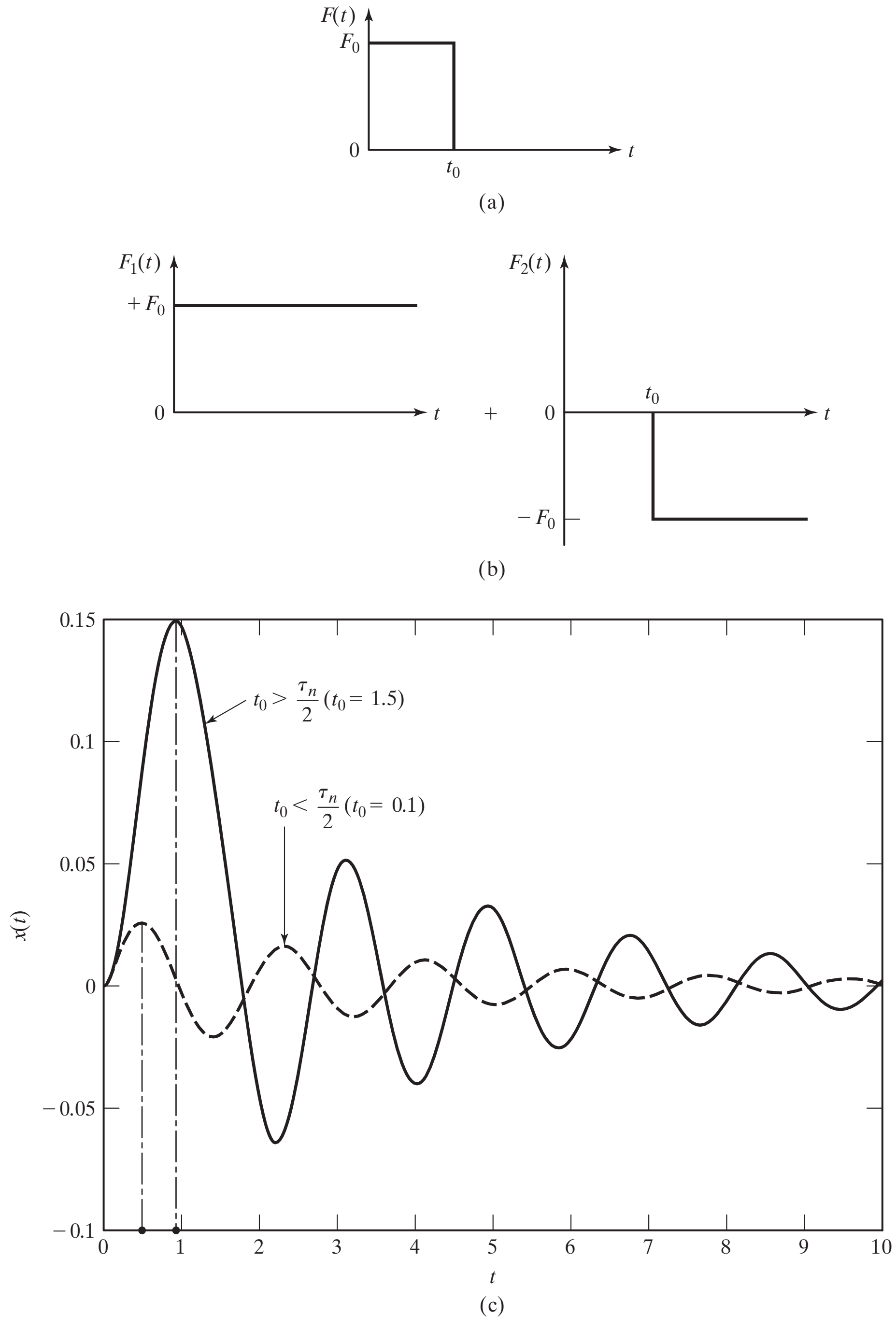


FIGURE 4.12 Response due to a pulse load.

■

EXAMPLE 4.12 Compacting Machine Under Linear Force

Determine the response of the compacting machine shown in Fig. 4.13(a) when a linearly varying force (shown in Fig. 4.13(b)) is applied due to the motion of the cam.

Solution: The linearly varying force shown in Fig. 4.13(b) is known as the ramp function. This forcing function can be represented as $F(\tau) = \delta F \cdot \tau$, where δF denotes the rate of increase of the force F per unit time. By substituting this into Eq. (4.31), we obtain

$$x(t) = \frac{\delta F}{m\omega_d}\int_0^t \tau e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau)\, d\tau$$

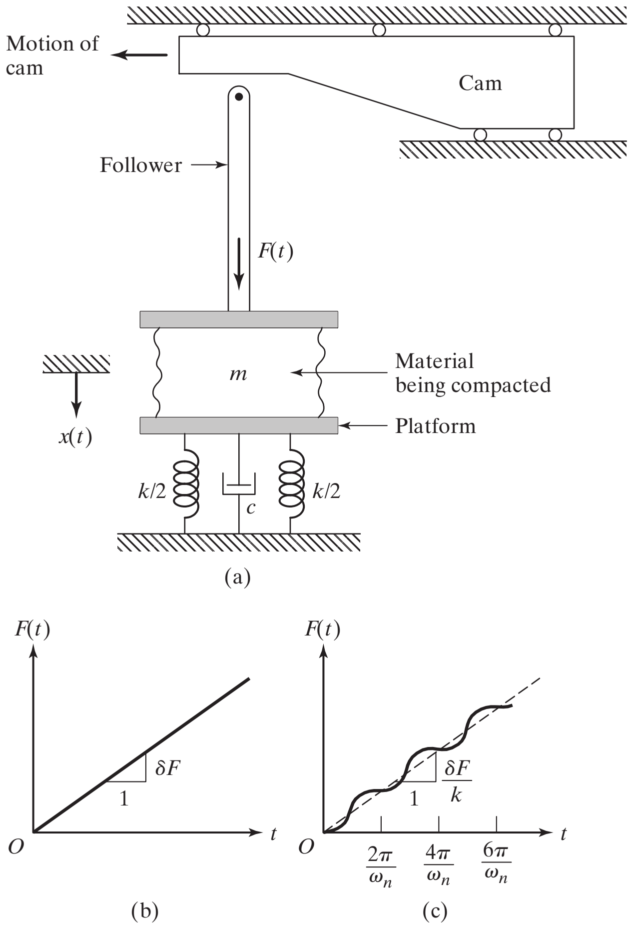


FIGURE 4.13 Compacting machine subjected to a linear force.

$$= \frac{\delta F}{m\omega_d}\int_0^t (t-\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,(-d\tau)$$

$$-\frac{\delta F\cdot t}{m\omega_d}\int_0^t e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,(-d\tau)$$

These integrals can be evaluated and the response expressed as follows:

$$x(t) = \frac{\delta F}{k}\left[t - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n t}\left(\frac{2\zeta}{\omega_n}\cos\omega_d t - \left\{\frac{\omega_d^2 - \zeta^2\omega_n^2}{\omega_n^2\omega_d}\right\}\sin\omega_d t\right)\right] \tag{E.1}$$

(See Problem 4.28.) For an undamped system, Eq. (E.1) reduces to

$$x(t) = \frac{\delta F}{\omega_n k}[\omega_n t - \sin\omega_n t] \tag{E.2}$$

Figure 4.13(c) shows the response given by Eq. (E.2).

■

EXAMPLE 4.13 Blast Load on a Building Frame

A building frame is modeled as an undamped single-degree-of-freedom system (Fig. 4.14(a)). Find the response of the frame if it is subjected to a blast loading represented by the triangular pulse shown in Fig. 4.14(b).

Solution: The forcing function is given by

$$F(\tau) = F_0\left(1 - \frac{\tau}{t_0}\right) \qquad \text{for } 0 \le \tau \le t_0 \tag{E.1}$$

$$F(\tau) = 0 \qquad \text{for } \tau > t_0 \tag{E.2}$$

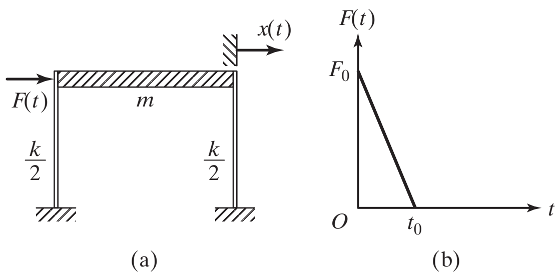


FIGURE 4.14 Building frame subjected to a blast load.

Equation (4.31) gives, for an undamped system,

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t - \tau)\, d\tau \tag{E.3}$$

Response during $0 \le t \le t_0$: Using Eq. (E.1) for $F(\tau)$ in Eq. (E.3) gives

$$\begin{aligned} x(t) &= \frac{F_0}{m\omega_n^2} \int_0^t \left(1 - \frac{\tau}{t_0}\right) [\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau]\, d(\omega_n \tau) \\ &= \frac{F_0}{k} \sin \omega_n t \int_0^t \left(1 - \frac{\tau}{t_0}\right) \cos \omega_n \tau \cdot d(\omega_n \tau) \\ &\quad - \frac{F_0}{k} \cos \omega_n t \int_0^t \left(1 - \frac{\tau}{t_0}\right) \sin \omega_n \tau \cdot d(\omega_n \tau) \end{aligned} \tag{E.4}$$

By noting that integration by parts gives

$$\int \tau \cos \omega_n \tau \cdot d(\omega_n \tau) = \tau \sin \omega_n \tau + \frac{1}{\omega_n} \cos \omega_n \tau \tag{E.5}$$

and

$$\int \tau \sin \omega_n \tau \cdot d(\omega_n \tau) = -\tau \cos \omega_n \tau + \frac{1}{\omega_n} \sin \omega_n \tau \tag{E.6}$$

Eq. (E.4) can be written as

$$\begin{aligned} x(t) = \frac{F_0}{k} \Bigg\{ &\sin \omega_n t \left[\sin \omega_n t - \frac{t}{t_0} \sin \omega_n t - \frac{1}{\omega_n t_0} \cos \omega_n t + \frac{1}{\omega_n t_0} \right] \\ &- \cos \omega_n t \left[-\cos \omega_n t + 1 + \frac{t}{t_0} \cos \omega_n t - \frac{1}{\omega_n t_0} \sin \omega_n t \right] \Bigg\} \end{aligned} \tag{E.7}$$

Simplifying this expression, we obtain

$$x(t) = \frac{F_0}{k} \left[1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t \right] \tag{E.8}$$

Response during $t > t_0$: Here also we use Eq. (E.1) for $F(\tau)$, but the upper limit of integration in Eq. (E.3) will be t_0, since $F(\tau) = 0$ for $\tau > t_0$. Thus the response can be found from Eq. (E.7) by setting $t = t_0$ within the square brackets. This results in

$$x(t) = \frac{F_0}{k\omega_n t_0} \left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t \right] \tag{E.9}$$

■

Daniel Bernoulli (1700–1782) was a Swiss who became a professor of mathematics at Saint Petersburg in 1725 after receiving his doctorate in medicine for his thesis on the action of lungs. He later became professor of anatomy and botany at Basel. He developed the theory of hydrostatics and hydrodynamics, and "Bernoulli's theorem" is well known to engineers. He derived the equation of motion for the vibration of beams (the Euler-Bernoulli theory) and studied the problem of vibrating strings. Bernoulli was the first person to propose the principle of superposition of harmonics in free vibration.
(A photo of a portrait courtesy of David Eugene Smith, *History of Mathematics*, Volume 1—*General Survey of the History of Elementary Mathematics*. Dover Publications, New York, 1958.)

CHAPTER 5

Two-Degree-of-Freedom Systems

Chapter Outline

This chapter deals with two-degree-of-freedom systems, which require two independent coordinates to describe their motion. The coupled equations of motion of the system are derived using Newton's second law of motion. By expressing these equations in matrix form, the mass, damping, and stiffness matrices of the system are identified. By assuming harmonic motion of the two masses, the eigenvalues or natural frequencies of vibration, the modal vectors, and the free-vibration solution of the undamped system are found. The method of incorporating the initial conditions is also outlined. The two-degrees-of-freedom torsional systems are considered in an analogous manner. The concepts of coordinate coupling, generalized coordinates, and principal coordinates are introduced with examples. The forced-vibration analysis of the system under the complex form of harmonic force is presented and the impedance matrix is identified. The semidefinite, unrestricted, or degenerate systems are introduced along with a method of finding their natural frequencies of vibration. The self-excitation and stability analysis of two-degrees-of-freedom systems are considered along with a derivation of the conditions of stability. The Routh-Hurwitz criterion, which can be used for deriving the conditions of stability of any n-degree-of-freedom system, is also introduced. The transfer-function approach, the computation of the response of two-degree-of-freedom systems using Laplace transform, and solutions using frequency transfer functions are also presented. Finally, the free- and forced-vibration solutions of two-degree-of-freedom systems using MATLAB are illustrated with examples.

Learning Objectives

After completing this chapter, you should be able to do the following:

- Formulate the equations of motion of two-degree-of-freedom systems.
- Identify the mass, damping, and stiffness matrices from the equations of motion.
- Compute the eigenvalues or natural frequencies of vibration and the modal vectors.
- Determine the free-vibration solution using the known initial conditions.
- Understand the concepts of coordinate coupling and principal coordinates.
- Determine the forced-vibration solutions under harmonic forces.
- Understand the concepts of self-excitation and stability of the system.
- Use the Laplace transform approach for solution of two-degree-of-freedom systems.
- Solve two-degree-of-freedom free- and forced-vibration problems using MATLAB.

5.1 Introduction

Systems that require two independent coordinates to describe their motion are called *two-degree-of-freedom systems*. Some examples of systems having two degrees of freedom were shown in Fig. 1.12. We shall consider only two-degree-of-freedom systems in this chapter, so as to provide a simple introduction to the behavior of systems with an arbitrarily large number of degrees of freedom, which is the subject of Chapter 6.

Consider a simplified model of a lathe shown in Fig. 5.1(a), in which the lathe bed, represented as an elastic beam, is supported on short elastic columns with the headstock and tailstock denoted as lumped masses attached to the beam [5.1–5.3]. For a simplified vibration analysis, the lathe can be treated as a rigid body of total mass m and mass moment of inertia J_0 about its center of gravity (C.G.), resting on springs of stiffness k_1 and k_2, as shown in Fig. 5.1(b). The displacement of the system at any time can be specified by a linear coordinate $x(t)$, indicating the vertical displacement of the C.G. of the mass,

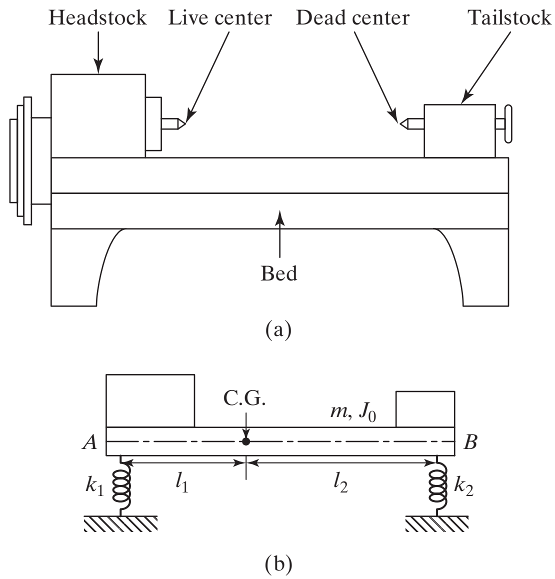


FIGURE 5.1 Lathe.

and an angular coordinate $\theta(t)$, denoting the rotation of the mass m about its C.G. Instead of $x(t)$ and $\theta(t)$, we can also use $x_1(t)$ and $x_2(t)$, the displacements of points A and B, as independent coordinates to specify the motion of the system. Thus the system has two degrees of freedom. It is important to note that in this case the mass m is treated not as a point mass but as a rigid body having two possible types of motion. (If it is a particle, there is no need to specify the rotation of the mass about its C.G.)

Similarly, consider the automobile shown in Fig. 5.2(a). For the vibration of the automobile in the vertical plane, a two-degree-of-freedom model shown in Fig. 5.2(b) can be used. Here the body is idealized as a bar of mass m and mass moment of inertia J_0, supported on the rear and front wheels (suspensions) of stiffness k_1 and k_2. The displacement of the automobile at any time can be specified by the linear coordinate $x(t)$ denoting the vertical displacement of the C.G. of the body and the angular coordinate $\theta(t)$ indicating the rotation (pitching) of the body about its C.G. Alternately, the motion of the automobile can be specified using the independent coordinates, $x_1(t)$ and $x_2(t)$, of points A and B.

Next, consider the motion of a multistory building under an earthquake. For simplicity, a two-degree-of-freedom model can be used as shown in Fig. 5.3. Here the building is modeled as a rigid bar having a mass m and mass moment of inertia J_0. The resistance offered to the motion of the building by the foundation and surrounding soil is approximated by a linear spring on stiffness k and a torsional spring of stiffness k_t. The displacement of the building at any time can be specified by the horizontal motion of the base $x(t)$ and the angular motion $\theta(t)$ about the point O. Finally, consider the system shown in Fig. 5.4(a), which illustrates the packaging of an instrument of mass m. Assuming that the motion of the instrument is confined to the xy-plane, the system can be modeled as a mass

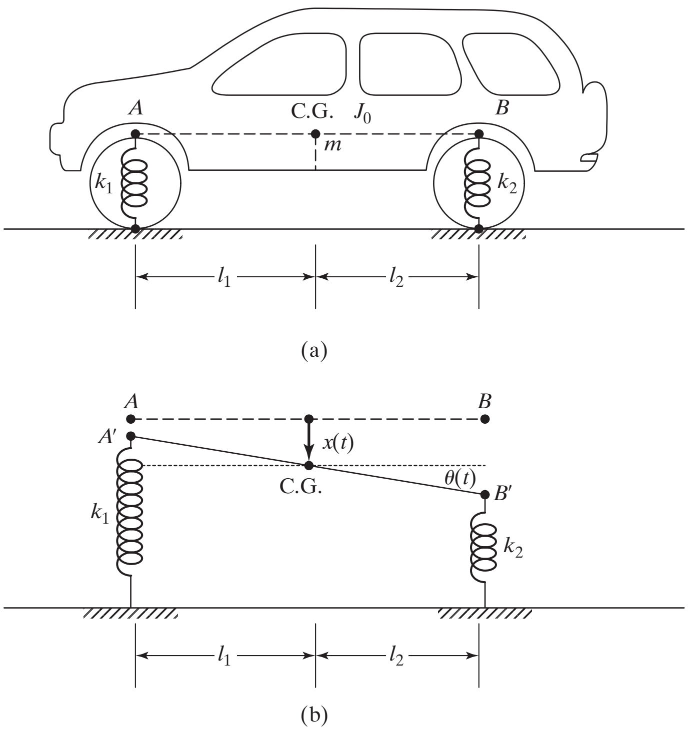


FIGURE 5.2 Automobile.

m supported by springs in the x and y directions, as indicated in Fig. 5.4(b). Thus the system has one point mass m and two degrees of freedom, because the mass has two possible types of motion (translations along the x and y directions). The general rule for the computation of the number of degrees of freedom can be stated as follows:

$$\begin{array}{c} \text{Number of} \\ \text{degrees of freedom} \\ \text{of the system} \end{array} = \begin{array}{c} \text{Number of masses in the system} \\ \times \text{ number of possible types} \\ \text{of motion of each mass} \end{array}$$

There are two equations of motion for a two-degree-of-freedom system, one for each mass (more precisely, for each degree of freedom). They are generally in the form of *coupled differential equations*—that is, each equation involves all the coordinates. If a harmonic solution is assumed for each coordinate, the equations of motion lead to a frequency equation that gives two natural frequencies for the system. If we give suitable initial excitation, the system vibrates at one of these natural frequencies. During free vibration at one of the natural frequencies, the amplitudes of the two degrees of freedom (coordinates) are related in a specific manner and the configuration is called a *normal mode, principal mode*, or

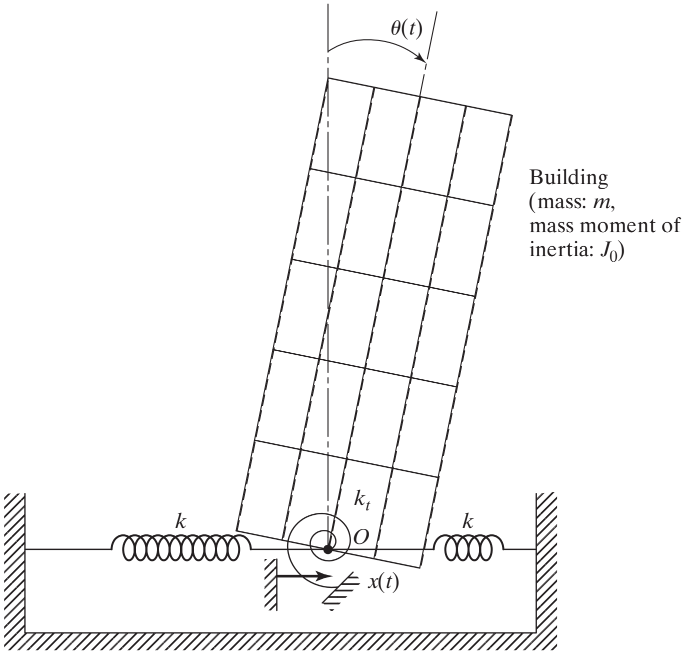


FIGURE 5.3 Multistory building subjected to an earthquake.

natural mode of vibration. Thus a two-degree-of-freedom system has two normal modes of vibration corresponding to the two natural frequencies.

If we give an arbitrary initial excitation to the system, the resulting free vibration will be a superposition of the two normal modes of vibration. However, if the system vibrates under the action of an external harmonic force, the resulting forced harmonic vibration takes place at the frequency of the applied force. Under harmonic excitation, resonance

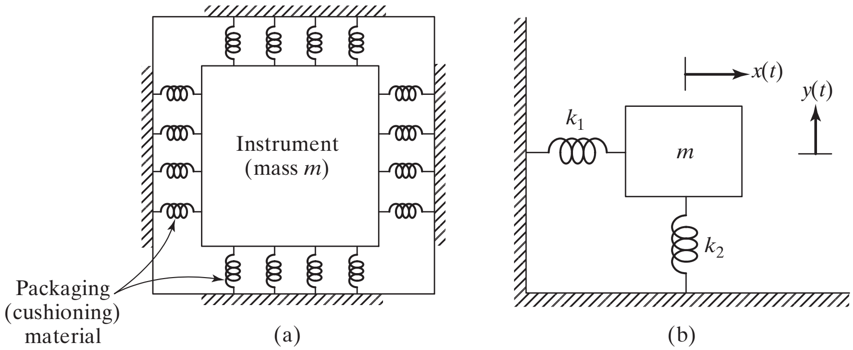


FIGURE 5.4 Packaging of an instrument.

occurs (i.e., the amplitudes of the two coordinates will be maximum) when the forcing frequency is equal to one of the natural frequencies of the system.

As is evident from the systems shown in Figs. 5.1–5.4, the configuration of a system can be specified by a set of independent coordinates such as length, angle, or some other physical parameters. Any such set of coordinates is called *generalized coordinates*. Although the equations of motion of a two-degree-of-freedom system are generally coupled so that each equation involves all the coordinates, it is always possible to find a particular set of coordinates such that each equation of motion contains only one coordinate. The equations of motion are then *uncoupled* and can be solved independently of each other. Such a set of coordinates, which leads to an uncoupled system of equations, is called *principal coordinates*.

5.2 Equations of Motion for Forced Vibration

Consider a viscously damped two-degree-of-freedom spring-mass system, shown in Fig. 5.5(a). The motion of the system is completely described by the coordinates $x_1(t)$ and $x_2(t)$, which define the positions of the masses m_1 and m_2 at any time t from the respective equilibrium positions. The external forces $F_1(t)$ and $F_2(t)$ act on the masses m_1 and m_2, respectively. The free-body diagrams of the masses m_1 and m_2 are shown in Fig. 5.5(b). The application of Newton's second law of motion to each of the masses gives the equations of motion:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1 \tag{5.1}$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2 \tag{5.2}$$

It can be seen that Eq. (5.1) contains terms involving x_2 (namely, $-c_2\dot{x}_2$ and $-k_2x_2$), whereas Eq. (5.2) contains terms involving x_1 (namely, $-c_2\dot{x}_1$ and $-k_2x_1$). Hence they

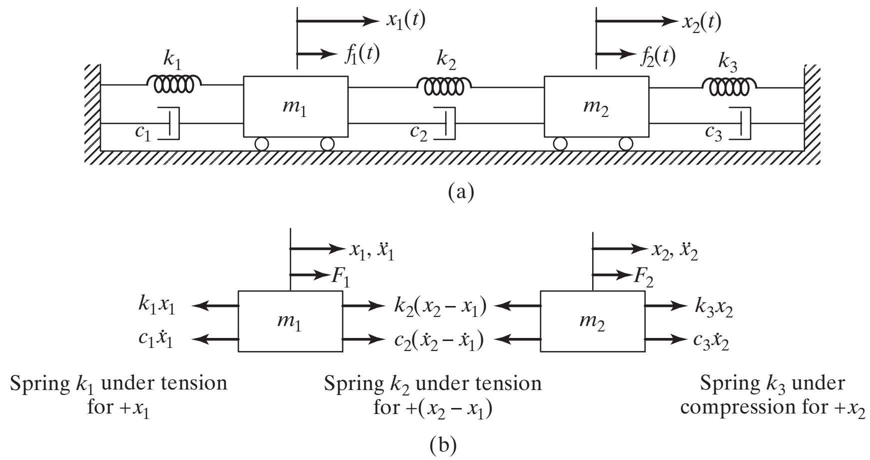


FIGURE 5.5 A two-degree-of-freedom spring-mass-damper system.

represent a system of two coupled second-order differential equations. We can therefore expect that the motion of the mass m_1 will influence the motion of the mass m_2, and vice versa. Equations (5.1) and (5.2) can be written in matrix form as

$$[m]\ \ddot{\vec{x}}(t) + [c]\ \dot{\vec{x}}(t) + [k]\ \vec{x}(t) = \vec{f}(t) \tag{5.3}$$

where $[m]$, $[c]$, and $[k]$ are called the *mass, damping*, and *stiffness matrices*, respectively, and are given by

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

and $\vec{x}(t)$ and $\vec{f}(t)$ are called the *displacement* and *force vectors*, respectively, and are given by

$$\vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

and

$$\vec{f}(t) = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

It can be seen that $[m]$, $[c]$, and $[k]$ are all 2×2 matrices whose elements are the known masses, damping coefficients, and stiffnesses of the system, respectively. Further, these matrices can be seen to be symmetric, so that

$$[m]^T = [m], \qquad [c]^T = [c], \qquad [k]^T = [k]$$

where the superscript T denotes the transpose of the matrix.

Notice that the equations of motion (5.1) and (5.2) become uncoupled (independent of one another) only when $c_2 = k_2 = 0$, which implies that the two masses m_1 and m_2 are not physically connected. In such a case, the matrices $[m]$, $[c]$, and $[k]$ become diagonal. The solution of the equations of motion (5.1) and (5.2) for any arbitrary forces $f_1(t)$ and $f_2(t)$ is difficult to obtain, mainly due to the coupling of the variables $x_1(t)$ and $x_2(t)$. The solution of Eqs. (5.1) and (5.2) involves four constants of integration (two for each equation). Usually the initial displacements and velocities of the two masses are specified as $x_1(t = 0) = x_1(0)$, $\dot{x}_1(t = 0) = \dot{x}_1(0)$, $x_2(t = 0) = x_2(0)$, and $\dot{x}_2(t = 0) = \dot{x}_2(0)$. We shall first consider the free-vibration solution of Eqs. (5.1) and (5.2).

5.3 Free-Vibration Analysis of an Undamped System

For the free-vibration analysis of the system shown in Fig. 5.5(a), we set $f_1(t) = f_2(t) = 0$. Further, if damping is disregarded, $c_1 = c_2 = c_3 = 0$, and the equations of motion (5.1) and (5.2) reduce to

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0 \tag{5.4}$$

$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0 \tag{5.5}$$

We are interested in knowing whether m_1 and m_2 can oscillate harmonically with the same frequency and phase angle but with different amplitudes. Assuming that it is possible to have harmonic motion of m_1 and m_2 at the same frequency ω and the same phase angle ϕ, we take the solutions of Eqs. (5.4) and (5.5) as

$$\begin{aligned} x_1(t) &= X_1 \cos(\omega t + \phi) \\ x_2(t) &= X_2 \cos(\omega t + \phi) \end{aligned} \tag{5.6}$$

where X_1 and X_2 are constants that denote the maximum amplitudes of $x_1(t)$ and $x_2(t)$, and ϕ is the phase angle. Substituting Eq. (5.6) into Eqs. (5.4) and (5.5), we obtain

$$\begin{aligned} [\{-m_1\omega^2 + (k_1 + k_2)\}X_1 - k_2X_2]\cos(\omega t + \phi) &= 0 \\ [-k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\}X_2]\cos(\omega t + \phi) &= 0 \end{aligned} \tag{5.7}$$

Since Eq. (5.7) must be satisfied for all values of the time t, the terms between brackets must be zero. This yields

$$\begin{aligned} \{-m_1\omega^2 + (k_1 + k_2)\}X_1 - k_2X_2 &= 0 \\ -k_2X_1 + \{-m_2\omega^2 + (k_2 + k_3)\}X_2 &= 0 \end{aligned} \tag{5.8}$$

which represent two simultaneous homogenous algebraic equations in the unknowns X_1 and X_2. It can be seen that Eq. (5.8) is satisfied by the trivial solution $X_1 = X_2 = 0$, which implies that there is no vibration. For a nontrivial solution of X_1 and X_2, the determinant of the coefficients of X_1 and X_2 must be zero:

$$\det\begin{bmatrix} \{-m_1\omega^2 + (k_1 + k_2)\} & -k_2 \\ -k_2 & \{-m_2\omega^2 + (k_2 + k_3)\} \end{bmatrix} = 0$$

or

$$\begin{aligned} (m_1m_2)\omega^4 &- \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}\omega^2 \\ &+ \{(k_1 + k_2)(k_2 + k_3) - k_2^2\} = 0 \end{aligned} \tag{5.9}$$

Equation (5.9) is called the *frequency* or *characteristic equation* because its solution yields the frequencies or the characteristic values of the system. The roots of Eq. (5.9) are given by

$$\begin{aligned}\omega_1^2, \omega_2^2 &= \frac{1}{2}\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1 m_2}\right\} \\ &\mp \frac{1}{2}\left[\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1 m_2}\right\}^2 \right. \\ &\left. - 4\left\{\frac{(k_1+k_2)(k_2+k_3)-k_2^2}{m_1 m_2}\right\}\right]^{1/2}\end{aligned} \tag{5.10}$$

This shows that it is possible for the system to have a nontrivial harmonic solution of the form of Eq. (5.6) when ω is equal to ω_1 and ω_2 given by Eq. (5.10). We call ω_1 and ω_2 the *natural frequencies* of the system.

The values of X_1 and X_2 remain to be determined. These values depend on the natural frequencies ω_1 and ω_2. We shall denote the values of X_1 and X_2 corresponding to ω_1 as $X_1^{(1)}$ and $X_2^{(1)}$ and those corresponding to ω_2 as $X_1^{(2)}$ and $X_2^{(2)}$. Further, since Eq. (5.8) is homogenous, only the ratios $r_1 = \{X_2^{(1)}/X_1^{(1)}\}$ and $r_2 = \{X_2^{(2)}/X_1^{(2)}\}$ can be found. For $\omega^2 = \omega_1^2$ and $\omega^2 = \omega_2^2$, Eq. (5.8) gives

$$\begin{aligned} r_1 &= \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2+(k_1+k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2+(k_2+k_3)} \\ r_2 &= \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2+(k_1+k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2+(k_2+k_3)}\end{aligned} \tag{5.11}$$

Notice that the two ratios given for each $r_i\,(i = 1, 2)$ in Eq. (5.11) are identical. The normal modes of vibration corresponding to ω_1^2 and ω_2^2 can be expressed, respectively, as

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \\ r_1 X_1^{(1)} \end{Bmatrix}$$

and

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \\ r_2 X_1^{(2)} \end{Bmatrix} \tag{5.12}$$

The vectors $\vec{X}^{(1)}$ and $\vec{X}^{(2)}$, which denote the normal modes of vibration, are known as the *modal vectors* of the system. The free-vibration solution or the motion in time can be expressed, using Eq. (5.6), as

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \text{first mode}$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} = \text{second mode} \tag{5.13}$$

where the constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 are determined by the initial conditions.

Initial Conditions. As stated earlier, each of the two equations of motion, Eqs. (5.1) and (5.2), involves second-order time derivatives; hence we need to specify two initial conditions for each mass. As stated in Section 5.1, the system can be made to vibrate in its ith normal mode $(i = 1, 2)$ by subjecting it to the specific initial conditions

$$x_1(t = 0) = X_1^{(i)} = \text{some constant}, \qquad \dot{x}_1(t = 0) = 0,$$

$$x_2(t = 0) = r_i X_1^{(i)}, \qquad \dot{x}_2(t = 0) = 0$$

However, for any other general initial conditions, both modes will be excited. The resulting motion, which is given by the general solution of Eqs. (5.4) and (5.5), can be obtained by a linear superposition of the two normal modes, Eq. (5.13):

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) \tag{5.14}$$

where c_1 and c_2 are constants. Since $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ already involve the unknown constants $X_1^{(1)}$ and $X_1^{(2)}$ (see Eq. (5.13)), we can choose $c_1 = c_2 = 1$ with no loss of generality. Thus the components of the vector $\vec{x}(t)$ can be expressed, using Eq. (5.14) with $c_1 = c_2 = 1$ and Eq. (5.13), as

$$\begin{aligned} x_1(t) &= x_1^{(1)}(t) + x_1^{(2)}(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ x_2(t) &= x_2^{(1)}(t) + x_2^{(2)}(t) \\ &= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{aligned} \tag{5.15}$$

where the unknown constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 can be determined from the initial conditions:

$$\begin{aligned} x_1(t = 0) &= x_1(0), \qquad \dot{x}_1(t = 0) = \dot{x}_1(0), \\ x_2(t = 0) &= x_2(0), \qquad \dot{x}_2(t = 0) = \dot{x}_2(0) \end{aligned} \tag{5.16}$$

Substitution of Eq. (5.16) into Eq. (5.15) leads to

$$x_1(0) = X_1^{(1)} \cos \phi_1 + X_1^{(2)} \cos \phi_2$$

$$\dot{x}_1(0) = -\omega_1 X_1^{(1)} \sin \phi_1 - \omega_2 X_1^{(2)} \sin \phi_2$$

$$x_2(0) = r_1 X_1^{(1)} \cos\phi_1 + r_2 X_1^{(2)} \cos\phi_2$$
$$\dot{x}_2(0) = -\omega_1 r_1 X_1^{(1)} \sin\phi_1 - \omega_2 r_2 X_1^{(2)} \sin\phi_2 \tag{5.17}$$

Equation (5.17) can be regarded as four algebraic equations in the unknowns $X_1^{(1)}\cos\phi_1$, $X_1^{(2)}\cos\phi_2$, $X_1^{(1)}\sin\phi_1$, and $X_1^{(2)}\sin\phi_2$. The solution of Eq. (5.17) can be expressed as

$$X_1^{(1)}\cos\phi_1 = \left\{\frac{r_2 x_1(0) - x_2(0)}{r_2 - r_1}\right\}, \qquad X_1^{(2)}\cos\phi_2 = \left\{\frac{-r_1 x_1(0) + x_2(0)}{r_2 - r_1}\right\}$$
$$X_1^{(1)}\sin\phi_1 = \left\{\frac{-r_2 \dot{x}_1(0) + \dot{x}_2(0)}{\omega_1(r_2 - r_1)}\right\}, \qquad X_1^{(2)}\sin\phi_2 = \left\{\frac{r_1 \dot{x}_1(0) - \dot{x}_2(0)}{\omega_2(r_2 - r_1)}\right\}$$

from which we obtain the desired solution:

$$X_1^{(1)} = [\{X_1^{(1)}\cos\phi_1\}^2 + \{X_1^{(1)}\sin\phi_1\}^2]^{1/2}$$
$$= \frac{1}{(r_2 - r_1)}\left[\{r_2 x_1(0) - x_2(0)\}^2 + \frac{\{-r_2\dot{x}_1(0) + \dot{x}_2(0)\}^2}{\omega_1^2}\right]^{1/2}$$
$$X_1^{(2)} = [\{X_1^{(2)}\cos\phi_2\}^2 + \{X_1^{(2)}\sin\phi_2\}^2]^{1/2}$$
$$= \frac{1}{(r_2 - r_1)}\left[\{-r_1 x_1(0) + x_2(0)\}^2 + \frac{\{r_1\dot{x}_1(0) - \dot{x}_2(0)\}^2}{\omega_2^2}\right]^{1/2}$$
$$\phi_1 = \tan^{-1}\left\{\frac{X_1^{(1)}\sin\phi_1}{X_1^{(1)}\cos\phi_1}\right\} = \tan^{-1}\left\{\frac{-r_2\dot{x}_1(0) + \dot{x}_2(0)}{\omega_1[r_2 x_1(0) - x_2(0)]}\right\}$$
$$\phi_2 = \tan^{-1}\left\{\frac{X_1^{(2)}\sin\phi_2}{X_1^{(2)}\cos\phi_2}\right\} = \tan^{-1}\left\{\frac{r_1\dot{x}_1(0) - \dot{x}_2(0)}{\omega_2[-r_1 x_1(0) + x_2(0)]}\right\} \tag{5.18}$$

EXAMPLE 5.1

Frequencies of Spring-Mass System

Find the natural frequencies and mode shapes of a spring-mass system, shown in Fig. 5.6, which is constrained to move in the vertical direction only. Take $n = 1$.

Solution: If we measure x_1 and x_2 from the static equilibrium positions of the masses m_1 and m_2, respectively, the equations of motion and the solution obtained for the system of Fig. 5.5(a) are also applicable to this case if we substitute $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. Thus the equations of motion, Eqs. (5.4) and (5.5), are given by

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 - kx_1 + 2kx_2 = 0 \tag{E.1}$$

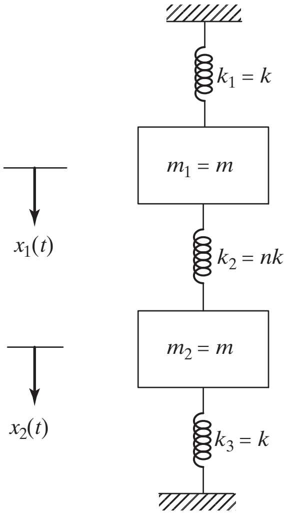


FIGURE 5.6 Two-degree-of-freedom system.

By assuming harmonic solution as

$$x_i(t) = X_i \cos(\omega t + \phi); \; i = 1, 2 \tag{E.2}$$

the frequency equation can be obtained by substituting Eq. (E.2) into Eq. (E.1):

$$\begin{vmatrix} (-m\omega^2 + 2k) & (-k) \\ (-k) & (-m\omega^2 + 2k) \end{vmatrix} = 0$$

or

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0 \tag{E.3}$$

The solution of Eq. (E.3) gives the natural frequencies

$$\omega_1 = \left\{ \frac{4km - [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2} \right\}^{1/2} = \sqrt{\frac{k}{m}} \tag{E.4}$$

$$\omega_2 = \left\{ \frac{4km + [16k^2m^2 - 12m^2k^2]^{1/2}}{2m^2} \right\}^{1/2} = \sqrt{\frac{3k}{m}} \tag{E.5}$$

From Eq. (5.11), the amplitude ratios are given by

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m\omega_1^2 + 2k}{k} = \frac{k}{-m\omega_1^2 + 2k} = 1 \tag{E.6}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m\omega_2^2 + 2k}{k} = \frac{k}{-m\omega_2^2 + 2k} = -1 \tag{E.7}$$

The natural modes are given by Eq. (5.13):

$$\text{First mode} = \vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\, t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\dfrac{k}{m}}\, t + \phi_1\right) \end{Bmatrix} \tag{E.8}$$

$$\text{Second mode} = \vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\, t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\, t + \phi_2\right) \end{Bmatrix} \tag{E.9}$$

It can be seen from Eq. (E.8) that when the system vibrates in its first mode, the amplitudes of the two masses remain the same. This implies that the length of the middle spring remains constant. Thus the motions of m_1 and m_2 are in phase (see Fig. 5.7(a)). When the system vibrates in its second mode, Eq. (E.9) shows that the displacements of the two masses have the same magnitude with opposite signs. Thus the motions of m_1 and m_2 are 180° out of phase (see Fig. 5.7(b)). In this case the midpoint of the middle spring remains stationary for all time t. Such a point is called a *node*. Using Eq. (5.15), the motion (general solution) of the system can be expressed as

$$\begin{aligned} x_1(t) &= X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \\ x_2(t) &= X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) - X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right) \end{aligned} \tag{E.10}$$

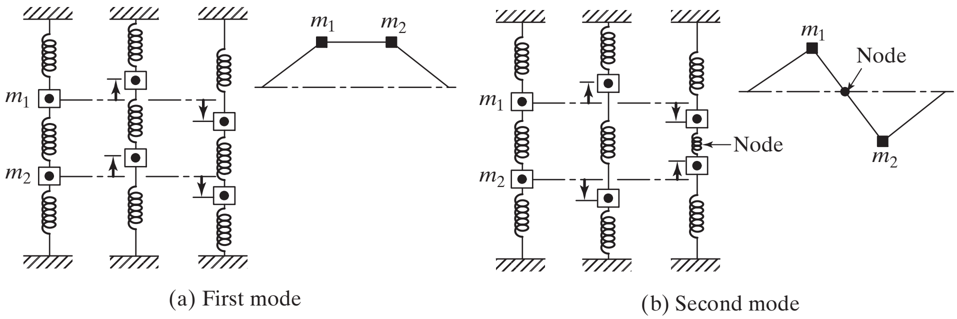


FIGURE 5.7 Modes of vibration.

■

Note: It can be seen that the computation of the natural frequencies and mode shapes is lengthy and tedious. Computer programs can be used conveniently for the numerical computation of the natural frequencies and mode shapes of multidegree-of-freedom systems (see Section 5.12).

EXAMPLE 5.2 Initial Conditions to Excite Specific Mode

Find the initial conditions that need to be applied to the system shown in Fig. 5.6 so as to make it vibrate in (a) the first mode, and (b) the second mode.

Solution:

Approach: Specify the solution to be obtained for the first or second mode from the general solution for arbitrary initial conditions and solve the resulting equations.

For arbitrary initial conditions, the motion of the masses is described by Eq. (5.15). In the present case, $r_1 = 1$ and $r_2 = -1$, so Eq. (5.15) reduces to Eq. (E.10) of Example 5.1:

$$\begin{aligned} x_1(t) &= X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_2\right) \\ x_2(t) &= X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) - X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\,t + \phi_2\right) \end{aligned} \tag{E.1}$$

Assuming the initial conditions as in Eq. (5.16), the constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 can be obtained from Eq. (5.18), using $r_1 = 1$ and $r_2 = -1$:

$$X_1^{(1)} = -\frac{1}{2}\left\{[x_1(0) + x_2(0)]^2 + \frac{m}{k}[\dot{x}_1(0) + \dot{x}_2(0)]^2\right\}^{1/2} \tag{E.2}$$

$$X_1^{(2)} = -\frac{1}{2}\left\{[-x_1(0) + x_2(0)]^2 + \frac{m}{3k}[\dot{x}_1(0) - \dot{x}_2(0)]^2\right\}^{1/2} \tag{E.3}$$

$$\phi_1 = \tan^{-1}\left\{\frac{-\sqrt{m}\,[\dot{x}_1(0) + \dot{x}_2(0)]}{\sqrt{k}\,[x_1(0) + x_2(0)]}\right\} \tag{E.4}$$

$$\phi_2 = \tan^{-1}\left\{\frac{\sqrt{m}\,[\dot{x}_1(0) - \dot{x}_2(0)]}{\sqrt{3k}\,[-x_1(0) + x_2(0)]}\right\} \tag{E.5}$$

a. The first normal mode of the system is given by Eq. (E.8) of Example 5.1:

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) \\ X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) \end{Bmatrix} \tag{E.6}$$

Comparison of Eqs. (E.1) and (E.6) shows that the motion of the system is identical with the first normal mode only if $X_1^{(2)} = 0$. This requires that (from Eq. E.3)

$$x_1(0) = x_2(0) \qquad \text{and} \qquad \dot{x}_1(0) = \dot{x}_2(0) \tag{E.7}$$

b. The second normal mode of the system is given by Eq. (E.9) of Example 5.1:

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\,t + \phi_2\right) \\ -X_1^{(2)} \cos\left(\sqrt{\dfrac{3k}{m}}\,t + \phi_2\right) \end{Bmatrix} \tag{E.8}$$

Comparison of Eqs. (E.1) and (E.8) shows that the motion of the system coincides with the second normal mode only if $X_1^{(1)} = 0$. This implies that (from Eq. E.2)

$$x_1(0) = -x_2(0) \qquad \text{and} \qquad \dot{x}_1(0) = -\dot{x}_2(0) \tag{E.9}$$

■

EXAMPLE 5.3

Free-Vibration Response of a Two-Degree-of-Freedom System

Find the free-vibration response of the system shown in Fig. 5.5(a) with $k_1 = 30$, $k_2 = 5$, $k_3 = 0$, $m_1 = 10$, $m_2 = 1$, and $c_1 = c_2 = c_3 = 0$ for the initial conditions $x_1(0) = 1$, $\dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$.

Solution: For the given data, the eigenvalue problem, Eq. (5.8), becomes

$$\begin{bmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or

$$\begin{bmatrix} -10\omega^2 + 35 & -5 \\ -5 & -\omega^2 + 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E.1}$$

By setting the determinant of the coefficient matrix in Eq. (E.1) to zero, we obtain the frequency equation (see Eq. (5.9)):

$$10\omega^4 - 85\omega^2 + 150 = 0 \tag{E.2}$$

from which the natural frequencies can be found as

$$\omega_1^2 = 2.5, \qquad \omega_2^2 = 6.0$$

or

$$\omega_1 = 1.5811, \qquad \omega_2 = 2.4495 \tag{E.3}$$

The substitution of $\omega^2 = \omega_1^2 = 2.5$ in Eq. (E.1) leads to $X_2^{(1)} = 2X_1^{(1)}$, while $\omega^2 = \omega_2^2 = 6.0$ in Eq. (E.1) yields $X_2^{(2)} = -5X_1^{(2)}$. Thus the normal modes (or eigenvectors) are given by

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} X_1^{(1)} \tag{E.4}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -5 \end{Bmatrix} X_1^{(2)} \tag{E.5}$$

The free-vibration responses of the masses m_1 and m_2 are given by (see Eq. (5.15)):

$$x_1(t) = X_1^{(1)} \cos(1.5811t + \phi_1) + X_1^{(2)} \cos(2.4495t + \phi_2) \tag{E.6}$$

$$x_2(t) = 2X_1^{(1)} \cos(1.5811t + \phi_1) - 5X_1^{(2)} \cos(2.4495t + \phi_2) \tag{E.7}$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 are constants to be determined from the initial conditions. By using the given intial conditions in Eqs. (E.6) and (E.7), we obtain

$$x_1(t = 0) = 1 = X_1^{(1)} \cos \phi_1 + X_1^{(2)} \cos \phi_2 \tag{E.8}$$

$$x_2(t = 0) = 0 = 2X_1^{(1)} \cos \phi_1 - 5X_1^{(2)} \cos \phi_2 \tag{E.9}$$

$$\dot{x}_1(t = 0) = 0 = -1.5811X_1^{(1)} \sin \phi_1 - 2.4495X_1^{(2)} \sin \phi_2 \tag{E.10}$$

$$\dot{x}_2(t = 0) = -3.1622X_1^{(1)} + 12.2475X_1^{(2)} \sin \phi_2 \tag{E.11}$$

The solution of Eqs. (E.8) and (E.9) yields

$$X_1^{(1)} \cos \phi_1 = \frac{5}{7}, \qquad X_1^{(2)} \cos \phi_2 = \frac{2}{7} \tag{E.12}$$

while the solution of Eqs. (E.10) and (E.11) leads to

$$X_1^{(1)} \sin \phi_1 = 0, \qquad X_1^{(2)} \sin \phi_2 = 0 \tag{E.13}$$

Equations (E.12) and (E.13) give

$$X_1^{(1)} = \frac{5}{7}, \qquad X_1^{(2)} = \frac{2}{7}, \qquad \phi_1 = 0, \qquad \phi_2 = 0 \tag{E.14}$$

Thus the free-vibration responses of m_1 and m_2 are given by

$$x_1(t) = \frac{5}{7} \cos 1.5811t + \frac{2}{7} \cos 2.4495t \tag{E.15}$$

$$x_2(t) = \frac{10}{7} \cos 1.5811t - \frac{10}{7} \cos 2.4495t \tag{E.16}$$

The graphical representation of Eqs. (E.15) and (E.16) is considered in Example 5.17.

■

5.4 Torsional System

Consider a torsional system consisting of two discs mounted on a shaft, as shown in Fig. 5.8. The three segments of the shaft have rotational spring constants k_{t1}, k_{t2}, and k_{t3}, as indicated in the figure. Also shown are the discs of mass moments of inertia J_1 and J_2, the applied torques M_{t1} and M_{t2}, and the rotational degrees of freedom θ_1 and θ_2. The differential equations of rotational motion for the discs J_1 and J_2 can be derived as

$$J_1\ddot{\theta}_1 = -k_{t1}\theta_1 + k_{t2}(\theta_2 - \theta_1) + M_{t1}$$
$$J_2\ddot{\theta}_2 = -k_{t2}(\theta_2 - \theta_1) - k_{t3}\theta_2 + M_{t2}$$

which upon rearrangement become

$$\begin{aligned} J_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 &= M_{t1} \\ J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2} + k_{t3})\theta_2 &= M_{t2} \end{aligned} \tag{5.19}$$

For the free-vibration analysis of the system, Eq. (5.19) reduces to

$$\begin{aligned} J_1\ddot{\theta}_1 + (k_{t1} + k_{t2})\theta_1 - k_{t2}\theta_2 &= 0 \\ J_2\ddot{\theta}_2 - k_{t2}\theta_1 + (k_{t2} + k_{t3})\theta_2 &= 0 \end{aligned} \tag{5.20}$$

Note that Eq. (5.20) is similar to Eqs. (5.4) and (5.5). In fact, Eq. (5.20) can be obtained by substituting θ_1, θ_2, J_1, J_2, k_{t1}, k_{t2}, and k_{t3} for x_1, x_2, m_1, m_2, k_1, k_2, and k_3, respectively. Thus the analysis presented in Section 5.3 is also applicable to torsional systems with proper substitutions. The following two examples illustrate the procedure.

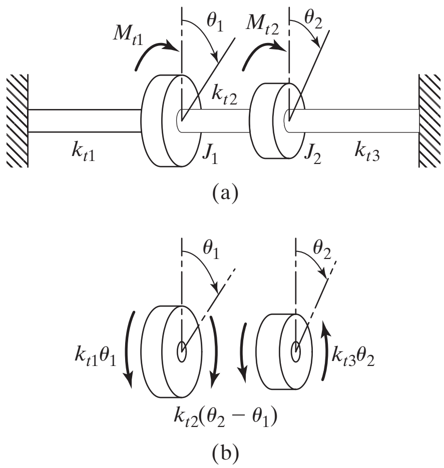


FIGURE 5.8 Torsional system with discs mounted on a shaft.

EXAMPLE 5.4 Natural Frequencies of a Torsional System

Find the natural frequencies and mode shapes for the torsional system shown in Fig. 5.9 for $J_1 = J_0$, $J_2 = 2J_0$, and $k_{t1} = k_{t2} = k_t$.

Solution: The differential equations of motion, Eq. (5.20), reduce to (with $k_{t3} = 0$, $k_{t1} = k_{t2} = k_t$, $J_1 = J_0$, and $J_2 = 2J_0$):

$$J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - k_t\theta_1 + k_t\theta_2 = 0 \tag{E.1}$$

Rearranging and substituting the harmonic solution

$$\theta_i(t) = \Theta_i \cos(\omega t + \phi); \qquad i = 1, 2 \tag{E.2}$$

gives the frequency equation:

$$2\omega^4 J_0^2 - 5\omega^2 J_0 k_t + k_t^2 = 0 \tag{E.3}$$

The solution of Eq. (E.3) gives the natural frequencies

$$\omega_1 = \sqrt{\frac{k_t}{4J_0}(5 - \sqrt{17})} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{k_t}{4J_0}(5 + \sqrt{17})} \tag{E.4}$$

The amplitude ratios are given by

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = 2 - \frac{(5 - \sqrt{17})}{4}$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = 2 - \frac{(5 + \sqrt{17})}{4} \tag{E.5}$$

Equations (E.4) and (E.5) can also be obtained by substituting $k_1 = k_{t1} = k_t$, $k_2 = k_{t2} = k_t$, $m_1 = J_1 = J_0$, $m_2 = J_2 = 2J_0$, and $k_3 = 0$ in Eqs. (5.10) and (5.11).

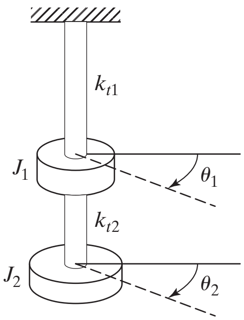


FIGURE 5.9
Torsional system.

■

Note: For a two-degree-of-freedom system, the two natural frequencies ω_1 and ω_2 are not equal to either of the natural frequencies of the two single-degree-of-freedom systems constructed from the same components. In Example 5.4, the single-degree-of-freedom systems k_{t1} and J_1

$$\left(\text{with } \overline{\omega}_1 = \sqrt{\frac{k_{t1}}{J_1}} = \sqrt{\frac{k_t}{J_0}}\right)$$

and k_{t2} and J_2

$$\left(\text{with } \overline{\omega}_2 = \sqrt{\frac{k_{t2}}{J_2}} = \frac{1}{\sqrt{2}}\sqrt{\frac{k_t}{J_0}}\right)$$

are combined to obtain the system shown in Fig. 5.9. It can be seen that ω_1 and ω_2 are different from $\overline{\omega}_1$ and $\overline{\omega}_2$.

EXAMPLE 5.5 Natural Frequencies of a Marine Engine Propeller

The schematic diagram of a marine engine connected to a propeller through gears is shown in Fig. 5.10(a). The mass moments of inertia of the flywheel, engine, gear 1, gear 2, and the propeller (in kg-m^2) are 9000, 1000, 250, 150, and 2000, respectively. Find the natural frequencies and mode shapes of the system in torsional vibration.

Solution

Approach: Find the equivalent mass moments of inertia of all rotors with respect to one rotor and use a two-degree-of-freedom model.

Assumptions:

1. The flywheel can be considered to be stationary (fixed), since its mass moment of inertia is very large compared to that of other rotors.
2. The engine and gears can be replaced by a single equivalent rotor.

Since gears 1 and 2 have 40 and 20 teeth, shaft 2 rotates at twice the speed of shaft 1. Thus the mass moments of inertia of gear 2 and the propeller, referred to the engine, are given by

$$(J_{G2})_{\text{eq}} = (2)^2(150) = 600 \text{ kg-m}^2$$

$$(J_P)_{\text{eq}} = (2)^2(2000) = 8000 \text{ kg-m}^2$$

Since the distance between the engine and the gear unit is small, the engine and the two gears can be replaced by a single rotor with a mass moment of inertia of

$$J_1 = J_E + J_{G1} + (J_{G2})_{\text{eq}} = 1000 + 250 + 600 = 1850 \text{ kg-m}^2$$

Assuming a shear modulus of 80×10^9 N/m^2 for steel, the torsional stiffnesses of shafts 1 and 2 can be determined as

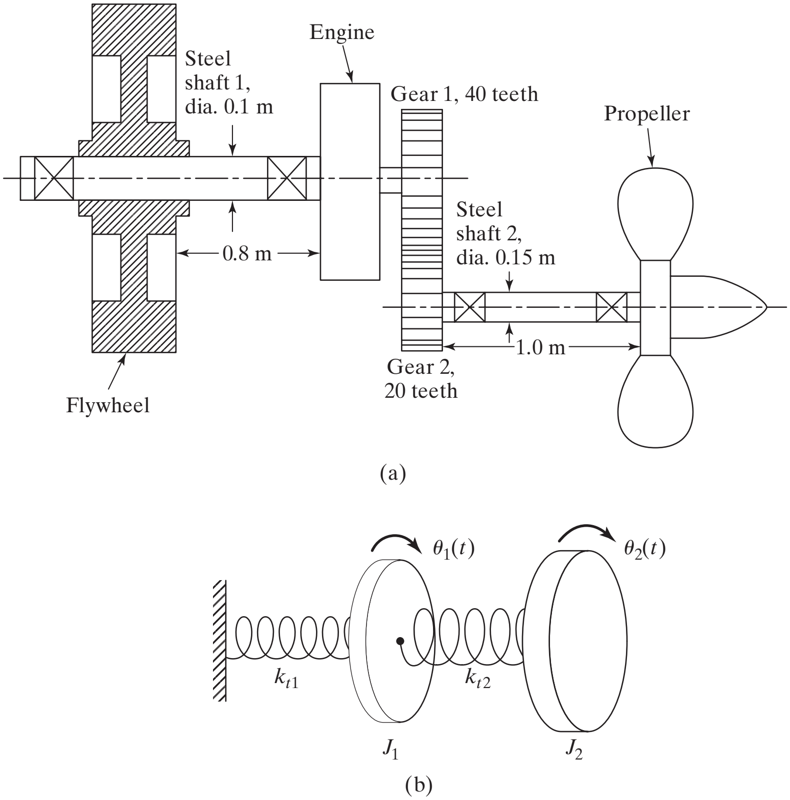


FIGURE 5.10 Marine engine propeller system.

$$k_{t1} = \frac{GI_{01}}{l_1} = \frac{G}{l_1}\left(\frac{\pi d_1^4}{32}\right) = \frac{(80 \times 10^9)(\pi)(0.10)^4}{(0.8)(32)} = 981{,}750.0 \text{ N-m/rad}$$

$$k_{t2} = \frac{GI_{02}}{l_2} = \frac{G}{l_2}\left(\frac{\pi d_2^4}{32}\right) = \frac{(80 \times 10^9)(\pi)(0.15)^4}{(1.0)(32)} = 3{,}976{,}087.5 \text{ N-m/rad}$$

Since the length of shaft 2 is not negligible, the propeller is assumed to be a rotor connected at the end of shaft 2. Thus the system can be represented as a two-degree-of-freedom torsional system, as indicated in Fig. 5.10(b). By setting $k_3 = 0$, $k_1 = k_{t1}$, $k_2 = k_{t2}$, $m_1 = J_1$, and $m_2 = J_2$ in Eq. (5.10), the natural frequencies of the system can be found as

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(k_{t1} + k_{t2})J_2 + k_{t2}J_1}{J_1 J_2}\right\}$$
$$\pm \left[\left\{\frac{(k_{t1} + k_{t2})J_2 + k_{t2}J_1}{J_1 J_2}\right\}^2 - 4\left\{\frac{(k_{t1} + k_{t2})k_{t2} - k_{t2}^2}{J_1 J_2}\right\}\right]^{1/2}$$

$$= \left\{ \frac{(k_{t1} + k_{t2})}{2J_1} + \frac{k_{t2}}{2J_2} \right\}$$

$$\pm \left[\left\{ \frac{(k_{t1} + k_{t2})}{2J_1} + \frac{k_{t2}}{2J_2} \right\}^2 - \frac{k_{t1}k_{t2}}{J_1 J_2} \right]^{1/2} \quad \text{(E.1)}$$

Since

$$\frac{(k_{t1} + k_{t2})}{2J_1} + \frac{k_{t2}}{2J_2} = \frac{(98.1750 + 397.6087) \times 10^4}{2 \times 1850} + \frac{397.6087 \times 10^4}{2 \times 8000}$$

$$= 1588.46$$

and

$$\frac{k_{t1}k_{t2}}{J_1 J_2} = \frac{(98.1750 \times 10^4)(397.6087 \times 10^4)}{(1850)(8000)} = 26.3750 \times 10^4$$

Eq. (E.1) gives

$$\omega_1^2, \omega_2^2 = 1588.46 \pm [(1588.46)^2 - 26.3750 \times 10^4]^{1/2}$$

$$= 1588.46 \pm 1503.1483$$

Thus

$$\omega_1^2 = 85.3117 \quad \text{or} \quad \omega_1 = 9.2364 \text{ rad/sec}$$

$$\omega_2^2 = 3091.6083 \quad \text{or} \quad \omega_2 = 55.6022 \text{ rad/sec}$$

For the mode shapes, we set $k_1 = k_{t1}$, $k_2 = k_{t2}$, $k_3 = 0$, $m_1 = J_1$, and $m_2 = J_2$ in Eq. (5.11) to obtain

$$r_1 = \frac{-J_1\omega_1^2 + (k_{t1} + k_{t2})}{k_{t2}}$$

$$= \frac{-(1850)(85.3117) + (495.7837 \times 10^4)}{397.6087 \times 10^4} = 1.2072$$

and

$$r_2 = \frac{-J_1\omega_2^2 + (k_{t1} + k_{t2})}{k_{t2}}$$

$$= \frac{-(1850)(3091.6083) + (495.7837 \times 10^4)}{397.6087 \times 10^4} = -0.1916$$

Thus the mode shapes can be determined from an equation similar to Eq. (5.12) as

$$\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ r_1 \end{Bmatrix} = \frac{1}{1.2072}$$

and

$$\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ r_2 \end{Bmatrix} = \frac{1}{-0.1916}$$

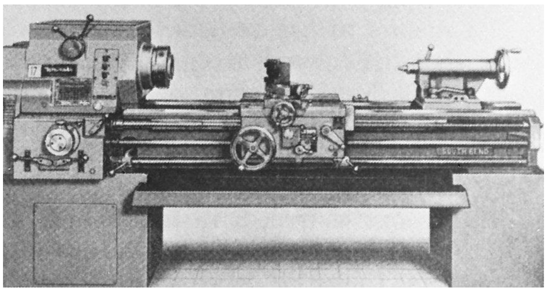
■

5.5 Coordinate Coupling and Principal Coordinates

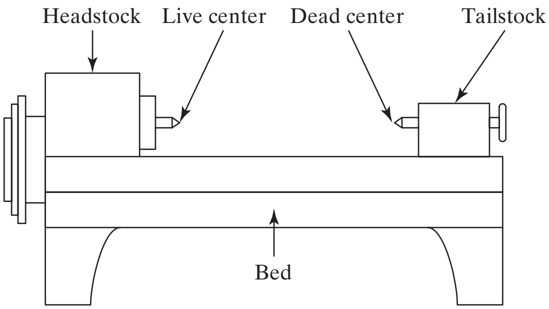
As stated earlier, an n-degree-of-freedom system requires n independent coordinates to describe its configuration. Usually, these coordinates are independent geometrical quantities measured from the equilibrium position of the vibrating body. However, it is possible to select some other set of n coordinates to describe the configuration of the system. The latter set may be, for example, different from the first set in that the coordinates may have their origin away from the equilibrium position of the body. There could be still other sets of coordinates to describe the configuration of the system. Each of these sets of n coordinates is called the *generalized coordinates.*

As an example, consider the lathe shown in Fig. 5.11(a). For simplicity, the lathe bed can be replaced by an elastic beam supported on short elastic columns and the headstock and tailstock can be replaced by two lumped masses as shown in Fig. 5.11(b). The modeling of the lathe as a two-degree-of-freedom system has been indicated in Section 5.1. As shown in Figs. 5.12(a) and (b), any of the following sets of coordinates can be used to describe the motion of this two-degree-of-freedom system:

1. Deflections $x_1(t)$ and $x_2(t)$ of the two ends of the lathe AB.
2. Deflection $x(t)$ of the C.G. and rotation $\theta(t)$.
3. Deflection $x_1(t)$ of the end A and rotation $\theta(t)$.
4. Deflection $y(t)$ of point P located at a distance e to the left of the C.G. and rotation $\theta(t)$.



(a)



(b)

FIGURE 5.11 Lathe. (Photo courtersy of South Bend Lathe Corp.)

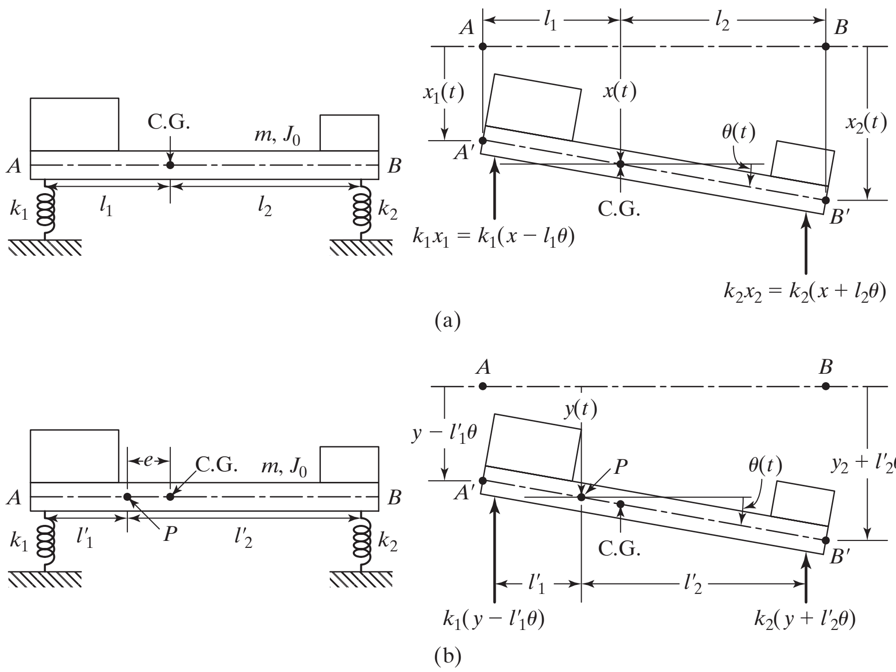


FIGURE 5.12 Modeling of a lathe.

Thus any set of these coordinates—(x_1, x_2), (x, θ), (x_1, θ), and (y, θ)—represents the generalized coordinates of the system. Now we shall derive the equations of motion of the lathe using two different sets of coordinates to illustrate the concept of coordinate coupling.

Equations of Motion Using $x(t)$ and $\theta(t)$. From the free-body diagram shown in Fig. 5.12(a), with the positive values of the motion variables as indicated, the force equilibrium equation in the vertical direction can be written as

$$m\ddot{x} = -k_1(x - l_1\theta) - k_2(x + l_2\theta) \tag{5.21}$$

and the moment equation about the C.G. can be expressed as

$$J_0\ddot{\theta} = k_1(x - l_1\theta)l_1 - k_2(x + l_2\theta)l_2 \tag{5.22}$$

Equations (5.21) and (5.22) can be rearranged and written in matrix form as

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix}$$
$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.23}$$

It can be seen that each of these equations contain x and θ. They become independent of each other if the coupling term $(k_1 l_1 - k_2 l_2)$ is equal to zero—that is, if $k_1 l_1 = k_2 l_2$. If $k_1 l_1 \neq k_2 l_2$, the resultant motion of the lathe AB is both translational and rotational when either a displacement or torque is applied through the C.G. of the body as an initial condition. In other words, the lathe rotates in the vertical plane and has vertical motion as well, unless $k_1 l_1 = k_2 l_2$. This is known as *elastic* or *static coupling*.

Equations of Motion Using $y(t)$ and $\theta(t)$. From Fig. 5.12(b), where $y(t)$ and $\theta(t)$ are used as the generalized coordinates of the system, the equations of motion for translation and rotation can be written as

$$
\begin{aligned}
m\ddot{y} &= -k_1(y - l_1'\theta) - k_2(y + l_2'\theta) - me\ddot{\theta} \\
J_p\ddot{\theta} &= k_1(y - l_1'\theta)l_1' - k_2(y + l_2'\theta)l_2' - me\ddot{y}
\end{aligned} \tag{5.24}
$$

These equations can be rearranged and written in matrix form as

$$
\begin{bmatrix} m & me \\ me & J_p \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_2 l_2' - k_1 l_1') \\ (-k_1 l_1' + k_2 l_2') & (k_1 l_1'^2 + k_2 l_2'^2) \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.25}
$$

Both the equations of motion represented by Eq. (5.25) contain y and θ, so they are coupled equations. They contain static (or elastic) as well as dynamic (or mass) coupling terms. If $k_1 l_1' = k_2 l_2'$, the system will have *dynamic* or *inertia coupling* only. In this case, if the lathe moves up and down in the y direction, the inertia force $m\ddot{y}$, which acts through the center of gravity of the body, induces a motion in the θ direction, by virtue of the moment $m\ddot{y}e$. Similarly, a motion in the θ direction induces a motion of the lathe in the y direction due to the force $me\ddot{\theta}$.

Note the following characteristics of these systems:

1. In the most general case, a viscously damped two-degree-of-freedom system has equations of motion in the following form:

$$
\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{Bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{Bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.26}
$$

   This equation reveals the type of coupling present. If the stiffness matrix is not diagonal, the system has elastic or static coupling. If the damping matrix is not diagonal, the system has damping or velocity coupling. Finally, if the mass matrix is not diagonal, the system has mass or inertial coupling. Both velocity and mass coupling come under the heading of dynamic coupling.
2. The system vibrates in its own natural way regardless of the coordinates used. The choice of the coordinates is a mere convenience.

3. From Eqs. (5.23) and (5.25), it is clear that the nature of the coupling depends on the coordinates used and is not an inherent property of the system. It is possible to choose a system of coordinates $q_1(t)$ and $q_2(t)$ which give equations of motion that are uncoupled both statically and dynamically. Such coordinates are called *principal* or *natural coordinates*. The main advantage of using principal coordinates is that the resulting uncoupled equations of motion can be solved independently of one another.

The following example illustrates the method of finding the principal coordinates in terms of the geometrical coordinates.

EXAMPLE 5.6 Principal Coordinates of Spring-Mass System

Determine the principal coordinates for the spring-mass system shown in Fig. 5.6.

Solution

Approach: Define two independent solutions as principal coordinates and express them in terms of the solutions $x_1(t)$ and $x_2(t)$.

The general motion of the system shown in Fig. 5.6 is given by Eq. (E.10) of Example 5.1:

$$x_1(t) = B_1 \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) + B_2 \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

$$x_2(t) = B_1 \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) - B_2 \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right) \tag{E.1}$$

where $B_1 = X_1^{(1)}$, $B_2 = X_1^{(2)}$, ϕ_1, and ϕ_2 are constants. We define a new set of coordinates $q_1(t)$ and $q_2(t)$ such that

$$q_1(t) = B_1 \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right)$$

$$q_2(t) = B_2 \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right) \tag{E.2}$$

Since $q_1(t)$ and $q_2(t)$ are harmonic functions, their corresponding equations of motion can be written as[1]

$$\ddot{q}_1 + \left(\frac{k}{m}\right)q_1 = 0$$

$$\ddot{q}_2 + \left(\frac{3k}{m}\right)q_2 = 0 \tag{E.3}$$

[1]Note that the equation of motion corresponding to the solution $q = B\cos(\omega t + \phi)$ is given by $\ddot{q} + \omega^2 q = 0$.

These equations represent a two-degree-of-freedom system whose natural frequencies are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. Because there is neither static nor dynamic coupling in the equations of motion (E.3), $q_1(t)$ and $q_2(t)$ are principal coordinates. From Eqs. (E.1) and (E.2), we can write

$$\begin{aligned} x_1(t) &= q_1(t) + q_2(t) \\ x_2(t) &= q_1(t) - q_2(t) \end{aligned} \tag{E.4}$$

The solution of Eqs. (E.4) gives the principal coordinates:

$$\begin{aligned} q_1(t) &= \frac{1}{2}[x_1(t) + x_2(t)] \\ q_2(t) &= \frac{1}{2}[x_1(t) - x_2(t)] \end{aligned} \tag{E.5}$$

■

EXAMPLE 5.7 Frequencies and Modes of an Automobile

Determine the pitch (angular motion) and bounce (up-and-down linear motion) frequencies and the location of oscillation centers (nodes) of an automobile with the following data (see Fig. 5.13):

Mass $(m) = 1000$ kg
Radius of gyration $(r) = 0.9$ m
Distance between front axle and C.G. $(l_1) = 1.0$ m
Distance between rear axle and C.G. $(l_2) = 1.5$ m
Front spring stiffness $(k_f) = 18$ kN/m
Rear spring stiffness $(k_r) = 22$ kN/m

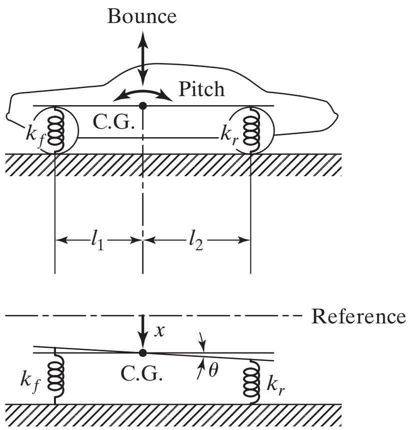


FIGURE 5.13 Pitch and bounce motions of an automobile.

Solution: If x and θ are used as independent coordinates, the equations of motion are given by Eq. (5.23) with $k_1 = k_f$, $k_2 = k_r$, and $J_0 = mr^2$. For free vibration, we assume a harmonic solution:

$$x(t) = X\cos(\omega t + \phi), \qquad \theta(t) = \Theta\cos(\omega t + \phi) \tag{E.1}$$

Using Eqs. (E.1) and (5.23), we obtain

$$\begin{bmatrix} (-m\omega^2 + k_f + k_r) & (-k_f l_1 + k_r l_2) \\ (-k_f l_1 + k_r l_2) & (-J_0\omega^2 + k_f l_1^2 + k_r l_2^2) \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E.2}$$

For the known data, Eq. (E.2) becomes

$$\begin{bmatrix} (-1000\omega^2 + 40{,}000) & 15{,}000 \\ 15{,}000 & (-810\omega^2 + 67{,}500) \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E.3}$$

from which the frequency equation can be derived:

$$8.1\omega^4 - 999\omega^2 + 24{,}750 = 0 \tag{E.4}$$

The natural frequencies can be found from Eq. (E.4):

$$\omega_1 = 5.8593 \text{ rad/s}, \qquad \omega_2 = 9.4341 \text{ rad/s} \tag{E.5}$$

With these values, the ratio of amplitudes can be found from Eq. (E.3):

$$\frac{X^{(1)}}{\Theta^{(1)}} = -2.6461, \qquad \frac{X^{(2)}}{\Theta^{(2)}} = 0.3061 \tag{E.6}$$

The node locations can be obtained by noting that the tangent of a small angle is approximately equal to the angle itself. Thus, from Fig. 5.14, we find that the distance between the C.G. and the node is -2.6461 m for ω_1 and 0.3061 m for ω_2. The mode shapes are shown by dashed lines in Fig. 5.14.

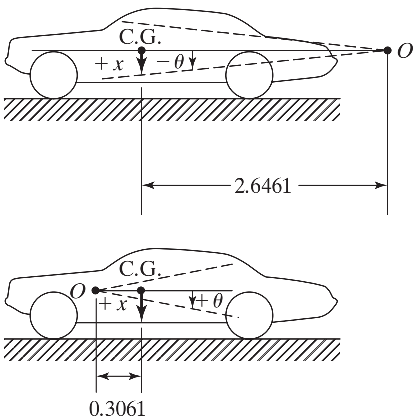


FIGURE 5.14 Mode shapes of an automobile.

■

5.6 Forced-Vibration Analysis

The equations of motion of a general two-degree-of-freedom system under external forces can be written as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{5.27}$$

Equations (5.1) and (5.2) can be seen to be special cases of Eq. (5.27), with $m_{11} = m_1$, $m_{22} = m_2$, and $m_{12} = 0$. We shall consider the external forces to be harmonic:

$$F_j(t) = F_{j0}e^{i\omega t}, \qquad j = 1, 2 \tag{5.28}$$

where ω is the forcing frequency. We can write the steady-state solutions as

$$x_j(t) = X_j e^{i\omega t}, \qquad j = 1, 2 \tag{5.29}$$

where X_1 and X_2 are, in general, complex quantities that depend on ω and the system parameters. Substitution of Eqs. (5.28) and (5.29) into Eq. (5.27) leads to

$$\begin{bmatrix} (-\omega^2 m_{11} + i\omega c_{11} + k_{11}) & (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) \\ (-\omega^2 m_{12} + i\omega c_{12} + k_{12}) & (-\omega_2 m_{22} + i\omega c_{22} + k_{22}) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} \tag{5.30}$$

As in Section 3.5, we define the mechanical impedance $Z_{rs}(i\omega)$ as

$$Z_{rs}(i\omega) = -\omega^2 m_{rs} + i\omega c_{rs} + k_{rs}, \qquad r, s = 1, 2 \tag{5.31}$$

and write Eq. (5.30) as

$$[Z(i\omega)]\vec{X} = \vec{F}_0 \tag{5.32}$$

where

$$[Z(i\omega)] = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{12}(i\omega) & Z_{22}(i\omega) \end{bmatrix} = \text{Impedance matrix}$$

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

and

$$\vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Equation (5.32) can be solved to obtain

$$\vec{X} = [Z(i\omega)]^{-1}\vec{F}_0 \tag{5.33}$$

where the inverse of the impedance matrix is given by

$$[Z(i\omega)]^{-1} = \frac{1}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)}\begin{bmatrix} Z_{22}(i\omega) & -Z_{12}(i\omega) \\ -Z_{12}(i\omega) & Z_{11}(i\omega) \end{bmatrix} \tag{5.34}$$

Equations (5.33) and (5.34) lead to the solution

$$\begin{aligned} X_1(i\omega) &= \frac{Z_{22}(i\omega)F_{10} - Z_{12}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)} \\ X_2(i\omega) &= \frac{-Z_{12}(i\omega)F_{10} + Z_{11}(i\omega)F_{20}}{Z_{11}(i\omega)Z_{22}(i\omega) - Z_{12}^2(i\omega)} \end{aligned} \tag{5.35}$$

By substituting Eq. (5.35) into Eq. (5.29) we can find the complete solution, $x_1(t)$ and $x_2(t)$.

The analysis of a two-degree-of-freedom system used as a vibration absorber is given in Section 9.11. Reference [5.4] deals with the impact response of a two-degree-of-freedom system, while Ref. [5.5] considers the steady-state response under harmonic excitation.

EXAMPLE 5.8 Steady-State Response of Spring-Mass System

Find the steady-state response of the system shown in Fig. 5.15 when the mass m_1 is excited by the force $F_1(t) = F_{10}\cos\omega t$. Also, plot its frequency-response curve.

Solution: The equations of motion of the system can be expressed as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_{10}\cos\omega t \\ 0 \end{Bmatrix} \tag{E.1}$$

Comparison of Eq. (E.1) with Eq. (5.27) shows that

$$\begin{aligned} &m_{11} = m_{22} = m, \quad m_{12} = 0, \quad c_{11} = c_{12} = c_{22} = 0, \\ &k_{11} = k_{22} = 2k, \quad k_{12} = -k, \quad F_1 = F_{10}\cos\omega t, \quad F_2 = 0 \end{aligned}$$

We assume the solution to be as follows:[2]

$$x_j(t) = X_j\cos\omega t, \qquad j = 1, 2 \tag{E.2}$$

Equation (5.31) gives

$$Z_{11}(\omega) = Z_{22}(\omega) = -m\omega^2 + 2k, \qquad Z_{12}(\omega) = -k \tag{E.3}$$

[2]Since $F_{10}\cos\omega t = \mathrm{Re}(F_{10}e^{i\omega t})$, we shall assume the solution also to be $x_j = \mathrm{Re}(X_j e^{i\omega t}) = X_j\cos\omega t$, $j = 1, 2$. It can be verified that X_j are real for an undamped system.

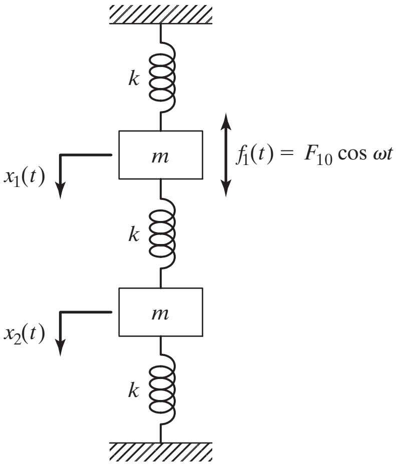


FIGURE 5.15 A two-mass system subjected to harmonic force.

Hence X_1 and X_2 are given by Eq. (5.35):

$$X_1(\omega) = \frac{(-\omega^2 m + 2k)\,F_{10}}{(-\omega^2 m + 2k)^2 - k^2} = \frac{(-\omega^2 m + 2k)\,F_{10}}{(-m\omega^2 + 3k)(-m\omega^2 + k)} \tag{E.4}$$

$$X_2(\omega) = \frac{kF_{10}}{(-m\omega^2 + 2k)^2 - k^2} = \frac{kF_{10}}{(-m\omega^2 + 3k)(-m\omega^2 + k)} \tag{E.5}$$

By defining $\omega_1^2 = k/m$ and $\omega_2^2 = 3k/m$, Eqs. (E.4) and (E.5) can be expressed as

$$X_1(\omega) = \frac{\left\{2 - \left(\dfrac{\omega}{\omega_1}\right)^2\right\}F_{10}}{k\left[\left(\dfrac{\omega_2}{\omega_1}\right)^2 - \left(\dfrac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\dfrac{\omega}{\omega_1}\right)^2\right]} \tag{E.6}$$

$$X_2(\omega) = \frac{F_{10}}{k\left[\left(\dfrac{\omega_2}{\omega_1}\right)^2 - \left(\dfrac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\dfrac{\omega}{\omega_1}\right)^2\right]} \tag{E.7}$$

The responses X_1 and X_2 are shown in Fig. 5.16 in terms of the dimensionless parameter ω/ω_1. In this parameter, ω_1 was selected arbitrarily; ω_2 could have been selected just as easily. It can be seen that the amplitudes X_1 and X_2 become infinite when $\omega^2 = \omega_1^2$ or $\omega^2 = \omega_2^2$. Thus there are two resonance conditions for the system: one at ω_1 and another at ω_2. At all other values of ω, the amplitudes of vibration are finite. It can be noted from Fig. 5.16 that there is a particular value of the frequency ω at

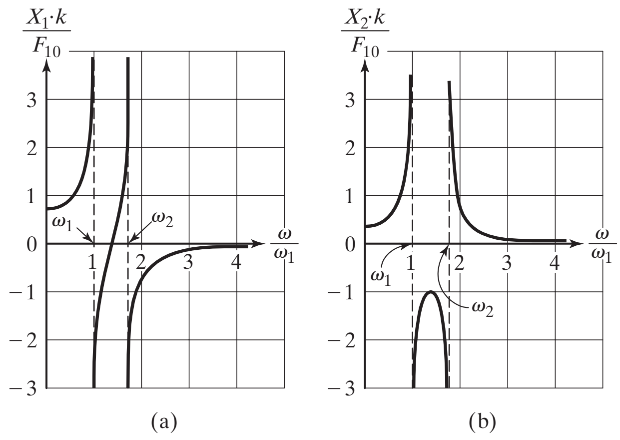


FIGURE 5.16 Frequency-response curves of Example 5.8.

which the vibration of the first mass m_1, to which the force $f_1(t)$ is applied, is reduced to zero. This characteristic forms the basis of the dynamic vibration absorber discussed in Chapter 9.

■

5.7 Semidefinite Systems

Semidefinite systems are also known as *unrestrained* or *degenerate systems.* Two examples of such systems are shown in Fig. 5.17. The arrangement in Fig. 5.17(a) may be considered to represent two railway cars of masses m_1 and m_2 with a coupling spring k. The arrangement in Fig. 5.17(c) may be considered to represent two rotors of mass moments of inertia J_1 and J_2 connected by a shaft of torsional stiffness k_t.

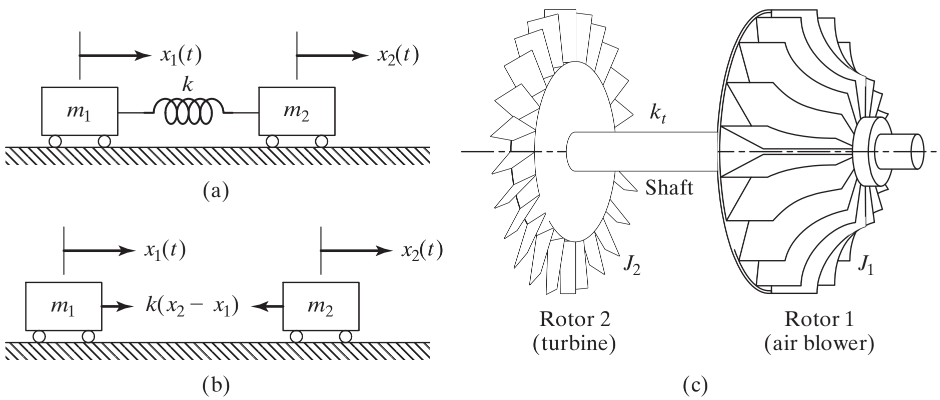
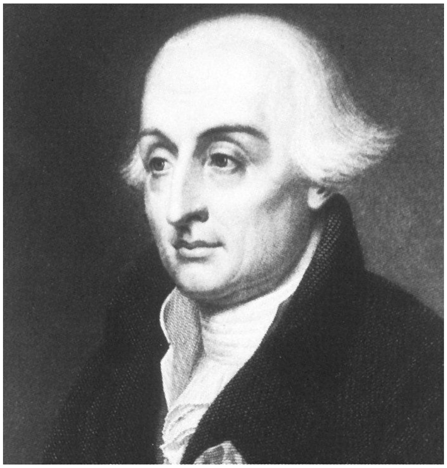


FIGURE 5.17 Semidefinite systems.



Joseph Louis Lagrange (1736–1813) was an Italian-born mathematician famous for his work on theoretical mechanics. He was made professor of mathematics in 1755 at the Artillery School in Turin. Lagrange's masterpiece, his *Méchanique*, contains what are now known as "Lagrange's equations," which are very useful in the study of vibrations. His work on elasticity and strength of materials, where he considered the strength and deflection of struts, is less well known.
(Courtesy of Dirk J. Struik, *A Concise History of Mathematics*, 2nd ed., Dover Publications, New York, 1948.)

CHAPTER 6

Multidegree-of-Freedom Systems

Chapter Outline

Multidegree-of-freedom systems are the topic of this chapter. The modeling of continuous systems as multidegree-of-freedom systems is presented. The equations of a general n-degree-of-freedom system are derived using Newton's second law of motion. Because the solution of the equations of motion in scalar form involve complicated algebraic manipulations, we use matrix representation for multidegree-of-freedom systems. By expressing the coupled set of n equations in matrix form, the mass, damping, and stiffness matrices are identified. The derivation of equations using influence coefficients is also presented. The stiffness, flexibility, and inertia influence coefficients are presented from first principles. The expressions for potential and kinetic energies and their use in deriving the equations of motion based on Lagrange's equations are presented. The concepts of generalized coordinates and generalized forces are presented. After expressing the free-vibration equations in matrix form, the eigenvalue problem is derived in matrix form. The solution of the eigenvalue problem using the solution of the characteristic (polynomial) equation is outlined to determine the natural frequencies and mode shapes (or normal modes) of the system. The concepts of orthogonality of normal modes, modal matrix, and orthonormalization of the mass and stiffness matrices are introduced. The expansion theorem and the unrestrained or semidefinite systems are also presented. The free vibration of undamped systems using modal vectors and the forced vibration of undamped systems using modal analysis are considered with illustrative examples. The equations of motion for the forced vibration of viscously damped systems are considered through the introduction of Rayleigh's dissipation function. The equations of motion are uncoupled for proportionally damped systems, and the solution of each of the uncoupled equations is outlined through the Duhamel integral. The self-excitation and stability analysis of multidegree-of-freedom systems is considered using Ruth-Hurwitz stability criterion. Finally, MATLAB solutions are presented for the free and forced vibration of multidegree-of-freedom systems.

Learning Objectives

After you have finished studying this chapter, you should be able to do the following:

- Formulate the equations of motion of multidegree-of-freedom systems using Newton's second law, influence coefficients, or Lagrange's equations.
- Express the equation of motion in matrix form.
- Find the natural frequencies of vibration and the modal vectors by solving the eigenvalue problem.
- Determine the free- and forced-vibration response of undamped systems using modal analysis.
- Use proportional damping to find the response of damped systems.
- Analyze the stability characteristics of multidegree-of-freedom systems using the Routh-Hurwitz criterion.
- Solve free- and forced-vibration problems using MATLAB.

6.1 Introduction

As stated in Chapter 1, most engineering systems are continuous and have an infinite number of degrees of freedom. The vibration analysis of continuous systems requires the solution of partial differential equations, which is quite difficult. For many partial differential equations, in fact, analytical solutions do not exist. The analysis of a multidegree-of-freedom system, on the other hand, requires the solution of a set of ordinary differential equations, which is relatively simple. Hence, for simplicity of analysis, continuous systems are often approximated as multidegree-of-freedom systems.

All the concepts introduced in the preceding chapter can be directly extended to the case of multidegree-of-freedom systems. For example, there is one equation of motion for each degree of freedom; if generalized coordinates are used, there is one generalized coordinate for each degree of freedom. The equations of motion can be obtained from Newton's second law of motion or by using the influence coefficients defined in Section 6.4. However, it is often more convenient to derive the equations of motion of a multidegree-of-freedom system by using Lagrange's equations.

There are n natural frequencies, each associated with its own mode shape, for a system having n degrees of freedom. The method of determining the natural frequencies from the characteristic equation obtained by equating the determinant to zero also applies to these systems. However, as the number of degrees of freedom increases, the solution of the characteristic equation becomes more complex. The mode shapes exhibit a property known as *orthogonality*, which can be utilized for the solution of undamped forced-vibration problems using a procedure known as modal analysis. The solution of forced-vibration problems associated with viscously damped systems can also be found conveniently by using a concept called *proportional damping*.

6.2 Modeling of Continuous Systems as Multidegree-of-Freedom Systems

Different methods can be used to approximate a continuous system as a multidegree-of-freedom system. A simple method involves replacing the distributed mass or inertia of the system by a finite number of lumped masses or rigid bodies. The lumped masses are assumed to be connected by massless elastic and damping members. Linear (or angular) coordinates are used to describe the motion of the lumped masses (or rigid bodies). Such models are called *lumped-parameter* or *lumped-mass* or *discrete-mass* systems. The minimum number of coordinates necessary to describe the motion of the lumped masses and rigid bodies defines the number of degrees of freedom of the system. Naturally, the larger the number of lumped masses used in the model, the higher the accuracy of the resulting analysis.

Some problems automatically indicate the type of lumped-parameter model to be used. For example, the three-story building shown in Fig. 6.1(a) automatically suggests using a three-lumped-mass model, as indicated in Fig. 6.1(b). In this model, the inertia of the system is assumed to be concentrated as three point masses located at the floor levels, and the elasticities of the columns are replaced by the springs. Similarly, the radial drilling

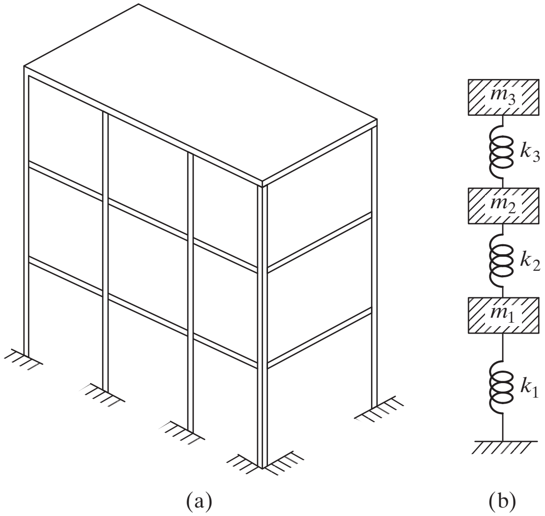


FIGURE 6.1 Three-story building.

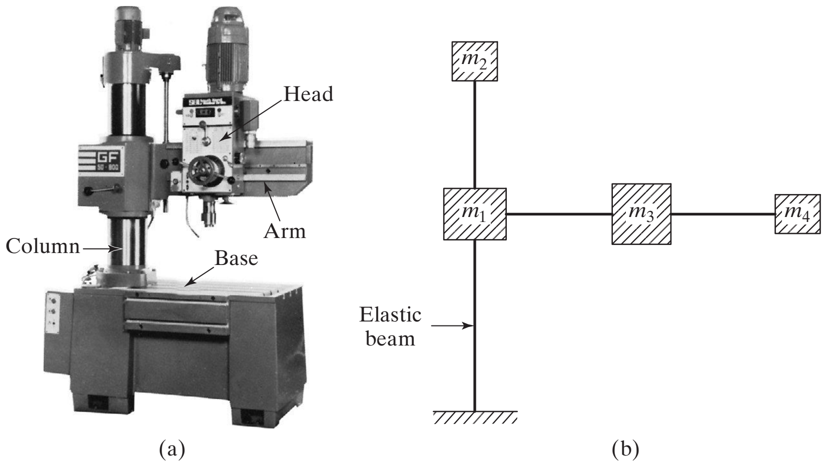


FIGURE 6.2 Radial drilling machine. (Photo courtesy of South Bend Lathe Corp.)

machine shown in Fig. 6.2(a) can be modeled using four lumped masses and four spring elements (elastic beams), as shown in Fig. 6.2(b).

Another popular method of approximating a continuous system as a multidegree-of-freedom system involves replacing the geometry of the system by a large number of small elements. By assuming a simple solution within each element, the principles of compatibility and equilibrium are used to find an approximate solution to the original system. This method, known as the *finite element method*, is considered in detail in Chapter 12.

6.3 Using Newton's Second Law to Derive Equations of Motion

The following procedure can be adopted to derive the equations of motion of a multidegree-of-freedom system using Newton's second law of motion:

1. Set up suitable coordinates to describe the positions of the various point masses and rigid bodies in the system. Assume suitable positive directions for the displacements, velocities, and accelerations of the masses and rigid bodies.
2. Determine the static equilibrium configuration of the system and measure the displacements of the masses and rigid bodies from their respective static equilibrium positions.
3. Draw the free-body diagram of each mass or rigid body in the system. Indicate the spring, damping, and external forces acting on each mass or rigid body when positive displacement and velocity are given to that mass or rigid body.
4. Apply Newton's second law of motion to each mass or rigid body shown by the free-body diagram as

$$m_i \ddot{x}_i = \sum_j F_{ij} \text{ (for mass } m_i) \tag{6.1}$$

or

$$J_i \ddot{\theta}_i = \sum_j M_{ij} \text{ (for rigid body of inertia } J_i) \tag{6.2}$$

where $\Sigma_j \mathrm{F}_{ij}$ denotes the sum of all forces acting on mass m_i and $\Sigma_j M_{ij}$ indicates the sum of moments of all forces (about a suitable axis) acting on the rigid body of mass moment of inertia J_i.

The procedure is illustrated in the following examples.

EXAMPLE 6.1 Equations of Motion of a Spring-Mass-Damper System

Derive the equations of motion of the spring-mass-damper system shown in Fig. 6.3(a).

Solution:

Approach: Draw free-body diagrams of masses and apply Newton's second law of motion. The coordinates describing the positions of the masses, $x_i(t)$, are measured from their respective static equilibrium positions, as indicated in Fig. 6.3(a). The free-body diagram of a typical interior mass m_i is shown in Fig. 6.3(b) along with the assumed positive directions for its displacement, velocity, and acceleration. The application of Newton's second law of motion to mass m_i gives

$$\begin{aligned} m_i \ddot{x}_i = &-k_i(x_i - x_{i-1}) + k_{i+1}(x_{i+1} - x_i) - c_i(\dot{x}_i - \dot{x}_{i-1}) \\ &+ c_{i+1}(\dot{x}_{i+1} - \dot{x}_i) + F_i; \qquad i = 2, 3, \ldots, n-1 \end{aligned}$$

or

$$\begin{aligned} m_i \ddot{x}_i &- c_i \dot{x}_{i-1} + (c_i + c_{i+1})\dot{x}_i - c_{i+1}\dot{x}_{i+1} - k_i x_{i-1} \\ &+ (k_i + k_{i+1})x_i - k_{i+1}x_{i+1} = F_i; \qquad i = 2, 3, \ldots, n-1 \end{aligned} \tag{E.1}$$

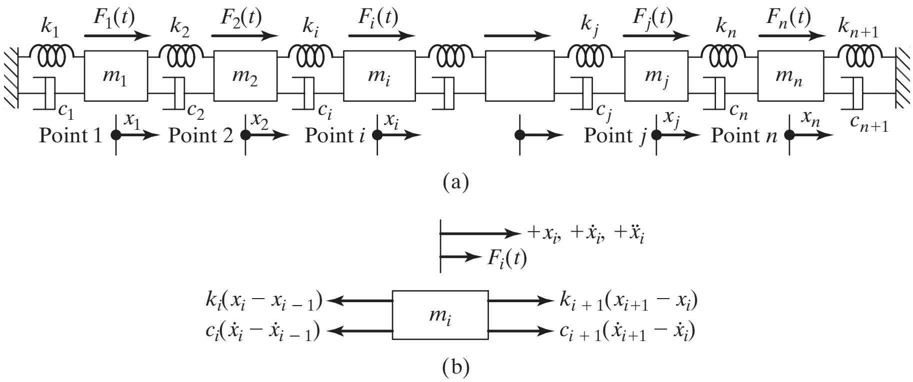


FIGURE 6.3 Spring-mass-damper system.

The equations of motion of the masses m_1 and m_n can be derived from Eq. (E.1) by setting $i = 1$ along with $x_0 = 0$ and $i = n$ along with $x_{n+1} = 0$, respectively:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1 \tag{E.2}$$

$$m_n\ddot{x}_n - c_n\dot{x}_{n-1} + (c_n + c_{n+1})\dot{x}_n - k_nx_{n-1} + (k_n + k_{n+1})x_n = F_n \tag{E.3}$$

Notes:

1. The equations of motion, Eqs. (E.1) to (E.3), of Example 6.1 can be expressed in matrix form as

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F} \tag{6.3}$$

where $[m]$, $[c]$, and $[k]$ are called the mass, damping, and stiffness matrices, respectively, and are given by

$$[m] = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & m_3 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & m_n \end{bmatrix} \tag{6.4}$$

$$[c] = \begin{bmatrix} (c_1 + c_2) & -c_2 & 0 & \cdots & 0 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 & \cdots & 0 & 0 \\ 0 & -c_3 & (c_3 + c_4) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -c_n & (c_n + c_{n+1}) \end{bmatrix} \tag{6.5}$$

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & (k_3 + k_4) & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdots & -k_n & (k_n + k_{n+1}) \end{bmatrix} \tag{6.6}$$

and $\vec{x}$, $\dot{\vec{x}}$, $\ddot{\vec{x}}$, and $\vec{F}$ are the displacement, velocity, acceleration, and force vectors, given by

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{Bmatrix}, \qquad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n(t) \end{Bmatrix},$$

$$\ddot{\vec{x}} = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \ddot{x}_n(t) \end{Bmatrix}, \qquad \vec{F} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ \cdot \\ \cdot \\ \cdot \\ F_n(t) \end{Bmatrix} \tag{6.7}$$

2. For an undamped system (with all $c_i = 0$, $i = 1, 2, \ldots, n + 1$), the equations of motion reduce to

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F} \tag{6.8}$$

3. The spring-mass-damper system considered above is a particular case of a general n-degree-of-freedom spring-mass-damper system. In their most general form, the mass, damping, and stiffness matrices are given by

$$[m] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{12} & m_{22} & m_{23} & \cdots & m_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ m_{1n} & m_{2n} & m_{3n} & \cdots & m_{nn} \end{bmatrix} \tag{6.9}$$

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{12} & c_{22} & c_{23} & \cdots & c_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ c_{1n} & c_{2n} & c_{3n} & \cdots & c_{nn} \end{bmatrix} \tag{6.10}$$

and

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{12} & k_{22} & k_{23} & \cdots & k_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ k_{1n} & k_{2n} & k_{3n} & \cdots & k_{nn} \end{bmatrix} \tag{6.11}$$

As stated in Section 5.5, if the mass matrix is not diagonal, the system is said to have mass or inertia coupling. If the damping matrix is not diagonal, the system is said to have damping or velocity coupling. Finally, if the stiffness matrix is not diagonal, the system is said to have elastic or static coupling. Both mass and damping coupling are also known as dynamic coupling.

4. The differential equations of the spring-mass system considered in Example 6.1 (Fig. 6.3(a)) can be seen to be coupled; each equation involves more than one coordinate. This means that the equations cannot be solved individually one at a time; they can only be solved simultaneously. In addition, the system can be seen to be statically coupled, since stiffnesses are coupled—that is, the stiffness matrix has at least one nonzero off-diagonal term. On the other hand, if the mass matrix has at least one off-diagonal term nonzero, the system is said to be dynamically coupled. Further, if both the stiffness and mass matrices have nonzero off-diagonal terms, the system is said to be coupled both statically and dynamically.

■

EXAMPLE 6.2 Equations of Motion of a Trailer–Compound Pendulum System

Derive the equations of motion of the trailer–compound pendulum system shown in Fig. 6.4(a).

Solution:

Approach: Draw the free-body diagrams and apply Newton's second law of motion.

The coordinates $x(t)$ and $\theta(t)$ are used to describe, respectively, the linear displacement of the trailer and the angular displacement of the compound pendulum from their respective static equilibrium positions. When positive values are assumed for the displacements $x(t)$ and $\theta(t)$, velocities $\dot{x}(t)$ and $\dot{\theta}(t)$, and accelerations $\ddot{x}(t)$ and $\ddot{\theta}(t)$, the external forces on the trailer will be the applied force $F(t)$, the spring forces k_1x and k_2x, and the damping forces $c_1\dot{x}$ and $c_2\dot{x}$, as shown in Fig. 6.4(b). The external forces on the compound pendulum will be the applied torque $M_t(t)$ and the gravitational force mg, as shown in Fig. 6.4(b). The inertia forces that act on the trailer and the compound pendulum are indicated by the dashed lines in Fig. 6.4(b). Note that the rotational motion of the compound pendulum about the hinge O induces a radially inward force (toward O) $m\frac{l}{2}\dot{\theta}^2$ and a normal force (perpendicular to OC) $m\frac{l}{2}\ddot{\theta}$ as shown in Fig. 6.4(b). The application of Newton's second law for translatory motion in the horizontal direction gives

$$M\ddot{x} + m\ddot{x} + m\frac{l}{2}\ddot{\theta}\cos\theta - m\frac{l}{2}\dot{\theta}^2\sin\theta = -k_1x - k_2x - c_1\dot{x} - c_2\dot{x} + F(t) \tag{E.1}$$

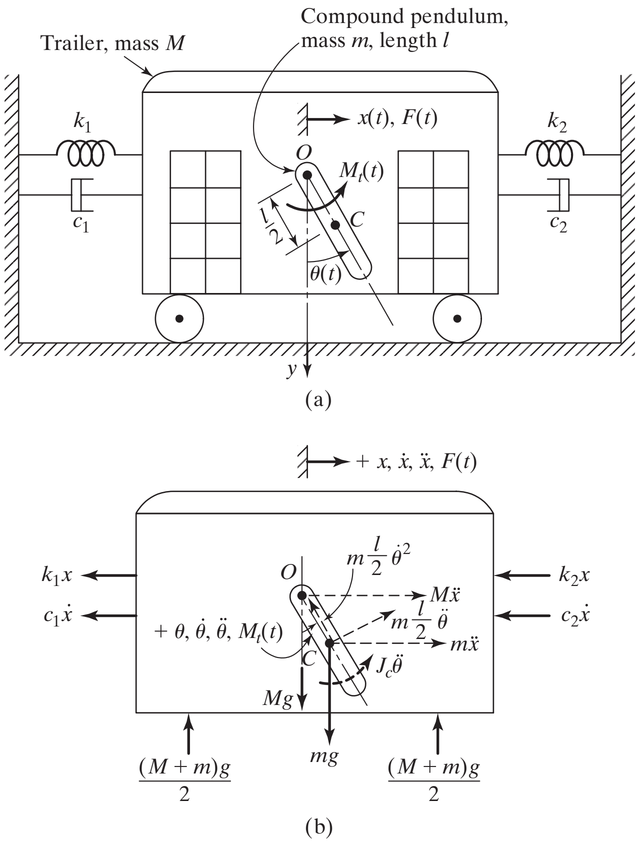


FIGURE 6.4 Compound pendulum and trailer system.

Similarly the application of Newton's second law for rotational motion about hinge O yields

$$\left(m\frac{l}{2}\ddot{\theta}\right)\frac{l}{2} + \left(m\frac{l^2}{12}\right)\ddot{\theta} + (m\ddot{x})\,\frac{l}{2}\cos\theta = -(mg)\,\frac{l}{2}\sin\theta + M_t(t) \tag{E.2}$$

Notes:

1. The equations of motion, Eqs. (E.1) and (E.2), can be seen to be nonlinear due to the presence of the terms involving $\sin\theta$, $\cos\theta$, and $(\dot{\theta})^2\sin\theta$.
2. Equations (E.1) and (E.2) can be linearized if the term involving $(\dot{\theta})^2\sin\theta$ is assumed negligibly small and the displacements are assumed small so that $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. The linearized equations can be derived as

$$(M + m)\ddot{x} + \left(m\frac{l}{2}\right)\ddot{\theta} + (k_1 + k_2)x + (c_1 + c_2)\,\dot{x} = F(t) \tag{E.3}$$

and

$$\left(\frac{ml}{2}\right)\ddot{x} + \left(\frac{ml^2}{3}\right)\ddot{\theta} + \left(\frac{mgl}{2}\right)\theta = M_t(t) \tag{E.4}$$

■

6.4 Influence Coefficients

The equations of motion of a multidegree-of-freedom system can also be written in terms of influence coefficients, which are extensively used in structural engineering. Basically, one set of influence coefficients can be associated with each of the matrices involved in the equations of motion. The influence coefficients associated with the stiffness and mass matrices are, respectively, known as the stiffness and inertia influence coefficients. In some cases, it is more convenient to rewrite the equations of motion using the inverse of the stiffness matrix (known as the flexibility matrix) or the inverse of the mass matrix. The influence coefficients corresponding to the inverse stiffness matrix are called the *flexibility influence coefficients*, and those corresponding to the inverse mass matrix are known as the *inverse inertia coefficients.*

6.4.1 Stiffness Influence Coefficients

For a simple linear spring, the force necessary to cause a unit elongation is called the stiffness of the spring. In more complex systems, we can express the relation between the displacement at a point and the forces acting at various other points of the system by means of stiffness influence coefficients. The stiffness influence coefficient, denoted as k_{ij}, is defined as the force at point i due to a unit displacement at point j when all the points other than the point j are fixed. Using this definition, for the spring-mass system shown in Fig. 6.5, the total force at point i, F_i, can be found by summing up the forces due to all displacements $x_j (j = 1, 2, \ldots, n)$ as

$$F_i = \sum_{j=1}^{n} k_{ij} x_j, \qquad i = 1, 2, \ldots, n \tag{6.12}$$

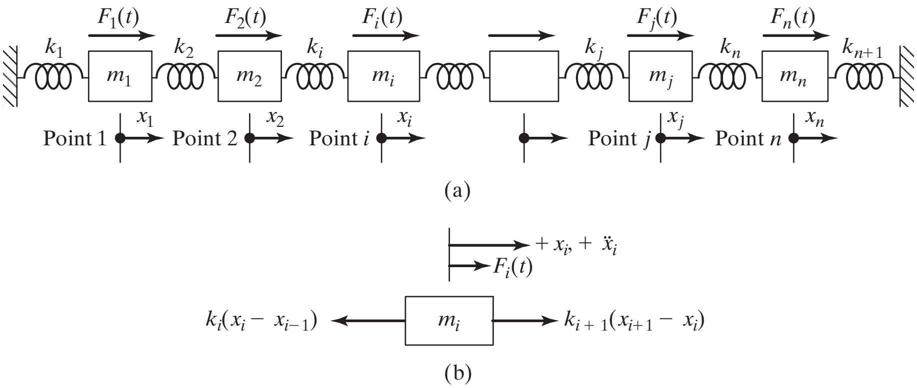


FIGURE 6.5 Multidegree-of-freedom spring-mass system.

Equation (6.12) can be stated in matrix form as

$$\vec{F} = [k]\vec{x} \tag{6.13}$$

where $\vec{x}$ and $\vec{F}$ are the displacement and force vectors defined in Eq. (6.7) and $[k]$ is the stiffness matrix given by

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \tag{6.14}$$

The following aspects of stiffness influence coefficients are to be noted:

1. Since the force required at point i to cause a unit deflection at point j and zero deflection at all other points is the same as the force required at point j to cause a unit deflection at point i and zero deflection at all other points (Maxwell's reciprocity theorem [6.1]), we have $k_{ij} = k_{ji}$.
2. The stiffness influence coefficients can be calculated by applying the principles of statics and solid mechanics.
3. The stiffness influence coefficients for torsional systems can be defined in terms of unit angular displacement and the torque that causes the angular displacement. For example, in a multirotor torsional system, k_{ij} can be defined as the torque at point i (rotor i) due to a unit angular displacement at point j and zero angular displacement at all other points.

The stiffness influence coefficients of a multidegree-of-freedom system can be determined as follows:

1. Assume a value of one for the displacement x_j ($j = 1$ to start with) and a value of zero for all other displacements $x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$. By definition, the set of forces $k_{ij}(i = 1, 2, \ldots, n)$ will maintain the system in the assumed configuration $(x_j = 1, x_1 = x_2 = \cdots = x_{j-1} = x_{j+1} = \cdots = x_n = 0)$. Then the static equilibrium equations are written for each mass and the resulting set of n equations solved to find the n influence coefficients $k_{ij}(i = 1, 2, \ldots, n)$.
2. After completing step 1 for $j = 1$, the procedure is repeated for $j = 2, 3, \ldots, n$.

The following examples illustrate the procedure.

EXAMPLE 6.3 Stiffness Influence Coefficients

Find the stiffness influence coefficients of the system shown in Fig. 6.6(a).

Solution:

Approach: Use the definition of k_{ij} and static equilibrium equations.

Let x_1, x_2, and x_3 denote the displacements of the masses m_1, m_2, and m_3, respectively. The stiffness influence coefficients k_{ij} of the system can be determined in terms of the spring stiffnesses

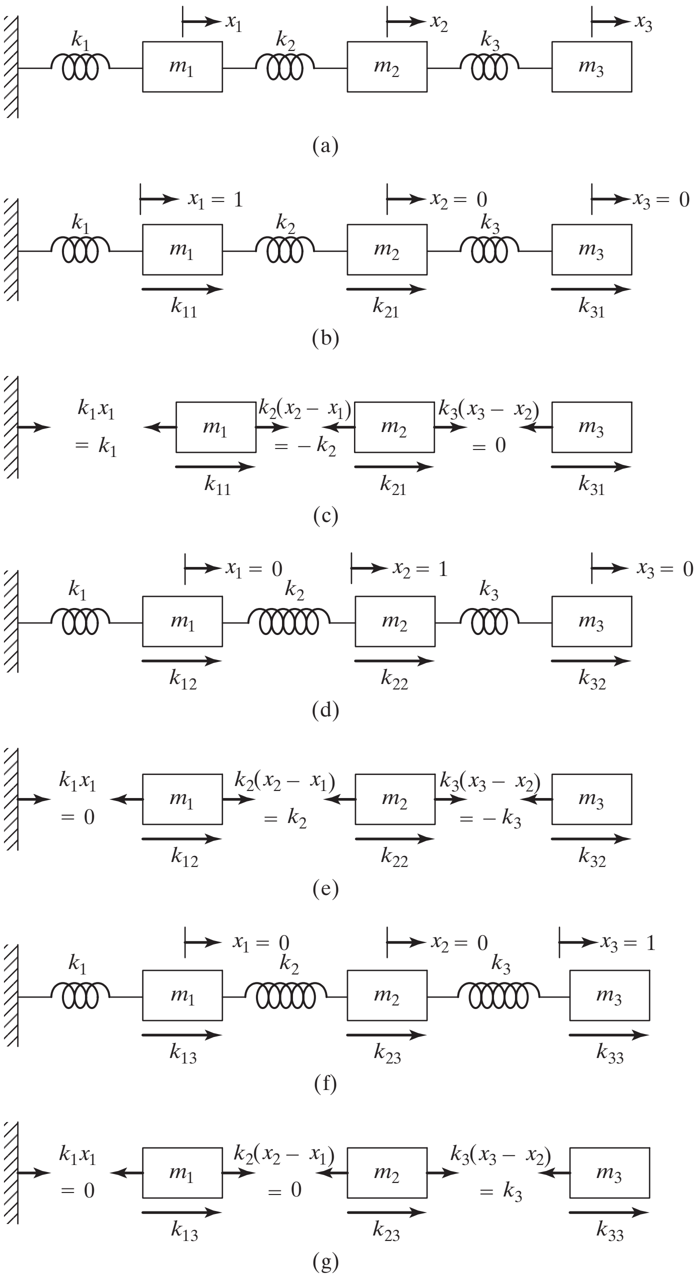


FIGURE 6.6 Determination of stiffness influence coefficients.

k_1, k_2, and k_3 as follows. First, we set the displacement of m_1 equal to one $(x_1 = 1)$ and the displacements of m_2 and m_3 equal to zero $(x_2 = x_3 = 0)$, as shown in Fig. 6.6(b). The set of forces $k_{i1}\ (i = 1, 2, 3)$ is assumed to maintain the system in this configuration. The free-body diagrams of the masses corresponding to the configuration of Fig. 6.6(b) are indicated in Fig. 6.6(c). The equilibrium of forces for the masses m_1, m_2, and m_3 in the horizontal direction yields

$$\text{Mass } m_1\text{: } k_1 = -k_2 + k_{11} \tag{E.1}$$

$$\text{Mass } m_2\text{: } k_{21} = -k_2 \tag{E.2}$$

$$\text{Mass } m_3\text{: } k_{31} = 0 \tag{E.3}$$

The solution of Eqs. (E.1) to (E.3) gives

$$k_{11} = k_1 + k_2, \qquad k_{21} = -k_2, \qquad k_{31} = 0 \tag{E.4}$$

Next the displacements of the masses are assumed as $x_1 = 0$, $x_2 = 1$, and $x_3 = 0$, as shown in Fig. 6.6(d). Since the forces $k_{i2}\,(i = 1, 2, 3)$ are assumed to maintain the system in this configuration, the free-body diagrams of the masses can be developed as indicated in Fig. 6.6(e). The force equilibrium equations of the masses are:

$$\text{Mass } m_1\text{: } k_{12} + k_2 = 0 \tag{E.5}$$

$$\text{Mass } m_2\text{: } k_{22} - k_3 = k_2 \tag{E.6}$$

$$\text{Mass } m_3\text{: } k_{32} = -k_3 \tag{E.7}$$

The solution of Eqs. (E.5) to (E.7) yields

$$k_{12} = -k_2, \qquad k_{22} = k_2 + k_3, \qquad k_{32} = -k_3 \tag{E.8}$$

Finally the set of forces $k_{i3}\,(i = 1, 2, 3)$ is assumed to maintain the system with $x_1 = 0$, $x_2 = 0$, and $x_3 = 1$ (Fig. 6.6(f)). The free-body diagrams of the various masses in this configuration are shown in Fig. 6.6(g), and the force equilibrium equations lead to

$$\text{Mass } m_1\text{: } k_{13} = 0 \tag{E.9}$$

$$\text{Mass } m_2\text{: } k_{23} + k_3 = 0 \tag{E.10}$$

$$\text{Mass } m_3\text{: } k_{33} = k_3 \tag{E.11}$$

The solution of Eqs. (E.9) to (E.11) yields

$$k_{13} = 0, \qquad k_{23} = -k_3, \qquad k_{33} = k_3 \tag{E.12}$$

Thus the stiffness matrix of the system is given by

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{E.13}$$

■

EXAMPLE 6.4 Stiffness Matrix of a Frame

Determine the stiffness matrix of the frame shown in Fig. 6.7(a). Neglect the effect of axial stiffness of the members *AB* and *BC*.

Solution: Since the segments *AB* and *BC* of the frame can be considered as beams, the beam force-deflection formulas can be used to generate the stiffness matrix of the frame. The forces necessary to cause a displacement along one coordinate while maintaining zero displacements along other coordinates of a beam are indicated in Fig. 6.7(b) [6.1, 6.8]. In Fig. 6.7(a), the ends *A* and *C* are fixed and hence the joint *B* will have three possible displacements—x, y, and θ, as indicated. The forces necessary to maintain a unit displacement along x direction and zero displacement along y and θ directions at the joint *B* are given by (from Fig. 6.7(b))

$$F_x = \left(\frac{12EI}{l^3}\right)_{BC} = \frac{3EI}{2\,l^3}, \qquad F_y = 0, \qquad M_\theta = \left(\frac{6EI}{l^2}\right)_{BC} = \frac{3EI}{2\,l^2}$$

Similarly, when a unit displacement is given along y direction at joint *B* with zero displacements along x and θ directions, the forces required to maintain the configuration can be found from Fig. 6.7(b) as

$$F_x = 0, \qquad F_y = \left(\frac{12EI}{l^3}\right)_{BA} = \frac{24EI}{l^3}, \qquad M_\theta = -\left(\frac{6EI}{l^2}\right)_{BA} = -\frac{12EI}{l^2}$$

Finally, the forces necessary to maintain a unit displacement along θ direction and zero displacements along x and y directions at joint *B* can be seen, from Fig. 6.7(b), as

$$F_x = \left(\frac{6EI}{l^2}\right)_{BC} = \frac{3EI}{2\,l^2}, \qquad F_y = -\left(\frac{6EI}{l^2}\right)_{BA} = -\frac{12EI}{l^3}$$

$$M_\theta = \left(\frac{4EI}{l}\right)_{BC} + \left(\frac{4EI}{l}\right)_{BA} = \frac{2EI}{l} + \frac{8EI}{l} = \frac{10EI}{l}$$

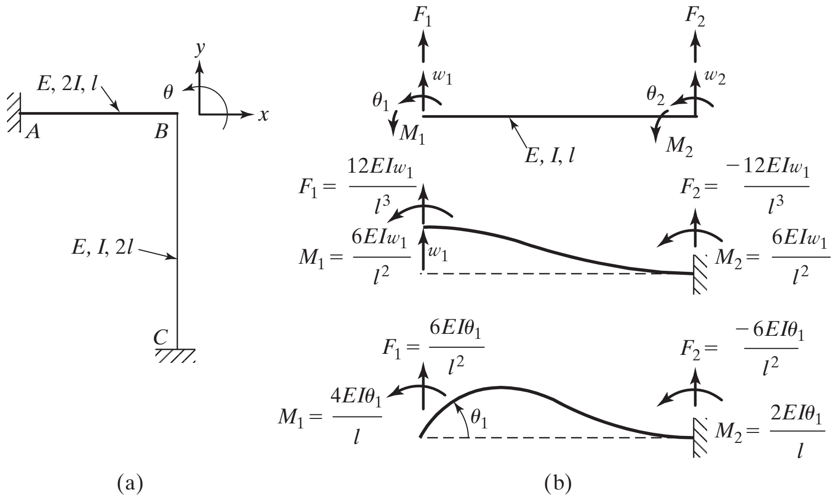


FIGURE 6.7 Stiffness matrix of a frame.

Thus the stiffness matrix, $[k]$, is given by

$$\vec{F} = [k]\vec{x}$$

where

$$\vec{F} = \begin{Bmatrix} F_x \\ F_y \\ M_\theta \end{Bmatrix}, \qquad \vec{x} = \begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix}, \qquad [k] = \frac{EI}{l^3}\begin{bmatrix} \frac{3}{2} & 0 & \frac{3l}{2} \\ 0 & 24 & -12l \\ \frac{3l}{2} & -12l & 10l^2 \end{bmatrix}$$

■

6.4.2 Flexibility Influence Coefficients

As seen in Examples 6.3 and 6.4, the computation of stiffness influence coefficients requires the application of the principles of statics and some algebraic manipulation. In fact, the generation of n stiffness influence coefficients $k_{1j}, k_{2j}, \ldots, k_{nj}$ for any specific j requires the solution of n simultaneous linear equations. Thus n sets of linear equations (n equations in each set) are to be solved to generate all the stiffness influence coefficients of an n-degree-of-freedom system. This implies a significant computational effort for large values of n. The generation of the flexibility influence coefficients, on the other hand, proves to be simpler and more convenient. To illustrate the concept, consider again the spring-mass system shown in Fig. 6.5.

Let the system be acted on by just one force F_j, and let the displacement at point i (i.e., mass m_i) due to F_j be x_{ij}. The flexibility influence coefficient, denoted by a_{ij}, is defined as the deflection at point i due to a unit load at point j. Since the deflection increases proportionately with the load for a linear system, we have

$$x_{ij} = a_{ij}F_j \tag{6.15}$$

If several forces F_j $(j = 1, 2, \ldots, n)$ act at different points of the system, the total deflection at any point i can be found by summing up the contributions of all forces F_j:

$$x_i = \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} a_{ij}F_j, \quad i = 1, 2, \ldots, n \tag{6.16}$$

Equation (6.16) can be expressed in matrix form as

$$\vec{x} = [a]\vec{F} \tag{6.17}$$

where $\vec{x}$ and $\vec{F}$ are the displacement and force vectors defined in Eq. (6.7) and $[a]$ is the flexibility matrix given by

$$[a] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{6.18}$$

The following characteristics of flexibility influence coefficients can be noted:

1. An examination of Eqs. (6.17) and (6.13) indicates that the flexibility and stiffness matrices are related. If we substitute Eq. (6.13) into Eq. (6.17), we obtain

$$\vec{x} = [a]\vec{F} = [a][k]\vec{x} \tag{6.19}$$

from which we can obtain the relation

$$[a][k] = [I] \tag{6.20}$$

where $[I]$ denotes the unit matrix. Equation (6.20) is equivalent to

$$[k] = [a]^{-1}, \qquad [a] = [k]^{-1} \tag{6.21}$$

That is, the stiffness and flexibility matrices are the inverse of one another. The use of dynamic stiffness influence coefficients in the vibration of nonuniform beams is discussed in reference [6.10].

2. Since the deflection at point i due to a unit load at point j is the same as the deflection at point j due to a unit load at point i for a linear system (Maxwell's reciprocity theorem [6.1]), we have $a_{ij} = a_{ji}$.
3. The flexibility influence coefficients of a torsional system can be defined in terms of unit torque and the angular deflection it causes. For example, in a multirotor torsional system, a_{ij} can be defined as the angular deflection of point i (rotor i) due to a unit torque at point j (rotor j).

The flexibility influence coefficients of a multidegree-of-freedom system can be determined as follows:

1. Assume a unit load at point j ($j = 1$ to start with). By definition, the displacements of the various points i ($i = 1, 2, \ldots, n$) resulting from this load give the flexibility influence coefficients, a_{ij}, $i = 1, 2, \ldots, n$. Thus a_{ij} can be found by applying the simple principles of statics and solid mechanics.
2. After completing Step 1 for $j = 1$, the procedure is repeated for $j = 2, 3, \ldots, n$.
3. Instead of applying Steps 1 and 2, the flexibility matrix, $[a]$, can be determined by finding the inverse of the stiffness matrix, $[k]$, if the stiffness matrix is available.

The following examples illustrate the procedure.

EXAMPLE 6.5 Flexibility Influence Coefficients

Find the flexibility influence coefficients of the system shown in Fig. 6.8(a).

Solution: Let x_1, x_2, and x_3 denote the displacements of the masses m_1, m_2, and m_3, respectively. The flexibility influence coefficients a_{ij} of the system can be determined in terms of the spring stiffnesses k_1, k_2, and k_3 as follows. Apply a unit force at mass m_1 and no force at other masses ($F_1 = 1, F_2 = F_3 = 0$), as shown in Fig. 6.8(b). The resulting deflections of the masses m_1, m_2, and

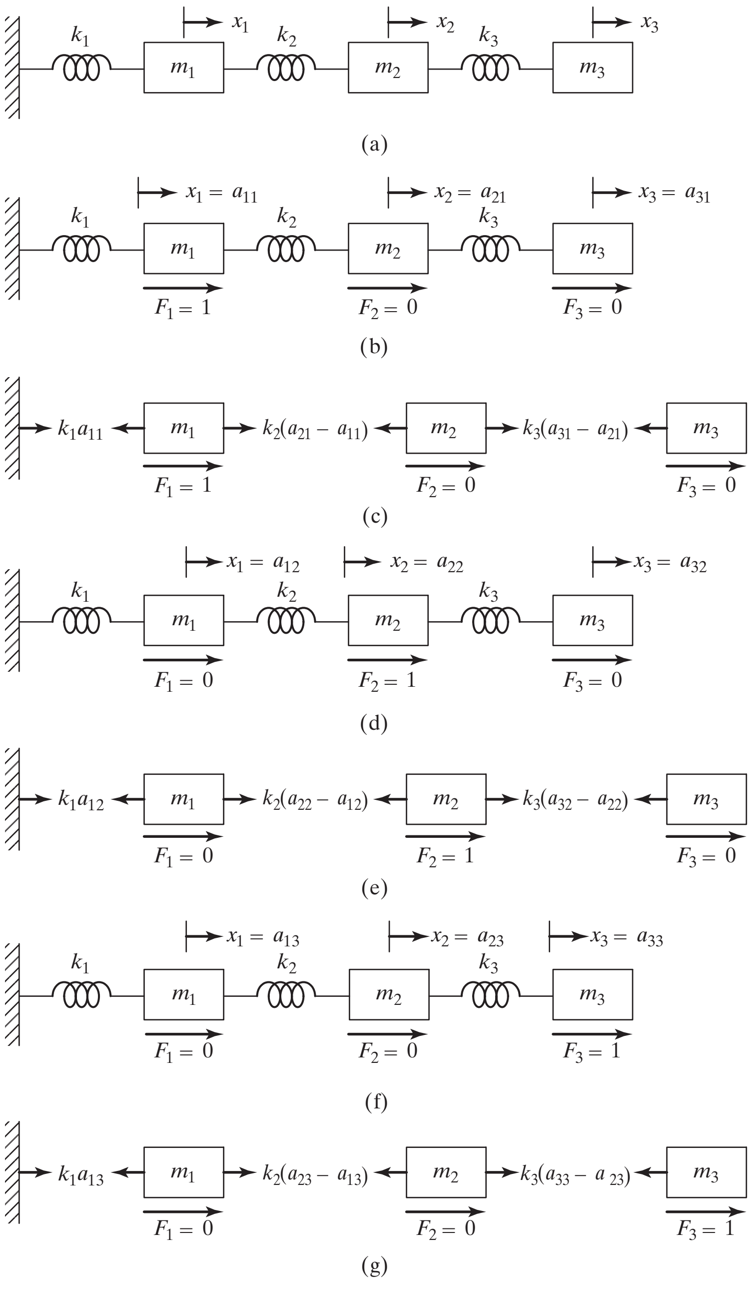


FIGURE 6.8 Determination of flexibility influence coefficients.

m_3 (x_1, x_2, and x_3) are, by definition, a_{11}, a_{21}, and a_{31}, respectively (see Fig. 6.8b). The free-body diagrams of the masses are shown in Fig. 6.8(c). The equilibrium of forces in the horizontal direction for the various masses gives the following:

$$\text{Mass } m_1\text{:}\ \ k_1 a_{11} = k_2(a_{21} - a_{11}) + 1 \tag{E.1}$$

$$\text{Mass } m_2\text{:}\ \ k_2(a_{21} - a_{11}) = k_3(a_{31} - a_{21}) \tag{E.2}$$

$$\text{Mass } m_3\text{:}\ \ k_3(a_{31} - a_{21}) = 0 \tag{E.3}$$

The solution of Eqs. (E.1) to (E.3) yields

$$a_{11} = \frac{1}{k_1}, \qquad a_{21} = \frac{1}{k_1}, \qquad a_{31} = \frac{1}{k_1} \tag{E.4}$$

Next, we apply a unit force at mass m_2 and no force at masses m_1 and m_3, as shown in Fig. 6.8(d). These forces cause the masses m_1, m_2, and m_3 to deflect by $x_1 = a_{12}$, $x_2 = a_{22}$, and $x_3 = a_{32}$, respectively (by definition of a_{i2}), as shown in Fig. 6.8(d). The free-body diagrams of the masses, shown in Fig. 6.8(e), yield the following equilibrium equations:

$$\text{Mass } m_1\text{:}\ \ k_1(a_{12}) = k_2(a_{22} - a_{12}) \tag{E.5}$$

$$\text{Mass } m_2\text{:}\ \ k_2(a_{22} - a_{12}) = k_3(a_{32} - a_{22}) + 1 \tag{E.6}$$

$$\text{Mass } m_3\text{:}\ \ k_3(a_{32} - a_{22}) = 0 \tag{E.7}$$

The solution of Eqs. (E.5) to (E.7) gives

$$a_{12} = \frac{1}{k_1}, \qquad a_{22} = \frac{1}{k_1} + \frac{1}{k_2}, \qquad a_{32} = \frac{1}{k_1} + \frac{1}{k_2} \tag{E.8}$$

Finally, when we apply a unit force to mass m_3 and no force to masses m_1 and m_2, the masses deflect by $x_1 = a_{13}$, $x_2 = a_{23}$, and $x_3 = a_{33}$, as shown in Fig. 6.8(f). The resulting free-body diagrams of the various masses (Fig. 6.8(g)) yield the following equilibrium equations:

$$\text{Mass } m_1\text{:}\ \ k_1 a_{13} = k_2(a_{23} - a_{13}) \tag{E.9}$$

$$\text{Mass } m_2\text{:}\ \ k_2(a_{23} - a_{13}) = k_3(a_{33} - a_{23}) \tag{E.10}$$

$$\text{Mass } m_3\text{:}\ \ k_3(a_{33} - a_{23}) = 1 \tag{E.11}$$

The solution of Eqs. (E.9) to (E.11) gives the flexibility influence coefficients a_{i3} as

$$a_{13} = \frac{1}{k_1}, \qquad a_{23} = \frac{1}{k_1} + \frac{1}{k_2}, \qquad a_{33} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \tag{E.12}$$

It can be verified that the stiffness matrix of the system, given by Eq. (E.13) of Example 6.3, can also be found from the relation $[k] = [a]^{-1}$.

■

EXAMPLE 6.6 Flexibility Matrix of a Beam

Derive the flexibility matrix of the weightless beam shown in Fig. 6.9(a). The beam is simply supported at both ends, and the three masses are placed at equal intervals. Assume the beam to be uniform with stiffness *EI*.

Solution: Let x_1, x_2, and x_3 denote the total transverse deflection of the masses m_1, m_2, and m_3, respectively. From the known formula for the deflection of a pinned-pinned beam [6.2], the influence coefficients $a_{1j}(j = 1, 2, 3)$ can be found by applying a unit load at the location of m_1 and zero load at the locations of m_2 and m_3 (see Fig. 6.9(b)):

$$a_{11} = \frac{9}{768}\frac{l^3}{EI}, \qquad a_{12} = \frac{11}{768}\frac{l^3}{EI}, \qquad a_{13} = \frac{7}{768}\frac{l^3}{EI} \tag{E.1}$$

Similarly, by applying a unit load at the locations of m_2 and m_3 separately (with zero load at other locations), we obtain

$$a_{21} = a_{12} = \frac{11}{768}\frac{l^3}{EI}, \qquad a_{22} = \frac{1}{48}\frac{l^3}{EI}, \qquad a_{23} = \frac{11}{768}\frac{l^3}{EI} \tag{E.2}$$

and

$$a_{31} = a_{13} = \frac{7}{768}\frac{l^3}{EI}, \qquad a_{32} = a_{23} = \frac{11}{768}\frac{l^3}{EI}, \qquad a_{33} = \frac{9}{768}\frac{l^3}{EI} \tag{E.3}$$

Thus the flexibility matrix of the system is given by

$$[a] = \frac{l^3}{768EI}\begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix} \tag{E.4}$$

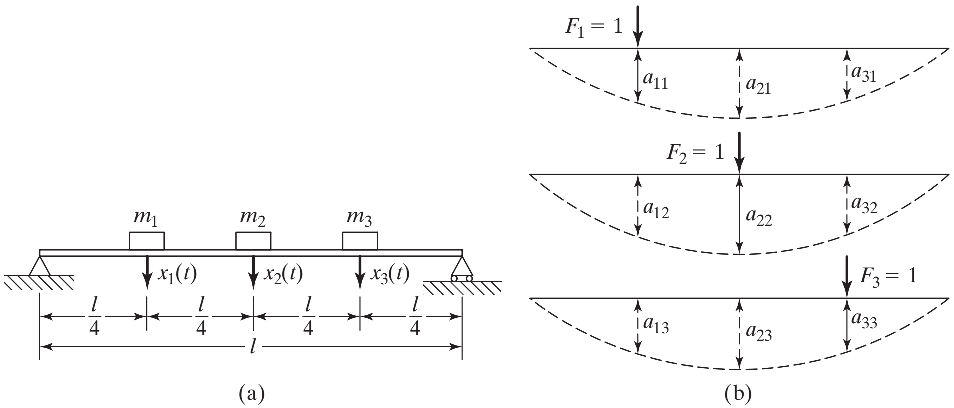


FIGURE 6.9 Beam deflections.

■

6.4.3 Inertia Influence Coefficients

The elements of the mass matrix, m_{ij}, are known as the inertia influence coefficients. Although it is more convenient to derive the inertia influence coefficients from the expression for kinetic energy of the system (see Section 6.5), the coefficients m_{ij} can be computed using the impulse-momentum relations. The inertia influence coefficients $m_{1j}, m_{2j}, \ldots, m_{nj}$ are defined as the set of impulses applied at points 1, 2, ..., n, respectively, to produce a unit velocity at point j and zero velocity at every other point (that is, $\dot{x}_j = 1$, $\dot{x}_1 = \dot{x}_2 = \cdots = \dot{x}_{j-1} = \dot{x}_{j+1} = \cdots = \dot{x}_n = 0$). Thus, for a multidegree-of-freedom system, the total impulse at point i, $\boldsymbol{F}_i$, can be found by summing up the impulses causing the velocities $\dot{x}_j (j = 1, 2, \ldots, n)$ as

$$\boldsymbol{F}_i = \sum_{j=1}^{n} m_{ij} \dot{x}_j \tag{6.22}$$

Equation (6.22) can be stated in matrix form as

$$\vec{\boldsymbol{F}} = [m]\vec{\dot{x}} \tag{6.23}$$

where $\vec{\dot{x}}$ and $\vec{\boldsymbol{F}}$ are the velocity and impulse vectors given by

$$\vec{\dot{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{Bmatrix}, \qquad \vec{\boldsymbol{F}} = \begin{Bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{F}_n \end{Bmatrix} \tag{6.24}$$

and $[m]$ is the mass matrix given by

$$[m] = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix} \tag{6.25}$$

It can be verified easily that the inertia influence coefficients are symmetric for a linear system—that is, $m_{ij} = m_{ji}$. The following procedure can be used to derive the inertia influence coefficients of a multidegree-of-freedom system.

1. Assume that a set of impulses $\boldsymbol{f}_{ij}$ are applied at various points $i (i = 1, 2, \ldots, n)$ so as to produce a unit velocity at point $j (\dot{x}_j = 1$ with $j = 1$ to start with) and a zero velocity at all other points $(\dot{x}_1 = \dot{x}_2 = \cdots \dot{x}_{j-1} = \dot{x}_{j+1} = \cdots = \dot{x}_n = 0)$. By definition, the set of impulses $\boldsymbol{f}_{ij} (i = 1, 2, \ldots, n)$ denote the inertia influence coefficients $m_{ij} (i = 1, 2, \ldots, n)$.

2. After completing step 1 for $j = 1$, the procedure is repeated for $j = 2, 3, \ldots, n$.

Note that if x_j denotes an angular coordinate, then $\dot{x}_j$ represents an angular velocity and $\boldsymbol{F}_j$ indicates an angular impulse. The following example illustrates the procedure of generating m_{ij}.

EXAMPLE 6.7 Inertia Influence Coefficients

Find the inertia influence coefficients of the system shown in Fig. 6.4(a).

Solution:

Approach: Use the definition of m_{ij} along with impulse-momentum relations.

Let $x(t)$ and $\theta(t)$ denote the coordinates to define the linear and angular positions of the trailer (M) and the compound pendulum (m). To derive the inertia influence coefficients, impulses of magnitudes m_{11} and m_{21} are applied along the directions $x(t)$ and $\theta(t)$ to result in the velocities $\dot{x} = 1$ and $\dot{\theta} = 0$. Then the linear impulse-linear momentum equation gives

$$m_{11} = (M + m)(1) \tag{E.1}$$

and the angular impulse-angular momentum equation (about O) yields

$$m_{21} = m(1)\frac{l}{2} \tag{E.2}$$

Next, impulses of magnitudes m_{12} and m_{22} are applied along the directions $x(t)$ and $\theta(t)$ to obtain the velocities $\dot{x} = 0$ and $\dot{\theta} = 1$. Then the linear impulse-linear momentum relation provides

$$m_{12} = m(1)\left(\frac{l}{2}\right) \tag{E.3}$$

and the angular impulse-angular momentum equation (about O) gives

$$m_{22} = \left(\frac{ml^2}{3}\right)(1) \tag{E.4}$$

Thus the mass or inertia matrix of the system is given by

$$[m] = \begin{bmatrix} M + m & \dfrac{ml}{2} \\ \dfrac{ml}{2} & \dfrac{ml^2}{3} \end{bmatrix} \tag{E.5}$$

■

6.13 Free Vibration of Undamped Systems

The equation of motion for the free vibration of an undamped system can be expressed in matrix form as

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{0} \tag{6.95}$$

The most general solution of Eq. (6.95) can be expressed as a linear combination of all possible solutions given by Eqs. (6.56) and (6.62) as

$$\vec{x}(t) = \sum_{i=1}^{n} \vec{X}^{(i)} A_i \cos(\omega_i t + \phi_i) \tag{6.96}$$

where $\vec{X}^{(i)}$ is the ith modal vector and ω_i is the corresponding natural frequency, and A_i and ϕ_i are constants. The constants A_i and $\phi_i (i = 1, 2, \ldots, n)$ can be evaluated from the specified initial conditions of the system. If

$$\vec{x}(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \\ \cdot \\ \cdot \\ \cdot \\ x_n(0) \end{Bmatrix} \quad \text{and} \quad \dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n(0) \end{Bmatrix} \tag{6.97}$$

denote the initial displacements and velocities given to the system, Eqs. (6.96) give

$$\vec{x}(0) = \sum_{i=1}^{n} \vec{X}^{(i)} A_i \cos \phi_i \tag{6.98}$$

$$\dot{\vec{x}}(0) = -\sum_{i=1}^{n} \vec{X}^{(i)} A_i \omega_i \sin \phi_i \tag{6.99}$$

Equations (6.98) and (6.99) represent, in scalar form, $2n$ simultaneous equations which can be solved to find the n values of $A_i (i = 1, 2, \ldots, n)$ and n values of $\phi_i (i = 1, 2, \ldots, n)$.

EXAMPLE 6.15 Free-Vibration Analysis of a Spring-Mass System

Find the free-vibration response of the spring-mass system shown in Fig. 6.12 corresponding to the initial conditions $\dot{x}_i(0) = 0 \, (i = 1, 2, 3)$, $x_1(0) = x_{10}$, $x_2(0) = x_3(0) = 0$. Assume that $k_i = k$ and $m_i = m$ for $i = 1, 2, 3$.

Solution:

Approach: Assume free-vibration response as a sum of natural modes.

The natural frequencies and mode shapes of the system are given by (see Example 6.11):

$$\omega_1 = 0.44504\sqrt{\frac{k}{m}}, \qquad \omega_2 = 1.2471\sqrt{\frac{k}{m}}, \qquad \omega_3 = 1.8025\sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.8019 \\ 2.2470 \end{Bmatrix}, \qquad \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ 0.4450 \\ -0.8020 \end{Bmatrix}, \qquad \vec{X}^{(3)} = \begin{Bmatrix} 1.0 \\ -1.2468 \\ 0.5544 \end{Bmatrix}$$

where the first component of each mode shape is assumed as unity for simplicity. The application of the initial conditions, Eqs. (6.98) and (6.99), leads to

$$A_1 \cos\phi_1 + A_2 \cos\phi_2 + A_3 \cos\phi_3 = x_{10} \tag{E.1}$$

$$1.8019 A_1 \cos\phi_1 + 0.4450 A_2 \cos\phi_2 - 1.2468 A_3 \cos\phi_3 = 0 \tag{E.2}$$

$$2.2470 A_1 \cos\phi_1 - 0.8020 A_2 \cos\phi_2 + 0.5544 A_3 \cos\phi_3 = 0 \tag{E.3}$$

$$-0.44504\sqrt{\frac{k}{m}} A_1 \sin\phi_1 - 1.2471\sqrt{\frac{k}{m}} A_2 \sin\phi_2 - 1.8025\sqrt{\frac{k}{m}} A_3 \sin\phi_3 = 0 \tag{E.4}$$

$$-0.80192\sqrt{\frac{k}{m}} A_1 \sin\phi_1 - 0.55496\sqrt{\frac{k}{m}} A_2 \sin\phi_2 + 2.2474\sqrt{\frac{k}{m}} A_3 \sin\phi_3 = 0 \tag{E.5}$$

$$-1.0\sqrt{\frac{k}{m}} A_1 \sin\phi_1 + 1.0\sqrt{\frac{k}{m}} A_2 \sin\phi_2 - 1.0\sqrt{\frac{k}{m}} A_3 \sin\phi_3 = 0 \tag{E.6}$$

The solution of Eqs. (E.1) to (E.6) is given by[9] $A_1 = 0.1076 x_{10}$, $A_2 = 0.5431 x_{10}$, $A_3 = 0.3493 x_{10}$, $\phi_1 = 0$, $\phi_2 = 0$, and $\phi_3 = 0$. Thus the free-vibration solution of the system can be expressed as

$$\begin{aligned} x_1(t) = x_{10}\Bigg[& 0.1076 \cos\left(0.44504\sqrt{\frac{k}{m}}\, t\right) \\ & + 0.5431 \cos\left(1.2471\sqrt{\frac{k}{m}}\, t\right) \\ & + 0.3493 \cos\left(1.8025\sqrt{\frac{k}{m}}\, t\right)\Bigg] \end{aligned} \tag{E.7}$$

[9]Note that Eqs. (E.1) to (E.3) can be considered as a system of linear equations in the unknowns $A_1 \cos\phi_1$, $A_2 \cos\phi_2$, and $A_3 \cos\phi_3$, while Eqs. (E.4) to (E.6) can be considered as a set of linear equations in the unknowns

$$\sqrt{\frac{k}{m}} A_1 \sin\phi_1, \quad \sqrt{\frac{k}{m}} A_2 \sin\phi_2, \quad \text{and} \quad \sqrt{\frac{k}{m}} A_3 \sin\phi_3.$$

$$x_2(t) = x_{10}\left[0.1939\cos\left(0.44504\sqrt{\frac{k}{m}}\,t\right) + 0.2417\cos\left(1.2471\sqrt{\frac{k}{m}}\,t\right) - 0.4355\cos\left(1.8025\sqrt{\frac{k}{m}}\,t\right)\right] \tag{E.8}$$

$$x_3(t) = x_{10}\left[0.2418\cos\left(0.44504\sqrt{\frac{k}{m}}\,t\right) - 0.4356\cos\left(1.2471\sqrt{\frac{k}{m}}\,t\right) + 0.1937\cos\left(1.8025\sqrt{\frac{k}{m}}\,t\right)\right] \tag{E.9}$$

■

6.14 Forced Vibration of Undamped Systems Using Modal Analysis

When external forces act on a multidegree-of-freedom system, the system undergoes forced vibration. For a system with n coordinates or degrees of freedom, the governing equations of motion are a set of n coupled ordinary differential equations of second order. The solution of these equations becomes more complex when the degree of freedom of the system (n) is large and/or when the forcing functions are nonperiodic.[10] In such cases, a more convenient method known as *modal analysis* can be used to solve the problem. In this method, the expansion theorem is used, and the displacements of the masses are expressed as a linear combination of the normal modes of the system. This linear transformation uncouples the equations of motion so that we obtain a set of n uncoupled differential equations of second order. The solution of these equations, which is equivalent to the solution of the equations of n single-degree-of-freedom systems, can be readily obtained. We shall now consider the procedure of modal analysis.

Modal Analysis. The equations of motion of a multidegree-of-freedom system under external forces are given by

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F} \tag{6.100}$$

[10]The dynamic response of multidegree-of-freedom systems with statistical properties is considered in reference [6.15].

where $\vec{F}$ is the vector of arbitrary external forces. To solve Eq. (6.100) by modal analysis, it is necessary first to solve the eigenvalue problem.

$$\omega^2[m]\vec{X} = [k]\vec{X} \tag{6.101}$$

and find the natural frequencies $\omega_1, \omega_2, \ldots, \omega_n$ and the corresponding normal modes $\vec{X}^{(1)}, \vec{X}^{(2)}, \ldots, \vec{X}^{(n)}$. According to the expansion theorem, the solution vector of Eq. (6.100) can be expressed by a linear combination of the normal modes

$$\vec{x}(t) = q_1(t)\vec{X}^{(1)} + q_2(t)\vec{X}^{(2)} + \cdots + q_n(t)\vec{X}^{(n)} \tag{6.102}$$

where $q_1(t), q_2(t), \ldots, q_n(t)$ are time-dependent generalized coordinates, also known as the *principal coordinates* or *modal participation coefficients*. By defining a modal matrix $[X]$ in which the jth column is the vector $\vec{X}^{(j)}$—that is,

$$[X] = [\vec{X}^{(1)}\vec{X}^{(2)} \cdots \vec{X}^{(n)}] \tag{6.103}$$

Eq. (6.102) can be rewritten as

$$\vec{x}(t) = [X]\vec{q}(t) \tag{6.104}$$

where

$$\vec{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{Bmatrix} \tag{6.105}$$

Since $[X]$ is not a function of time, we obtain from Eq. (6.104)

$$\ddot{\vec{x}}(t) = [X]\ddot{\vec{q}}(t) \tag{6.106}$$

Using Eqs. (6.104) and (6.106), we can write Eq. (6.100) as

$$[m][X]\ddot{\vec{q}} + [k][X]\vec{q} = \vec{F} \tag{6.107}$$

Premultiplying Eq. (6.107) throughout by $[X]^T$, we obtain

$$[X]^T[m][X]\ddot{\vec{q}} + [X]^T[k][X]\vec{q} = [X]^T\vec{F} \tag{6.108}$$

If the normal modes are normalized according to Eqs. (6.74) and (6.75), we have

$$[X]^T[m][X] = [I] \tag{6.109}$$

$$[X]^T[k][X] = [\nwarrow \omega^2 \searrow] \tag{6.110}$$

By defining the vector of generalized forces $\vec{Q}(t)$ associated with the generalized coordinates $\vec{q}(t)$ as

$$\vec{Q}(t) = [X]^T \vec{F}(t) \tag{6.111}$$

Eq. (6.108) can be expressed, using Eqs. (6.109) and (6.110), as

$$\ddot{\vec{q}}(t) + [\nwarrow \omega^2 \searrow] \vec{q}(t) = \vec{Q}(t) \tag{6.112}$$

Equation (6.112) denotes a set of n uncoupled differential equations of second order[11]

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t), \qquad i = 1, 2, \ldots, n \tag{6.113}$$

It can be seen that Eqs. (6.113) have precisely the form of the differential equation describing the motion of an undamped single-degree-of-freedom system. The solution of Eqs. (6.113) can be expressed (see Eq. (4.31)) as

$$\begin{aligned} q_i(t) &= q_i(0) \cos \omega_i t + \left(\frac{\dot{q}(0)}{\omega_i} \right) \sin \omega_i t \\ &\quad + \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i (t - \tau)\, d\tau, \\ &\quad i = 1, 2, \ldots, n \end{aligned} \tag{6.114}$$

The initial generalized displacements $q_i(0)$ and the initial generalized velocities $\dot{q}_i(0)$ can be obtained from the initial values of the physical displacements $x_i(0)$ and physical velocities $\dot{x}_i(0)$ as (see Problem 6.94):

$$\vec{q}(0) = [X]^T [m] \vec{x}(0) \tag{6.115}$$

$$\dot{\vec{q}}(0) = [X]^T [m] \dot{\vec{x}}(0) \tag{6.116}$$

---

[11] It is possible to approximate the solution vector $\vec{x}(t)$ by only the first r $(r < n)$ modal vectors (instead of n vectors as in Eq. (6.102)):

$$\underset{n\times 1}{\vec{x}(t)} = \underset{n\times r}{[X]}\, \underset{r\times 1}{\vec{q}(t)}$$

where

$$[X] = [\vec{X}^{(1)}\ \vec{X}^{(2)}\ \cdots\ \vec{X}^{(r)}] \qquad \text{and} \qquad \vec{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_r(t) \end{Bmatrix}$$

This leads to only r uncoupled differential equations

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t), \qquad i = 1, 2, \ldots, r$$

instead of n equations. The resulting solution $\vec{x}(t)$ will be an approximate solution. This procedure is called the *mode displacement method*. An alternate procedure, *mode acceleration method*, for finding an approximate solution is indicated in Problem 6.92.

where

$$\vec{q}(0) = \begin{Bmatrix} q_1(0) \\ q_2(0) \\ \vdots \\ q_n(0) \end{Bmatrix},$$

$$\dot{\vec{q}}(0) = \begin{Bmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \\ \vdots \\ \dot{q}_n(0) \end{Bmatrix},$$

$$\vec{x}(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{Bmatrix},$$

$$\dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \vdots \\ \dot{x}_n(0) \end{Bmatrix}$$

Once the generalized displacements $q_i(t)$ are found, using Eqs. (6.114) to (6.116), the physical displacements $x_i(t)$ can be found with the help of Eq. (6.104).

EXAMPLE 6.16 Free-Vibration Response Using Modal Analysis

Using modal analysis, find the free-vibration response of a two-degree-of-freedom system with equations of motion

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \vec{F} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E.1}$$

Assume the following data: $m_1 = 10$, $m_2 = 1$, $k_1 = 30$, $k_2 = 5$, $k_3 = 0$, and

$$\vec{x}(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \qquad \dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E.2}$$

Solution: The natural frequencies and normal modes of the system are given by (see Example 5.3)

$$\omega_1 = 1.5811, \qquad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} X_1^{(1)}$$

$$\omega_2 = 2.4495, \qquad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -5 \end{Bmatrix} X_1^{(2)}$$

where $X_1^{(1)}$ and $X_1^{(2)}$ are arbitrary constants. By orthogonalizing the normal modes with respect to the mass matrix, we can find the values of $X_1^{(1)}$ and $X_1^{(2)}$ as

$$\vec{X}^{(1)^T}[m]\vec{X}^{(1)} = 1 \Rightarrow (X_1^{(1)})^2 \{1 \quad 2\} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = 1$$

or $X_1^{(1)} = 0.2673$

$$\vec{X}^{(2)^T}[m]\vec{X}^{(2)} = 1 \Rightarrow (X_1^{(2)})^2 \{1 \quad -5\} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -5 \end{Bmatrix} = 1$$

or $X_1^{(2)} = 0.1690$

Thus the modal matrix becomes

$$[X] = \begin{bmatrix} \vec{X}^{(1)} & \vec{X}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.2673 & 0.1690 \\ 0.5346 & -0.8450 \end{bmatrix} \tag{E.3}$$

Using

$$\vec{x}(t) = [X]\vec{q}(t) \tag{E.4}$$

Equation (E.1) can be expressed as (see Eq. (6.112)):

$$\ddot{\vec{q}}(t) + [\nwarrow \omega^2 \searrow]\vec{q}(t) = \vec{Q}(t) = \vec{0} \tag{E.5}$$

where $\vec{Q}(t) = [X]^T \vec{F} = \vec{0}$. Equation (E.5) can be written in scalar form as

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = 0, \qquad i = 1, 2 \tag{E.6}$$

The solution of Eq. (E.6) is given by (see Eq. 2.18):

$$q_i(t) = q_{i0} \cos \omega_i t + \frac{\dot{q}_{i0}}{\omega_i} \sin \omega_i t \tag{E.7}$$

where q_{i0} and $\dot{q}_{i0}$ denote the initial values of $q_i(t)$ and $\dot{q}_i(t)$, respectively. Using the initial conditions of Eq. (E.2), we can find (see Eqs. (6.115) and (6.116)):

$$\begin{aligned} \vec{q}(0) &= \begin{Bmatrix} q_{10}(0) \\ q_{20}(0) \end{Bmatrix} = [X]^T[m]\ \vec{x}(0) \\ &= \begin{bmatrix} 0.2673 & 0.5346 \\ 0.1690 & -0.8450 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2.673 \\ 1.690 \end{Bmatrix} \end{aligned} \tag{E.8}$$

$$\dot{\vec{q}}(0) = \begin{Bmatrix} \dot{q}_{10}(0) \\ \dot{q}_{20}(0) \end{Bmatrix} = [X]^T[m]\dot{\vec{x}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E.9}$$

Equations (E.7) to (E.9) lead to

$$q_1(t) = 2.673 \cos 1.5811t \tag{E.10}$$

$$q_2(t) = 1.690 \cos 2.4495t \tag{E.11}$$

Using Eqs. (E.4), we obtain the displacements of the masses m_1 and m_2 as

$$\vec{x}(t) = \begin{bmatrix} 0.2673 & 0.1690 \\ 0.5346 & -0.8450 \end{bmatrix} \begin{Bmatrix} 2.673 \cos 1.5811t \\ 1.690 \cos 2.4495t \end{Bmatrix}$$

or

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} 0.7145 \cos 1.5811t + 0.2856 \cos 2.4495t \\ 1.4280 \cos 1.5811t - 1.4280 \cos 2.4495t \end{Bmatrix} \tag{E.12}$$

It can be seen that this solution is identical to the one obtained in Example 5.3 and plotted in Example 5.17.

■

EXAMPLE 6.17 Forced-Vibration Response of a Forging Hammer

The force acting on the workpiece of the forging hammer shown in Fig. 5.51 due to impact by the hammer can be approximated as a rectangular pulse, as shown in Fig. 6.15(a). Find the resulting vibration of the system for the following data: mass of the workpiece, anvil and frame $(m_1) = 200$ Mg, mass of the foundation block $(m_2) = 250$ Mg, stiffness of the elastic pad $(k_1) = 150$ MN/m, and stiffness of the soil $(k_2) = 75$ MN/m. Assume the initial displacements and initial velocities of the masses as zero.

Solution: The forging hammer can be modeled as a two-degree-of-freedom system as indicated in Fig. 6.15(b). The equations of motion of the system can be expressed as

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F}(t) \tag{E.1}$$

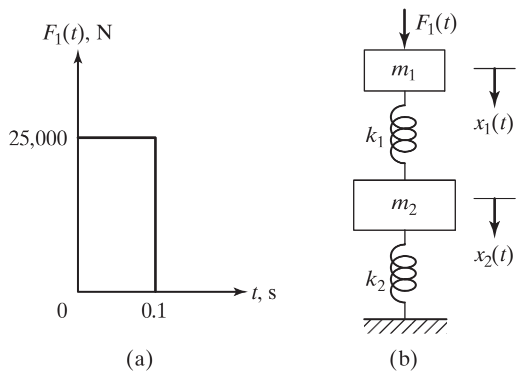


FIGURE 6.15 Impact caused by forging hammer.

where

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 200 & 0 \\ 0 & 250 \end{bmatrix} \text{Mg}$$

$$[k] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} = \begin{bmatrix} 150 & -150 \\ -150 & 225 \end{bmatrix} \text{MN/m}$$

$$\vec{F}(t) = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$

Natural Frequencies and Mode Shapes: The natural frequencies of the system can be found by solving the frequency equation

$$|-\omega^2[m] + [k]| = \left| -\omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 2.5 \end{bmatrix} 10^5 + \begin{bmatrix} 150 & -150 \\ -150 & 225 \end{bmatrix} 10^6 \right| = 0 \tag{E.2}$$

as

$$\omega_1 = 12.2474 \text{ rad/s} \qquad \text{and} \qquad \omega_2 = 38.7298 \text{ rad/s}$$

The mode shapes can be found as

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 0.8 \end{Bmatrix} \qquad \text{and} \qquad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Orthonormalization of Mode Shapes: The mode shapes are assumed as

$$\vec{X}^{(1)} = a \begin{Bmatrix} 1 \\ 0.8 \end{Bmatrix} \qquad \text{and} \qquad \vec{X}^{(2)} = b \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

where a and b are constants. The constants a and b can be determined by normalizing the vectors $\vec{X}^{(1)}$ and $\vec{X}^{(2)}$ as

$$[X]^T [m] [X] = [I] \tag{E.3}$$

where $[X] = [\vec{X}^{(1)} \vec{X}^{(2)}]$ denotes the modal matrix. Equation (E.3) gives $a = 1.6667 \times 10^{-3}$ and $b = 1.4907 \times 10^{-3}$, which means that the new modal matrix (with normalized mode shapes) becomes

$$[X] = [\vec{X}^{(1)} \vec{X}^{(2)}] = \begin{bmatrix} 1.6667 & 1.4907 \\ 1.3334 & -1.4907 \end{bmatrix} \times 10^{-3}$$

Response in Terms of Generalized Coordinates: Since the two masses m_1 and m_2 are at rest at $t = 0$, the initial conditions are $x_1(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$, hence Eqs. (6.115) and (6.116) give $q_1(0) = q_2(0) = \dot{q}_1(0) = \dot{q}_2(0) = 0$. Thus the generalized coordinates are given by the solution of the equations

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i(t - \tau)\, d\tau, \; i = 1, 2 \tag{E.4}$$

where

$$\vec{Q}(t) = [X]^T \vec{F}(t) \tag{E.5}$$

or

$$\begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix} = \begin{bmatrix} 1.6667 & 1.3334 \\ 1.4907 & -1.4907 \end{bmatrix} 10^{-3} \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$
$$= \begin{Bmatrix} 1.6667 \times 10^{-3} F_1(t) \\ 1.4907 \times 10^{-3} F_1(t) \end{Bmatrix} \tag{E.6}$$

with $F_1(t) = 25000$ N for $0 \leq t \leq 0.1$ s and 0 for $t > 0.1$ s. Using Eq. (6.104), the displacements of the masses can be found as

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = [X]\ \vec{q}(t) = \begin{Bmatrix} 1.6667 q_1(t) + 1.4907 q_2(t) \\ 1.3334 q_1(t) - 1.4907 q_2(t) \end{Bmatrix} 10^{-3} \text{ m} \tag{E.7}$$

where

$$q_1(t) = 3.4021 \int_0^t \sin 12.2474\,(t - \tau)\, d\tau = 0.2778\,(1 - \cos 12.2474\, t)$$

$$q_2(t) = 0.9622 \int_0^t \sin 38.7298\,(t - \tau)\, d\tau = 0.02484\,(1 - \cos 38.7298\, t) \tag{E.8}$$

Note that the solution given by Eqs. (E.8) is valid for $0 \leq 0.1$ s. For $t > 0.1$ s, there is no applied force, hence the response is given by the free-vibration solution of an undamped single-degree-of-freedom system (Eq. (2.18)) for $q_1(t)$ and $q_2(t)$ with $q_1(0.1)$ and $\dot{q}_1(0.1)$, and $q_2(0.1)$ and $\dot{q}_2(0.1)$ as initial conditions for $q_1(t)$ and $q_2(t)$, respectively.

■

6.15 Forced Vibration of Viscously Damped Systems

Modal analysis, as presented in Section 6.14, applies only to undamped systems. In many cases, the influence of damping upon the response of a vibratory system is minor and can be disregarded. However, it must be considered if the response of the system is required for a relatively long period of time compared to the natural periods of the system. Further, if the frequency of excitation (in the case of a periodic force) is at or near one of the natural frequencies of the system, damping is of primary importance and must be taken into account. In general, since the effects are not known in advance, damping must be considered in the vibration analysis of any system. In this section, we shall consider the equations of motion of a damped multidegree-of-freedom system and their solution using Lagrange's equations. If the system has viscous damping, its motion will be resisted by a force whose magnitude is proportional to that of the velocity but in the opposite direction. It is convenient to introduce a function R, known as Rayleigh's dissipation function, in deriving the equations of motion by means of Lagrange's equations [6.7]. This function is defined as

$$R = \frac{1}{2}\dot{\vec{x}}^T[c]\dot{\vec{x}} \tag{6.117}$$

where the matrix $[c]$ is called the *damping matrix* and is positive definite, like the mass and stiffness matrices. Lagrange's equations, in this case [6.8], can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial R}{\partial \dot{x}_i} + \frac{\partial V}{\partial x_i} = F_i, \qquad i = 1, 2, \ldots, n \tag{6.118}$$

where F_i is the force applied to mass m_i. By substituting Eqs. (6.30), (6.34), and (6.117) into Eq. (6.118), we obtain the equations of motion of a damped multidegree-of-freedom system in matrix form:

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F} \tag{6.119}$$

For simplicity, we shall consider a special system for which the damping matrix can be expressed as a linear combination of the mass and stiffness matrices:

$$[c] = \alpha[m] + \beta[k] \tag{6.120}$$

where α and β are constants. This is known as *proportional damping* because $[c]$ is proportional to a linear combination of $[m]$ and $[k]$. By substituting Eq. (6.120) into Eq. (6.119), we obtain

$$[m]\ddot{\vec{x}} + [\alpha[m] + \beta[k]]\dot{\vec{x}} + [k]\vec{x} = \vec{F} \tag{6.121}$$

By expressing the solution vector $\vec{x}$ as a linear combination of the natural modes of the undamped system, as in the case of Eq. (6.104),

$$\vec{x}(t) = [X]\vec{q}(t) \tag{6.122}$$

Eq. (6.121) can be rewritten as

$$[m][X]\ddot{\vec{q}}(t) + [\alpha[m] + \beta[k]][X]\,\dot{\vec{q}}(t) + [k][X]\vec{q}(t) = \vec{F}(t) \tag{6.123}$$

Premultiplication of Eq. (6.123) by $[X]^T$ leads to

$$[X]^T[m][X]\ddot{\vec{q}} + [\alpha[X]^T[m][X] + \beta[X]^T[k][X]]\dot{\vec{q}} + [X]^T[k][X]\vec{q} = [X]^T\vec{F} \tag{6.124}$$

If the eigenvectors $\vec{X}^{(j)}$ are normalized according to Eqs. (6.74) and (6.75), Eq. (6.124) reduces to

$$[I]\ddot{\vec{q}}(t) + \left[\alpha[I] + \beta\left[\nwarrow\omega^2\searrow\right]\right]\dot{\vec{q}}(t) + \left[\nwarrow\omega^2\searrow\right]\vec{q}(t) = \vec{Q}(t)$$

—that is,

$$\ddot{q}_i(t) + (\alpha + \omega_i^2\beta)\dot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t), \quad i = 1, 2, \ldots, n \tag{6.125}$$

where ω_i is the ith natural frequency of the undamped system and

$$\vec{Q}(t) = [X]^T \vec{F}(t) \tag{6.126}$$

By writing

$$\alpha + \omega_i^2\beta = 2\zeta_i\omega_i \tag{6.127}$$

where ζ_i is called the *modal damping ratio* for the ith normal mode, Eqs. (6.125) can be rewritten as

$$\ddot{q}_i(t) + 2\zeta_i\omega_i\dot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t), \qquad i = 1, 2, \ldots, n \tag{6.128}$$

It can be seen that each of the n equations represented by this expression is uncoupled from all of the others. Hence we can find the response of the ith mode in the same manner as that of a viscously damped single-degree-of-freedom system. The solution of Eqs. (6.128), when $\zeta_i < 1$, can be expressed as

$$\begin{aligned} q_i(t) &= e^{-\zeta_i\omega_i t}\left\{\cos\omega_{di}t + \frac{\zeta_i}{\sqrt{1-\zeta_i^2}}\sin\omega_{di}t\right\}q_i(0) \\ &\quad + \left\{\frac{1}{\omega_{di}}e^{-\zeta_i\omega_i t}\sin\omega_{di}t\right\}\dot{q}_0(0) \\ &\quad + \frac{1}{\omega_{di}}\int_0^t Q_i(\tau)e^{-\zeta_i\omega_i(t-\tau)}\sin\omega_{di}(t-\tau)\,d\tau, \\ & i = 1, 2, \ldots, n \end{aligned} \tag{6.129}$$

where

$$\omega_{di} = \omega_i\sqrt{1-\zeta_i^2} \tag{6.130}$$

Note the following aspects of these systems:

1. The identification of the sources and magnitude of damping is difficult in most practical problems. More than one type of damping—Coulomb, viscous, and hysteretic—may be present in the system. In addition, the exact nature of damping, such as linear, quadratic, cubic or other type of variation, is not known. Even when the source and nature of damping are known, obtaining the precise magnitude is very difficult. For some practical systems, experimentally determined damping values may be available for use in vibration analysis. Some damping, in the form of structural damping, is present in automobile, aerospace, and machine structures. Damping is introduced deliberately in certain practical applications such as vehicle suspension systems, aircraft landing gear, and machine isolation systems. Because the analysis of damped systems involves lengthy mathematical manipulations, in many vibration studies damping is either neglected or assumed to be proportional.
2. It has been shown by Caughey [6.9] that the condition given by Eq. (6.120) is sufficient but not necessary for the existence of normal modes in damped systems. The necessary condition is that the transformation that diagonalizes the damping matrix also uncouples the coupled equations of motion. This condition is less restrictive than Eq. (6.120) and covers more possibilities.
3. In the general case of damping, the damping matrix cannot be diagonalized simultaneously with the mass and stiffness matrices. In this case, the eigenvalues of the system are either real and negative or complex with negative real parts. The complex eigenvalues exist as conjugate pairs: the associated eigenvectors also consist of complex conjugate pairs. A common procedure for finding the solution of the eigenvalue problem of a damped system involves the transformation of the n coupled second-order equations of motion into $2n$ uncoupled first-order equations [6.6].
4. The error bounds and numerical methods in the modal analysis of dynamic systems are discussed in references [6.11, 6.12].

EXAMPLE 6.18 Equations of Motion of a Dynamic System

Derive the equations of motion of the system shown in Fig. 6.16.

Solution:

Approach: Use Lagrange's equations in conjunction with Rayleigh's dissipation function.

The kinetic energy of the system is

$$T = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2) \tag{E.1}$$

The potential energy has the form

$$V = \frac{1}{2}[k_1x_1^2 + k_2(x_2 - x_1)^2 + k_3(x_3 - x_2)^2] \tag{E.2}$$

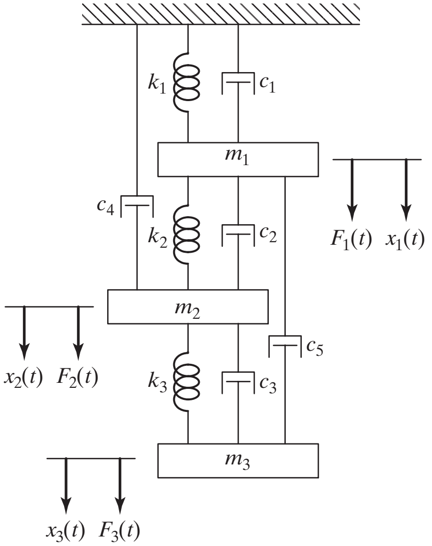


FIGURE 6.16 Three-degree-of-freedom dynamic system.

and Rayleigh's dissipation function is

$$R = \frac{1}{2}[c_1\dot{x}_1^2 + c_2(\dot{x}_2 - \dot{x}_1)^2 + c_3(\dot{x}_3 - \dot{x}_2)^2 + c_4\dot{x}_2^2 + c_5(\dot{x}_3 - \dot{x}_1)^2] \tag{E.3}$$

Lagrange's equations can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial R}{\partial \dot{x}_i} + \frac{\partial V}{\partial x_i} = F_i, \qquad i = 1, 2, 3 \tag{E.4}$$

By substituting Eqs. (E.1) to (E.3) into Eq. (E.4), we obtain the differential equations of motion

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F} \tag{E.5}$$

where

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{E.6}$$

$$[c] = \begin{bmatrix} c_1 + c_2 + c_5 & -c_2 & -c_5 \\ -c_2 & c_2 + c_3 + c_4 & -c_3 \\ -c_5 & -c_3 & c_3 + c_5 \end{bmatrix} \tag{E.7}$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{E.8}$$

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} \quad \text{and} \quad \vec{F} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix} \tag{E.9}$$

■

EXAMPLE 6.19 Steady-State Response of a Forced System

Find the steady-state response of the system shown in Fig. 6.16 when the masses are subjected to the simple harmonic forces $F_1 = F_2 = F_3 = F_0 \cos \omega t$, where $\omega = 1.75\sqrt{k/m}$. Assume that $m_1 = m_2 = m_3 = m$, $k_1 = k_2 = k_3 = k$, $c_4 = c_5 = 0$, and the damping ratio in each normal mode is given by $\zeta_i = 0.01$, $i = 1, 2, 3$.

Solution: The (undamped) natural frequencies of the system (see Example 6.11) are given by

$$\omega_1 = 0.44504\sqrt{\frac{k}{m}}$$

$$\omega_2 = 1.2471\sqrt{\frac{k}{m}}$$

$$\omega_3 = 1.8025\sqrt{\frac{k}{m}} \tag{E.1}$$

and the corresponding $[m]$-orthonormal mode shapes (see Example 6.12) are given by

$$\vec{X}^{(1)} = \frac{0.3280}{\sqrt{m}}\begin{Bmatrix} 1.0 \\ 1.8019 \\ 2.2470 \end{Bmatrix}, \qquad \vec{X}^{(2)} = \frac{0.7370}{\sqrt{m}}\begin{Bmatrix} 1.0 \\ 1.4450 \\ -0.8020 \end{Bmatrix}$$

$$\vec{X}^{(3)} = \frac{0.5911}{\sqrt{m}}\begin{Bmatrix} 1.0 \\ -1.2468 \\ 0.5544 \end{Bmatrix} \tag{E.2}$$

Thus the modal vector can be expressed as

$$[X] = [\vec{X}^{(1)}\vec{X}^{(2)}\vec{X}^{(3)}] = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.3280 & 0.7370 & 0.5911 \\ 0.5911 & 0.3280 & -0.7370 \\ 0.7370 & -0.5911 & 0.3280 \end{bmatrix} \tag{E.3}$$

The generalized force vector

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.3280 & 0.5911 & 0.7370 \\ 0.7370 & 0.3280 & -0.5911 \\ 0.5911 & -0.7370 & 0.3280 \end{bmatrix} \begin{Bmatrix} F_0 \cos \omega t \\ F_0 \cos \omega t \\ F_0 \cos \omega t \end{Bmatrix}$$

$$= \begin{Bmatrix} Q_{10} \\ Q_{20} \\ Q_{30} \end{Bmatrix} \cos \omega t \tag{E.4}$$

can be obtained where

$$Q_{10} = 1.6561 \frac{F_0}{\sqrt{m}}, \qquad Q_{20} = 0.4739 \frac{F_0}{\sqrt{m}}, \qquad Q_{30} = 0.1821 \frac{F_0}{\sqrt{m}} \tag{E.5}$$

If the generalized coordinates or the modal participation factors for the three principal modes are denoted as $q_1(t)$, $q_2(t)$, and $q_3(t)$, the equations of motion can be expressed as

$$\ddot{q}_i(t) + 2\zeta_i \omega_i \dot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t), \qquad i = 1, 2, 3 \tag{E.6}$$

The steady-state solution of Eqs. (E.6) can be written as

$$q_i(t) = q_{i0} \cos(\omega t - \phi), \qquad i = 1, 2, 3 \tag{E.7}$$

where

$$q_{i0} = \frac{Q_{i0}}{\omega_i^2} \frac{1}{\left[\left\{ 1 - \left(\frac{\omega}{\omega_i} \right)^2 \right\}^2 + \left(2\zeta_i \frac{\omega}{\omega_i} \right)^2 \right]^{1/2}} \tag{E.8}$$

and

$$\phi_i = \tan^{-1} \left\{ \frac{2\zeta_i \frac{\omega}{\omega_i}}{1 - \left(\frac{\omega}{\omega_i} \right)^2} \right\} \tag{E.9}$$

By substituting the values given in Eqs. (E.5) and (E.1) into Eqs. (E.8) and (E.9), we obtain

$$\begin{aligned} q_{10} &= 0.57815 \frac{F_0 \sqrt{m}}{k}, & \phi_1 &= \tan^{-1}(-0.00544) \\ q_{20} &= 0.31429 \frac{F_0 \sqrt{m}}{k}, & \phi_2 &= \tan^{-1}(-0.02988) \\ q_{30} &= 0.92493 \frac{F_0 \sqrt{m}}{k}, & \phi_3 &= \tan^{-1}(0.33827) \end{aligned} \tag{E.10}$$

Finally the steady-state response can be found using Eq. (6.122).

■

John William Strutt, Lord Rayleigh (1842–1919), was an English physicist who held the positions of professor of experimental physics at Cambridge University, professor of natural philosophy at the Royal Institution in London, president of the Royal Society, and chancellor of Cambridge University. His works in optics and acoustics are well known, with *Theory of Sound* (1877) considered as a standard reference even today. The method of computing approximate natural frequencies of vibrating bodies using an energy approach has become known as "Rayleigh's method." (Courtesy of *Applied Mechanics Reviews.*)

CHAPTER 7

Determination of Natural Frequencies and Mode Shapes

Chapter Outline

Several methods of determining the natural frequencies and mode shapes of multidegree-of-freedom systems are outlined in this chapter. Specifically, Dunkerley's formula, Rayleigh's method, Holzer's method, the matrix iteration method, and Jacobi's method are presented. Derivation of Dunkerley's formula is based on the fact that higher natural frequencies of most systems are large compared to their fundamental frequencies. It gives an approximate value, always smaller than the exact value, of the fundamental natural frequency. Rayleigh's method, which is based on Rayleigh's principle, also gives an approximate value of the fundamental natural frequency, which is always larger than the exact value. Proof is given of Rayleigh's quotient and its stationariness in the neighborhood of an eigenvalue. It is also shown that the Rayleigh's quotient is never lower than the first eigenvalue and never higher than the highest eigenvalue. Use of the static deflection curve in estimating the fundamental natural frequencies of beams and shafts using Rayleigh's method is presented. Holzer's method, based on a trial-and-error scheme, is presented to find the natural frequencies of undamped, damped, semidefinite, or branched translational and torsional systems. The matrix iteration method and its extensions for finding the smallest, highest, and intermediate natural frequencies are presented. A proof for the convergence of the method to the smallest frequency is given. Jacobi's method, which finds all the eigenvalues and eigenvectors of real symmetric matrices, is outlined. The standard eigenvalue problem is defined and the method of deriving it from the general eigenvalue problem, based on the Choleski decomposition method, is presented. Finally, the use of MATLAB in finding the eigenvalues and eigenvectors of multidegree-of-freedom systems is illustrated with several numerical examples.

Learning Objectives

After you have finished studying this chapter, you should be able to do the following:

- Find the approximate fundamental frequency of a composite system in terms of the natural frequencies of component parts using Dunkerley's formula.
- Understand Rayleigh's principle, and the properties of Rayleigh's quotient, and compute the fundamental natural frequency of a system using Rayleigh's method.
- Find the approximate natural frequencies of vibration and the modal vectors by using Holzer's method.
- Determine the smallest, intermediate, and highest natural frequencies of a system by using matrix iteration method and its extensions (using matrix deflation procedure).
- Find all the eigenvalues and eigenvectors of a multidegree-of-freedom system using Jacobi's method.
- Convert a general eigenvalue problem into a standard eigenvalue problem based on the Choleski decomposition method.
- Solve eigenvalue problems using MATLAB.

7.1 Introduction

In the preceding chapter, the natural frequencies (eigenvalues) and the natural modes (eigenvectors) of a multidegree-of-freedom system were found by setting the characteristic determinant equal to zero. Although this is an exact method, the expansion of the characteristic determinant and the solution of the resulting nth-degree polynomial equation to obtain the natural frequencies can become quite tedious for large values of n. Several analytical and numerical methods have been developed to compute the natural frequencies and mode shapes of multidegree-of-freedom systems. In this chapter, we shall consider Dunkerley's formula, Rayleigh's method, Holzer's method, the matrix iteration method, and Jacobi's method. Dunkerley's formula and Rayleigh's method are useful only for estimating the fundamental natural frequency. Holzer's method is essentially a tabular method that can be used to find partial or full solutions to eigenvalue problems. The matrix iteration method finds one natural frequency at a time, usually starting from the lowest value. The method can thus be terminated after finding the required number of natural frequencies and mode shapes. When all the natural frequencies and mode shapes are required, Jacobi's method can be used; it finds all the eigenvalues and eigenvectors simultaneously.

7.2 Dunkerley's Formula

Dunkerley's formula gives the approximate value of the fundamental frequency of a composite system in terms of the natural frequencies of its component parts. It is derived by making use of the fact that the higher natural frequencies of most vibratory systems are large compared to their fundamental frequencies [7.1–7.3]. To derive Dunkerley's formula, consider a general n-degree-of-freedom system whose eigenvalues can be determined by solving the frequency equation, Eq. (6.63):

$$|-[k]+\omega^2[m]|=0$$

or

$$\left|-\frac{1}{\omega^2}[I]+[a][m]\right|=0 \tag{7.1}$$

For a lumped-mass system with a diagonal mass matrix, Eq. (7.1) becomes

$$\left|-\frac{1}{\omega^2}\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & & & \\ 0 & 0 & \cdots & 1\end{bmatrix}+\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix}m_1 & 0 & \cdots & 0\\ 0 & m_2 & \cdots & 0\\ \vdots & & & \\ 0 & 0 & \cdots & m_n\end{bmatrix}\right|=0$$

—that is,

$$\begin{vmatrix}\left(-\frac{1}{\omega^2}+a_{11}m_1\right) & a_{12}m_2 & \cdots & a_{1n}m_n\\ a_{21}m_1 & \left(-\frac{1}{\omega^2}+a_{22}m_2\right) & \cdots & a_{2n}m_n\\ \vdots & \vdots & & \vdots\\ a_{n1}m_1 & a_{n2}m_2 & \cdots & \left(-\frac{1}{\omega^2}+a_{nn}m_n\right)\end{vmatrix}=0 \tag{7.2}$$

The expansion of Eq. (7.2) leads to

$$\begin{aligned}\left(\frac{1}{\omega^2}\right)^n &- (a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n)\left(\frac{1}{\omega^2}\right)^{n-1} \\ &+ (a_{11}a_{22}m_1m_2 + a_{11}a_{33}m_1m_3 + \cdots + a_{n-1,n-1}a_{nn}m_{n-1}m_n \\ &- a_{12}a_{21}m_1m_2 - \cdots - a_{n-1,n}a_{n,n-1}m_{n-1}m_n)\left(\frac{1}{\omega_2}\right)^{n-2} \\ &- \cdots = 0 \end{aligned} \tag{7.3}$$

This is a polynomial equation of nth degree in $(1/\omega^2)$. Let the roots of Eq. (7.3) be denoted as $1/\omega_1^2, 1/\omega_2^2, \ldots, 1/\omega_n^2$. Thus

$$\begin{aligned}&\left(\frac{1}{\omega^2}-\frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega^2}-\frac{1}{\omega_2^2}\right)\cdots\left(\frac{1}{\omega^2}-\frac{1}{\omega_n^2}\right) \\ &= \left(\frac{1}{\omega^2}\right)^n - \left(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}+\cdots+\frac{1}{\omega_n^2}\right)\left(\frac{1}{\omega^2}\right)^{n-1} - \cdots = 0\end{aligned} \tag{7.4}$$

Equating the coefficient of $(1/\omega^2)^{n-1}$ in Eqs. (7.4) and (7.3) gives

$$\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}+\cdots+\frac{1}{\omega_n^2} = a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n \tag{7.5}$$

In most cases, the higher frequencies $\omega_2, \omega_3, \ldots, \omega_n$ are considerably larger than the fundamental frequency ω_1, and so

$$\frac{1}{\omega_i^2} \ll \frac{1}{\omega_1^2}, \qquad i = 2, 3, \ldots, n$$

Thus, Eq. (7.5) can be approximately written as

$$\frac{1}{\omega_1^2} \simeq a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n \tag{7.6}$$

This equation is known as *Dunkerley's formula.* The fundamental frequency given by Eq. (7.6) will always be smaller than the exact value. In some cases, it will be more convenient to rewrite Eq. (7.6) as

$$\frac{1}{\omega_1^2} \simeq \frac{1}{\omega_{1n}^2}+\frac{1}{\omega_{2n}^2}+\cdots+\frac{1}{\omega_{nn}^2} \tag{7.7}$$

where $\omega_{in} = (1/a_{ii}m_i)^{1/2} = (k_{ii}/m_i)^{1/2}$ denotes the natural frequency of a single-degree-of-freedom system consisting of mass m_i and spring of stiffness k_{ii}, $i = 1, 2, \ldots, n$. The use of Dunkerley's formula for finding the lowest frequency of elastic systems is presented in references [7.4, 7.5].

EXAMPLE 7.1 Fundamental Frequency of a Beam

Estimate the fundamental natural frequency of a simply supported beam carrying three identical equally spaced masses, as shown in Fig. 7.1.

Solution: The flexibility influence coefficients (see Example 6.6) required for the application of Dunkerley's formula are given by

$$a_{11} = a_{33} = \frac{3}{256}\frac{l^3}{EI}, \qquad a_{22} = \frac{1}{48}\frac{l^3}{EI} \tag{E.1}$$

Using $m_1 = m_2 = m_3 = m$, Eq. (7.6) thus gives

$$\frac{1}{\omega_1^2} \simeq \left(\frac{3}{256} + \frac{1}{48} + \frac{3}{256}\right)\frac{ml^3}{EI} = 0.04427\,\frac{ml^3}{EI}$$

$$\omega_1 \simeq 4.75375\sqrt{\frac{EI}{ml^3}}$$

This value can be compared with the exact value of the fundamental frequency, $4.9326\sqrt{\frac{EI}{ml^3}}$ (see Problem 6.54)

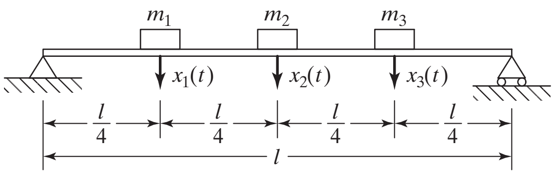


FIGURE 7.1 Beam carrying masses.

■

7.3 Rayleigh's Method

Rayleigh's method was presented in Section 2.5 to find the natural frequencies of single-degree-of-freedom systems. The method can be extended to find the approximate value of the fundamental natural frequency of a discrete system.[1] The method is based on *Rayleigh's principle*, which can be stated as follows [7.6]:

> *The frequency of vibration of a conservative system vibrating about an equilibrium position has a stationary value in the neighborhood of a natural mode. This stationary value, in fact, is a minimum value in the neighborhood of the fundamental natural mode.*

We shall now derive an expression for the approximate value of the first natural frequency of a multidegree-of-freedom system according to Rayleigh's method.

[1]Rayleigh's method for continuous systems is presented in Section 8.7

The kinetic and potential energies of an n-degree-of-freedom discrete system can be expressed as

$$T = \frac{1}{2}\dot{\vec{x}}^T[m]\dot{\vec{x}} \tag{7.8}$$

$$V = \frac{1}{2}\vec{x}^{\,T}[k]\,\vec{x} \tag{7.9}$$

To find the natural frequencies, we assume harmonic motion to be

$$\vec{x} = \vec{X}\cos\omega t \tag{7.10}$$

where $\vec{X}$ denotes the vector of amplitudes (mode shape) and ω represents the natural frequency of vibration. If the system is conservative, the maximum kinetic energy is equal to the maximum potential energy:

$$T_{\max} = V_{\max} \tag{7.11}$$

By substituting Eq. (7.10) into Eqs. (7.8) and (7.9), we find

$$T_{\max} = \frac{1}{2}\vec{X}^{\,T}[m]\vec{X}\omega^2 \tag{7.12}$$

$$V_{\max} = \frac{1}{2}\vec{X}^{\,T}[k]\vec{X} \tag{7.13}$$

By equating $T_{\max}$ and $V_{\max}$, we obtain[2]

$$\omega^2 = \frac{\vec{X}^{\,T}[k]\vec{X}}{\vec{X}^{\,T}[m]\vec{X}} \tag{7.14}$$

The right-hand side of Eq. (7.14) is known as *Rayleigh's quotient* and is denoted as $R(\vec{X})$.

7.3.1 Properties of Rayleigh's Quotient

As stated earlier, $R(\vec{X})$ has a stationary value when the arbitrary vector $\vec{X}$ is in the neighborhood of any eigenvector $\vec{X}^{(r)}$. To prove this, we express the arbitrary vector $\vec{X}$ in terms of the normal modes of the system, $\vec{X}^{(i)}$, as

$$\vec{X} = c_1\vec{X}^{(1)} + c_2\vec{X}^{(2)} + c_3\vec{X}^{(3)} + \cdots \tag{7.15}$$

Then

$$\begin{aligned}\vec{X}^{\,T}[k]\vec{X} &= c_1^2\,\vec{X}^{(1)^T}[k]\vec{X}^{(1)} + c_2^2\vec{X}^{(2)^T}[k]\vec{X}^{(2)} \\ &\quad + c_3^2\vec{X}^{(3)^T}[k]\vec{X}^{(3)} + \cdots \end{aligned} \tag{7.16}$$

[2]Equation (7.14) can also be obtained from the relation $[k]\vec{X} = \omega^2[m]\vec{X}$. Premultiplying this equation by $\vec{X}^T$ and solving the resulting equation gives Eq. (7.14).

and

$$\vec{X}^T[m]\vec{X} = c_1^2\vec{X}^{(1)^T}[m]\vec{X}^{(1)} + c_2^2\vec{X}^{(2)^T}[m]\vec{X}^{(2)} + c_3^2\vec{X}^{(3)^T}[m]\vec{X}^{(3)} + \cdots \tag{7.17}$$

as the cross terms of the form $c_i c_j \vec{X}^{(i)^T}[k]\vec{X}^{(j)}$ and $c_i c_j \vec{X}^{(i)^T}[m]\vec{X}^{(j)}$, $i \neq j$, are zero by the orthogonality property. Using Eqs. (7.16) and (7.17) and the relation

$$\vec{X}^{(i)^T}[k]\vec{X}^{(i)} = \omega_i^2\vec{X}^{(i)^T}[m]\vec{X}^{(i)} \tag{7.18}$$

the Rayleigh's quotient of Eq. (7.14) can be expressed as

$$\omega^2 = R(\vec{X}) = \frac{c_1^2\,\omega_1^2\vec{X}^{(1)^T}[m]\vec{X}^{(1)} + c_2^2\omega_2^2\vec{X}^{(2)^T}[m]\vec{X}^{(2)} + \cdots}{c_1^2\vec{X}^{(1)^T}[m]\vec{X}^{(1)} + c_2^2\,\vec{X}^{(2)^T}[m]\vec{X}^{(2)} + \cdots} \tag{7.19}$$

If the normal modes are normalized, this equation becomes

$$\omega^2 = R(\vec{X}) = \frac{c_1^2\omega_1^2 + c_2^2\omega_2^2 + \cdots}{c_1^2 + c_2^2 + \cdots} \tag{7.20}$$

If $\vec{X}$ differs little from the eigenvector $\vec{X}^{(r)}$, the coefficient c_r will be much larger than the remaining coefficients $c_i\,(i \neq r)$, and Eq. (7.20) can be written as

$$R(\vec{X}) = \frac{c_r^2\omega_r^2 + c_r^2 \displaystyle\sum_{\substack{i=1,2,\ldots\\ i\neq r}} \left(\frac{c_i}{c_r}\right)^2 \omega_i^2}{c_r^2 + c_r^2 \displaystyle\sum_{\substack{i=1,2,\ldots\\ i\neq r}} \left(\frac{c_i}{c_r}\right)^2} \tag{7.21}$$

Since $|c_i/c_r| = \varepsilon_i \ll 1$, where ε_i is a small number for all $i \neq r$, Eq. (7.21) gives

$$R(\vec{X}) = \omega_r^2\{1 + 0(\varepsilon^2)\} \tag{7.22}$$

where $0(\varepsilon^2)$ represents an expression in ε of the second order or higher. Equation (7.22) indicates that if the arbitrary vector $\vec{X}$ differs from the eigenvector $\vec{X}^{(r)}$ by a small quantity of the first order, $R(\vec{X})$ differs from the eigenvalue ω_r^2 by a small quantity of the second order. This means that Rayleigh's quotient has a stationary value in the neighborhood of an eigenvector.

The stationary value is actually a minimum value in the neighborhood of the fundamental mode, $\vec{X}^{(1)}$. To see this, let $r = 1$ in Eq. (7.21) and write

$$R(\vec{X}) = \frac{\omega_1^2 + \sum_{i=2,3,\ldots} \left(\frac{c_i}{c_1}\right)^2 \omega_i^2}{\left\{1 + \sum_{i=2,3,\ldots} \left(\frac{c_i}{c_1}\right)^2\right\}}$$

$$\simeq \omega_1^2 + \sum_{i=2,3,\ldots} \varepsilon_i^2 \omega_i^2 - \omega_1^2 \sum_{i=2,3,\ldots} \varepsilon_i^2$$

$$\simeq \omega_1^2 + \sum_{i=2,3,\ldots} (\omega_i^2 - \omega_1^2)\varepsilon_i^2 \tag{7.23}$$

Since, in general, $\omega_i^2 > \omega_1^2$ for $i = 2, 3, \ldots$, Eq. (7.23) leads to

$$R(\vec{X}) \geq \omega_1^2 \tag{7.24}$$

which shows that Rayleigh's quotient is never lower than the first eigenvalue. By proceeding in a similar manner, we can show that

$$R(\vec{X}) \leq \omega_n^2 \tag{7.25}$$

which means that Rayleigh's quotient is never higher than the highest eigenvalue. Thus Rayleigh's quotient provides an upper bound for ω_1^2 and a lower bound for ω_n^2.

7.3.2 Computation of the Fundamental Natural Frequency

Equation (7.14) can be used to find an approximate value of the first natural frequency (ω_1) of the system. For this, we select a trial vector $\vec{X}$ to represent the first natural mode $\vec{X}^{(1)}$ and substitute it on the right-hand side of Eq. (7.14). This yields the approximate value of ω_1^2. Because Rayleigh's quotient is stationary, remarkably good estimates of ω_1^2 can be obtained even if the trial vector $\vec{X}$ deviates greatly from the true natural mode $\vec{X}^{(1)}$. Obviously, the estimated value of the fundamental frequency ω_1 is more accurate if the trial vector ($\vec{X}$) chosen resembles the true natural mode $\vec{X}^{(1)}$ closely. Rayleigh's method is compared with Dunkerley's and other methods in Refs. [7.7–7.9].

EXAMPLE 7.2 Fundamental Frequency of a Three-Degree-of-Freedom System

Estimate the fundamental frequency of vibration of the system shown in Fig. 7.2. Assume that $m_1 = m_2 = m_3 = m$, $k_1 = k_2 = k_3 = k$, and the mode shape is

$$\vec{X} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

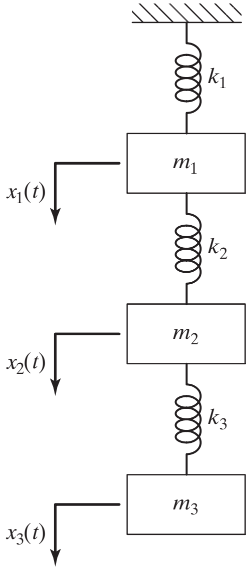


FIGURE 7.2 Three-degree-of-freedom spring-mass system.

Solution: The stiffness and mass matrices of the system are

$$[k] = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{E.1}$$

$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{E.2}$$

By substituting the assumed mode shape in the expression for Rayleigh's quotient, we obtain

$$R(\vec{X}) = \omega^2 = \frac{\vec{X}^T[k]\vec{X}}{\vec{X}^T[m]\vec{X}} = \frac{(1\ 2\ 3)k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1\ 2\ 3)m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = 0.2143\,\frac{k}{m} \tag{E.3}$$

$$\omega_1 = 0.4629\sqrt{\frac{k}{m}} \tag{E.4}$$

This value is 4.0225 percent larger than the exact value of $0.4450\sqrt{k/m}$. The exact fundamental mode shape (see Example 6.10) in this case is

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0000 \\ 1.8019 \\ 2.2470 \end{Bmatrix} \tag{E.5}$$

■

7.3.3 Fundamental Frequency of Beams and Shafts

Although the procedure outlined above is applicable to all discrete systems, a simpler equation can be derived for the fundamental frequency of the lateral vibration of a beam or a shaft carrying several masses such as pulleys, gears, or flywheels. In these cases, the static deflection curve is used as an approximation of the dynamic deflection curve.

Consider a shaft carrying several masses, as shown in Fig. 7.3. The shaft is assumed to have negligible mass. The potential energy of the system is the strain energy of the deflected shaft, which is equal to the work done by the static loads. Thus

$$V_{\max} = \frac{1}{2}(m_1 g w_1 + m_2 g w_2 + \cdots) \tag{7.26}$$

where $m_i g$ is the static load due to the mass m_i, and w_i is the total static deflection of mass m_i due to all the masses. For harmonic oscillation (free vibration), the maximum kinetic energy due to the masses is

$$T_{\max} = \frac{\omega^2}{2}(m_1 w_1^2 + m_2 w_2^2 + \cdots) \tag{7.27}$$

where ω is the frequency of oscillation. Equating $V_{\max}$ and $T_{\max}$, we obtain

$$\omega = \left\{ \frac{g(m_1 w_1 + m_2 w_2 + \cdots)}{(m_1 w_1^2 + m_2 w_2^2 + \cdots)} \right\}^{1/2} \tag{7.28}$$

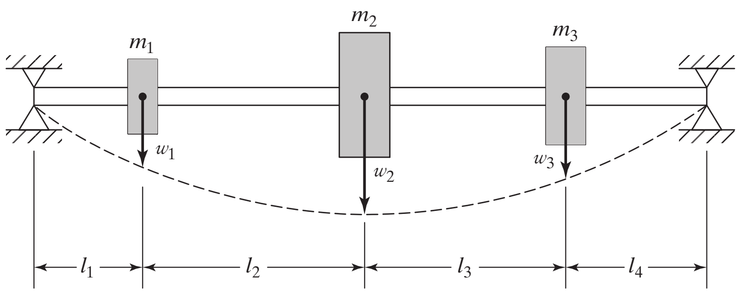


FIGURE 7.3 Shaft carrying masses.

EXAMPLE 7.3 Fundamental Frequency of a Shaft with Rotors

Estimate the fundamental frequency of the lateral vibration of a shaft carrying three rotors (masses), as shown in Fig. 7.3, with $m_1 = 20$ kg, $m_2 = 50$ kg, $m_3 = 40$ kg, $l_1 = 1$ m, $l_2 = 3$ m, $l_3 = 4$ m, and $l_4 = 2$ m. The shaft is made of steel with a solid circular cross section of diameter 10 cm.

Solution: From strength of materials, the deflection of the beam shown in Fig. 7.4 due to a static load P [7.10] is given by

$$w(x) = \begin{cases} \dfrac{Pbx}{6EIl}(l^2 - b^2 - x^2); & 0 \le x \le a \qquad \text{(E.1)} \\ -\dfrac{Pa(l-x)}{6EIl}[a^2 + x^2 - 2lx]; & a \le x \le l \qquad \text{(E.2)} \end{cases}$$

Deflection Due to the Weight of m_1: At the location of mass m_1 (with $x = 1$ m, $b = 9$ m, and $l = 10$ m in Eq. (E.1)):

$$w_1' = \frac{(20 \times 9.81)(9)(1)}{6EI(10)}(100 - 81 - 1) = \frac{529.74}{EI} \tag{E.3}$$

At the location of m_2 (with $a = 1$ m, $x = 4$ m, and $l = 10$ m in Eq. (E.2)):

$$w_2' = -\frac{(20 \times 9.81)(1)(6)}{6EI(10)}[1 + 16 - 2(10)(4)] = \frac{1236.06}{EI} \tag{E.4}$$

At the location of m_3 (with $a = 1$ m, $x = 8$ m, and $l = 10$ m in Eq. (E.2)):

$$w_3' = -\frac{(20 \times 9.81)(1)(2)}{6EI(10)}[1 + 64 - 2(10)(8)] = \frac{621.3}{EI} \tag{E.5}$$

Deflection Due to the Weight of m_2: At the location of m_1 (with $x = 1$ m, $b = 6$ m, and $l = 10$ m in Eq. (E.1)):

$$w_1'' = \frac{(50 \times 9.81)(6)(1)}{6EI(10)}(100 - 36 - 1) = \frac{3090.15}{EI} \tag{E.6}$$

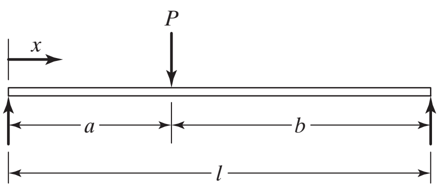


FIGURE 7.4 Beam under static load.

At the location of m_2 (with $x = 4$ m, $b = 6$ m, and $l = 10$ m in Eq. (E.1)):

$$w_2'' = \frac{(50 \times 9.81)(6)(4)}{6EI(10)}(100 - 36 - 16) = \frac{9417.6}{EI} \tag{E.7}$$

At the location of m_3 (with $a = 4$ m, $x = 8$ m, and $l = 10$ m in Eq. (E.2)):

$$w_3'' = -\frac{(50 \times 9.81)(4)(2)}{6EI(10)}[16 + 64 - 2(10)(8)] = \frac{5232.0}{EI} \tag{E.8}$$

Deflection Due to the Weight of m_3: At the location of m_1 (with $x = 1$ m, $b = 2$ m, and $l = 10$ m in Eq. (E.1)):

$$w_1''' = \frac{(40 \times 9.81)(2)(1)}{6EI(10)}(100 - 4 - 1) = \frac{1242.6}{EI} \tag{E.9}$$

At the location of m_2 (with $x = 4$ m, $b = 2$ m, and $l = 10$ m in Eq. (E.1)):

$$w_2''' = \frac{(40 \times 9.81)(2)(4)}{6EI(10)}(100 - 4 - 16) = \frac{4185.6}{EI} \tag{E.10}$$

At the location of m_3 (with $x = 8$ m, $b = 2$ m, and $l = 10$ m in Eq. (E.1)):

$$w_3''' = \frac{(40 \times 9.81)(2)(8)}{6EI(10)}(100 - 4 - 64) = \frac{3348.48}{EI} \tag{E.11}$$

The total deflections of the masses m_1, m_2, and m_3 are

$$w_1 = w_1' + w_1'' + w_1''' = \frac{4862.49}{EI}$$

$$w_2 = w_2' + w_2'' + w_2''' = \frac{14839.26}{EI}$$

$$w_3 = w_3' + w_3'' + w_3''' = \frac{9201.78}{EI}$$

Substituting into Eq. (7.28), we find the fundamental natural frequency:

$$\omega = \left\{\frac{9.81(20 \times 4862.49 + 50 \times 14839.26 + 40 \times 9201.78)EI}{20 \times (4862.49)^2 + 50 \times (14839.26)^2 + 40 \times (9201.78)^2}\right\}^{1/2}$$

$$= 0.028222\sqrt{EI} \tag{E.12}$$

For the shaft, $E = 2.07 \times 10^{11}$ N/m^2 and $I = \pi(0.1)^4/64 = 4.90875 \times 10^{-6}$ m^4 and hence Eq. (E.12) gives

$$\omega = 28.4482 \text{ rad/s}$$

■

7.4 Holzer's Method

Holzer's method is essentially a trial-and-error scheme to find the natural frequencies of undamped, damped, semidefinite, fixed, or branched vibrating systems involving linear and angular displacements [7.11, 7.12]. The method can also be programmed for computer applications. A trial frequency of the system is first assumed, and a solution is found when the assumed frequency satisfies the constraints of the system. This generally requires several trials. Depending on the trial frequency used, the fundamental as well as the higher frequencies of the system can be determined. The method also gives the mode shapes.

7.4.1 Torsional Systems

Consider the undamped torsional semidefinite system shown in Fig. 7.5. The equations of motion of the discs can be derived as follows:

$$J_1\ddot{\theta}_1 + k_{t1}(\theta_1 - \theta_2) = 0 \tag{7.29}$$

$$J_2\ddot{\theta}_2 + k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) = 0 \tag{7.30}$$

$$J_3\ddot{\theta}_3 + k_{t2}(\theta_3 - \theta_2) = 0 \tag{7.31}$$

Since the motion is harmonic in a natural mode of vibration, we assume that $\theta_i = \Theta_i \cos(\omega t + \phi)$ in Eqs. (7.29) to (7.31) and obtain

$$\omega^2 J_1 \Theta_1 = k_{t1}(\Theta_1 - \Theta_2) \tag{7.32}$$

$$\omega^2 J_2 \Theta_2 = k_{t1}(\Theta_2 - \Theta_1) + k_{t2}(\Theta_2 - \Theta_3) \tag{7.33}$$

$$\omega^2 J_3 \Theta_3 = k_{t2}(\Theta_3 - \Theta_2) \tag{7.34}$$

Summing these equations gives

$$\sum_{i=1}^{3} \omega^2 J_i \Theta_i = 0 \tag{7.35}$$

Equation (7.35) states that the sum of the inertia torques of the semidefinite system must be zero. This equation can be treated as another form of the frequency equation, and the trial frequency must satisfy this requirement.

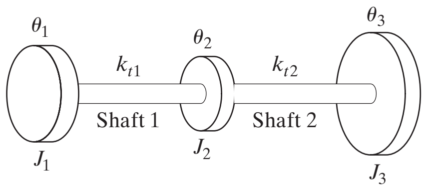


FIGURE 7.5 Torsional semidefinite system.

In Holzer's method, a trial frequency ω is assumed, and Θ_1 is arbitrarily chosen as unity. Next, Θ_2 is computed from Eq. (7.32), and then Θ_3 is found from Eq. (7.33). Thus we obtain

$$\Theta_1 = 1 \tag{7.36}$$

$$\Theta_2 = \Theta_1 - \frac{\omega^2 J_1 \Theta_1}{k_{t1}} \tag{7.37}$$

$$\Theta_3 = \Theta_2 - \frac{\omega^2}{k_{t2}}(J_1\Theta_1 + J_2\Theta_2) \tag{7.38}$$

These values are substituted in Eq. (7.35) to verify whether the constraint is satisfied. If Eq. (7.35) is not satisfied, a new trial value of ω is assumed and the process repeated. Equations (7.35), (7.37), and (7.38) can be generalized for an n-disc system as follows:

$$\sum_{i=1}^{n} \omega^2 J_i \Theta_i = 0 \tag{7.39}$$

$$\Theta_i = \Theta_{i-1} - \frac{\omega^2}{k_{ti-1}}\left(\sum_{k=1}^{i-1} J_k \Theta_k\right), \qquad i = 2, 3, \cdots, n \tag{7.40}$$

Thus the method uses Eqs. (7.39) and (7.40) repeatedly for different trial frequencies. If the assumed trial frequency is not a natural frequency of the system, Eq. (7.39) is not satisfied. The resultant torque in Eq. (7.39) represents a torque applied at the last disc. This torque M_t is then plotted for the chosen ω. When the calculation is repeated with other values of ω, the resulting graph appears as shown in Fig. 7.6. From this graph, the natural frequencies of the system can be identified as the values of ω at which $M_t = 0$. The amplitudes Θ_i $(i = 1, 2, \cdots, n)$ corresponding to the natural frequencies are the mode shapes of the system.

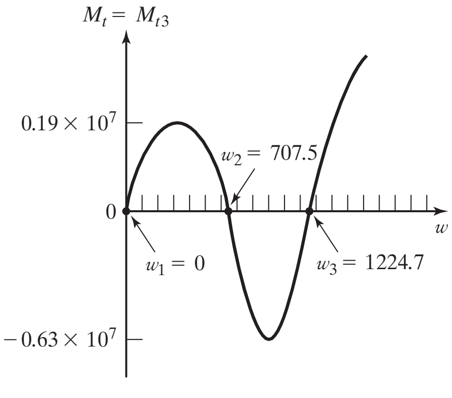


FIGURE 7.6 Resultant torque versus frequency.

Holzer's method can also be applied to systems with fixed ends. At a fixed end, the amplitude of vibration must be zero. In this case, the natural frequencies can be found by plotting the resulting amplitude (instead of the resultant torque) against the assumed frequencies. For a system with one end free and the other end fixed, Eq. (7.40) can be used for checking the amplitude at the fixed end. An improvement of Holzer's method is presented in references [7.13, 7.14].

EXAMPLE 7.4 Natural Frequencies of a Torsional System

The arrangement of the compressor, turbine, and generator in a thermal power plant is shown in Fig. 7.7. Find the natural frequencies and mode shapes of the system.

Solution: This system represents an unrestrained or free-free torsional system. Table 7.1 shows its parameters and the sequence of computations. The calculations for the trial frequencies $\omega = 0, 10,$ 20, 700, and 710 are shown in this table. The quantity M_{t3} denotes the torque to the right of Station 3

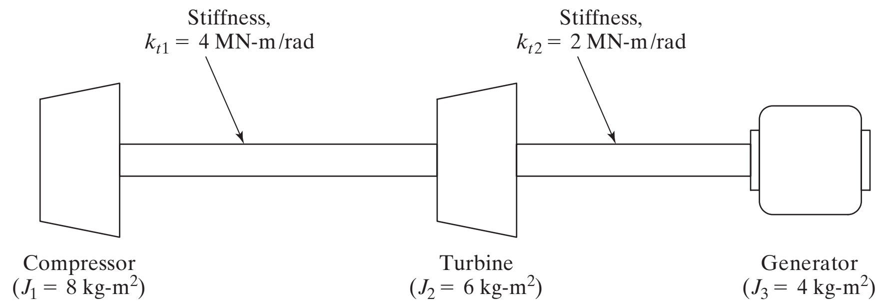


FIGURE 7.7 Free-free torsional system.

TABLE 7.1

Parameters of the System	Quantity	Trial 1	2	3	···	71	72
		0	10	20		700	710
	ω^2	0	100	400		490000	504100
Station 1:							
$J_1 = 8$	Θ_1	1.0	1.0	1.0		1.0	1.0
$k_{t1} = 4 \times 10^6$	$M_{t1} = \omega^2 J_1 \Theta_1$	0	800	3200		$0.392E7$	$0.403E7$
Station 2:							
$J_2 = 6$	$\Theta_2 = 1 - \dfrac{M_{t1}}{k_{t1}}$	1.0	0.9998	0.9992		0.0200	−0.0082
$k_{t2} = 2 \times 10^6$	$M_{t2} = M_{t1} + \omega^2 J_2 \Theta_2$	0	1400	5598		$0.398E7$	$0.401E7$
Station 3:							
$J_3 = 4$	$\Theta_3 = \Theta_2 - \dfrac{M_{t2}}{k_{t2}}$	1.0	0.9991	0.9964		−1.9690	−2.0120
$K_{t3} = 0$	$M_{t3} = M_{t2} + \omega^2 J_3 \Theta_3$	0	1800	7192		$0.119E6$	$-0.494E5$

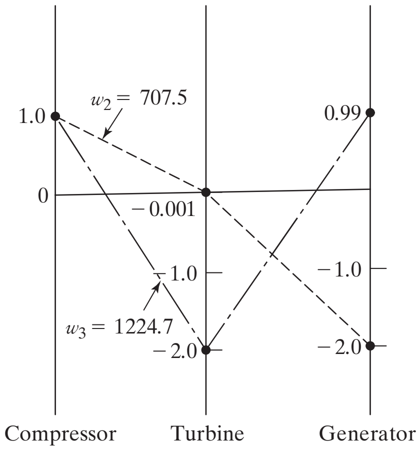


FIGURE 7.8 First two flexible modes.

(generator), which must be zero at the natural frequencies. Figure 7.6 shows the graph of M_{t3} versus ω. Closely spaced trial values of ω are used in the vicinity of $M_{t3} = 0$ to obtain accurate values of the first two flexible mode shapes, shown in Fig. 7.8. Note that the value $\omega = 0$ corresponds to the rigid-body rotation.

■

7.4.2 Spring-Mass Systems

Although Holzer's method has been extensively applied to torsional systems, the procedure is equally applicable to the vibration analysis of spring-mass systems. The equations of motion of a spring-mass system (see Fig. 7.9) can be expressed as

$$m_1 \ddot{x}_1 + k_1(x_1 - x_2) = 0 \tag{7.41}$$

$$m_2 \ddot{x}_2 + k_1(x_2 - x_1) + k_2(x_2 - x_3) = 0$$
$$\cdots \tag{7.42}$$

For harmonic motion, $x_i(t) = X_i \cos \omega t$, where X_i is the amplitude of mass m_i, and Eqs. (7.41) and (7.42) can be written as

$$\omega^2 m_1 X_1 = k_1(X_1 - X_2) \tag{7.43}$$

$$\begin{aligned} \omega^2 m_2 X_2 &= k_1(X_2 - X_1) + k_2(X_2 - X_3) \\ &= -\omega^2 m_1 X_1 + k_2(X_2 - X_3) \\ &\cdots \end{aligned} \tag{7.44}$$

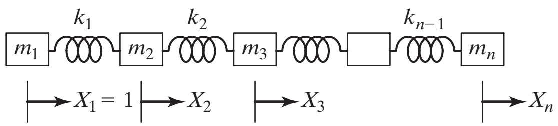


FIGURE 7.9 Free-free spring mass system.

The procedure for Holzer's method starts with a trial frequency ω and the amplitude of mass m_1 as $X_1 = 1$. Equations (7.43) and (7.44) can then be used to obtain the amplitudes of the masses $m_2, m_3, \cdots, m_i$:

$$X_2 = X_1 - \frac{\omega^2 m_1 X_1}{k_1} \tag{7.45}$$

$$X_3 = X_2 - \frac{\omega^2}{k_2}(m_1 X_1 + m_2 X_2) \tag{7.46}$$

$$X_i = X_{i-1} - \frac{\omega^2}{k_{i-1}}\left(\sum_{k=1}^{i-1} m_k X_k\right), \qquad i = 2, 3, \ldots, n \tag{7.47}$$

As in the case of torsional systems, the resultant force applied to the last (nth) mass can be computed as follows:

$$F = \sum_{i=1}^{n} \omega^2 m_i X_i \tag{7.48}$$

The calculations are repeated with several other trial frequencies ω. The natural frequencies are identified as those values of ω that give $F = 0$ for a free-free system. For this, it is convenient to plot a graph between F and ω, using the same procedure for spring-mass systems as for torsional systems.

7.5 Matrix Iteration Method

The matrix iteration method assumes that the natural frequencies are distinct and well separated such that $\omega_1 < \omega_2 < \cdots < \omega_n$. The iteration is started by selecting a trial vector $\vec{X}_1$, which is then premultiplied by the dynamical matrix $[D]$. The resulting column vector is then normalized, usually by making one of its components equal to unity. The normalized column vector is premultiplied by $[D]$ to obtain a third column vector, which is normalized in the same way as before and becomes still another trial column vector. The process is repeated until the successive normalized column vectors converge to a common vector: the fundamental eigenvector. The normalizing factor gives the largest value of $\lambda = 1/\omega^2$—that is, the smallest or the fundamental natural frequency [7.15]. The convergence of the process can be explained as follows.

According to the expansion theorem, any arbitrary n-dimensional vector $\vec{X}_1$ can be expressed as a linear combination of the n orthogonal eigenvectors of the system $\vec{X}^{(i)}$, $i = 1, 2, \ldots, n$:

$$\vec{X}_1 = c_1\vec{X}^{(1)} + c_2\vec{X}^{(2)} + \cdots + c_n\vec{X}^{(n)} \tag{7.49}$$

where $c_1, c_2, \ldots, c_n$ are constants. In the iteration method, the trial vector $\vec{X}_1$ is selected arbitrarily and is therefore a known vector. The modal vectors $\vec{X}^{(i)}$, although unknown, are constant vectors because they depend upon the properties of the system. The constants c_i

are unknown numbers to be determined. According to the iteration method, we premultiply $\vec{X}_1$ by the matrix $[D]$. In view of Eq. (7.49), this gives

$$[D]\vec{X}_1 = c_1[D]\vec{X}^{(1)} + c_2[D]\vec{X}^{(2)} + \cdots + c_n[D]\vec{X}^{(n)} \tag{7.50}$$

Now, according to Eq. (6.66), we have

$$[D]\vec{X}^{(i)} = \lambda_i[I]\vec{X}^{(i)} = \frac{1}{\omega_i^2}\vec{X}^{(i)}; \qquad i = 1, 2, \ldots, n \tag{7.51}$$

Substitution of Eq. (7.51) into Eq. (7.50) yields

$$\begin{aligned}[D]\vec{X}_1 &= \vec{X}_2 \\ &= \frac{c_1}{\omega_1^2}\vec{X}^{(1)} + \frac{c_2}{\omega_2^2}\vec{X}^{(2)} + \cdots + \frac{c_n}{\omega_n^2}\vec{X}^{(n)}\end{aligned} \tag{7.52}$$

where $\vec{X}_2$ is the second trial vector. We now repeat the process and premultiply $\vec{X}_2$ by $[D]$ to obtain, by Eqs. (7.49) and (6.66),

$$\begin{aligned}[D]\vec{X}_2 &= \vec{X}_3 \\ &= \frac{c_1}{\omega_1^4}\vec{X}^{(1)} + \frac{c_2}{\omega_2^4}\vec{X}^{(2)} + \cdots + \frac{c_n}{\omega_n^4}\vec{X}^{(n)}\end{aligned} \tag{7.53}$$

By repeating the process we obtain, after the rth iteration,

$$\begin{aligned}[D]\vec{X}_r &= \vec{X}_{r+1} \\ &= \frac{c_1}{\omega_1^{2r}}\vec{X}^{(1)} + \frac{c_2}{\omega_2^{2r}}\vec{X}^{(2)} + \cdots + \frac{c_n}{\omega_n^{2r}}\vec{X}^{(n)}\end{aligned} \tag{7.54}$$

Since the natural frequencies are assumed to be $\omega_1 < \omega_2 < \cdots < \omega_n$, a sufficiently large value of r yields

$$\frac{1}{\omega_1^{2r}} \gg \frac{1}{\omega_2^{2r}} \gg \cdots \gg \frac{1}{\omega_n^{2r}} \tag{7.55}$$

Thus the first term on the right-hand side of Eq. (7.54) becomes the only significant one. Hence we have

$$\vec{X}_{r+1} = \frac{c_1}{\omega_1^{2r}}\vec{X}^{(1)} \tag{7.56}$$

which means that the $(r + 1)$th trial vector becomes identical to the fundamental modal vector to within a multiplicative constant. Since

$$\vec{X}_r = \frac{c_1}{\omega_1^{2(r-1)}}\vec{X}^{(1)} \tag{7.57}$$

the fundamental natural frequency ω_1 can be found by taking the ratio of any two corresponding components in the vectors $\vec{X}_r$ and $\vec{X}_{r+1}$:

$$\omega_1^2 \simeq \frac{X_{i,r}}{X_{i,r+1}}, \qquad \text{for any } i = 1, 2, \ldots, n \tag{7.58}$$

where $X_{i,r}$ and $X_{i,r+1}$ are the ith elements of the vectors $\vec{X}_r$ and $\vec{X}_{r+1}$, respectively.

Discussion

1. In the above proof, nothing has been said about the normalization of the successive trial vectors $\vec{X}_i$. Actually, it is not necessary to establish the proof of convergence of the method. The normalization amounts to a readjustment of the constants $c_1, c_2, \ldots, c_n$ in each iteration.
2. Although it is theoretically necessary to have $r \to \infty$ for the convergence of the method, in practice only a finite number of iterations suffices to obtain a reasonably good estimate of ω_1.
3. The actual number of iterations necessary to find the value of ω_1 to within a desired degree of accuracy depends on how closely the arbitrary trial vector $\vec{X}_1$ resembles the fundamental mode $\vec{X}^{(1)}$ and on how well ω_1 and ω_2 are separated. The required number of iterations is less if ω_2 is very large compared to ω_1.
4. The method has a distinct advantage in that any computational errors made do not yield incorrect results. Any error made in premultiplying $\vec{X}_i$ by $[D]$ results in a vector other than the desired one, $\vec{X}_{i+1}$. But this wrong vector can be considered as a new trial vector. This may delay the convergence but does not produce wrong results.
5. One can take any set of n numbers for the first trial vector $\vec{X}_1$ and still achieve convergence to the fundamental modal vector. Only in the unusual case in which the trial vector $\vec{X}_1$ is exactly proportional to one of the modes $\vec{X}^{(i)}$ $(i \neq 1)$ does the method fail to converge to the first mode. In such a case, the premultiplication of $\vec{X}^{(i)}$ by $[D]$ results in a vector proportional to $\vec{X}^{(i)}$ itself.

7.5.1 Convergence to the Highest Natural Frequency

To obtain the highest natural frequency ω_n and the corresponding mode shape or eigenvector $\vec{X}^{(n)}$ by the matrix iteration method, we first rewrite Eq. (6.66) as

$$[D]^{-1}\vec{X} = \omega^2[I]\vec{X} = \omega^2\vec{X} \tag{7.59}$$

where $[D]^{-1}$ is the inverse of the dynamical matrix $[D]$ given by

$$[D]^{-1} = [m]^{-1}[k] \tag{7.60}$$

Now we select any arbitrary trial vector $\vec{X}_1$ and premultiply it by $[D]^{-1}$ to obtain an improved trial vector $\vec{X}_2$. The sequence of trial vectors $\vec{X}_{i+1}$ $(i = 1, 2, \ldots)$ obtained by premultiplying by $[D]^{-1}$ converges to the highest normal mode $\vec{X}^{(n)}$. It can be seen that the procedure is similar to the one already described. The constant of proportionality in this case is ω^2 instead of $1/\omega^2$.

7.5.2 Computation of Intermediate Natural Frequencies

Once the first natural frequency ω_1 (or the largest eigenvalue $\lambda_1 = 1/\omega_1^2$) and the corresponding eigenvector $\vec{X}^{(1)}$ are determined, we can proceed to find the higher natural frequencies and the corresponding mode shapes by the matrix iteration method. Before we proceed, it should be remembered that any arbitrary trial vector premultiplied by $[D]$ would lead again to the largest eigenvalue. It is thus necessary to remove the largest eigenvalue from the matrix $[D]$. The succeeding eigenvalues and eigenvectors can be obtained by eliminating the root λ_1 from the characteristic or frequency equation

$$|[D] - \lambda[I]| = 0 \tag{7.61}$$

A procedure known as *matrix deflation* can be used for this purpose [7.16]. To find the eigenvector $\vec{X}^{(i)}$ by this procedure, the previous eigenvector $\vec{X}^{(i-1)}$ is normalized with respect to the mass matrix such that

$$\vec{X}^{(i-1)T}[m]\vec{X}^{(i-1)} = 1 \tag{7.62}$$

The deflated matrix $[D_i]$ is then constructed as

$$[D_i] = [D_{i-1}] - \lambda_{i-1}\vec{X}^{(i-1)}\vec{X}^{(i-1)T}[m], \qquad i = 2, 3, \ldots, n \tag{7.63}$$

where $[D_1] = [D]$. Once $[D_i]$ is constructed, the iterative scheme

$$\vec{X}_{r+1} = [D_i]\vec{X}_r \tag{7.64}$$

is used, where $\vec{X}_1$ is an arbitrary trial eigenvector.

EXAMPLE 7.5 Natural Frequencies of a Three-Degree-of-Freedom System

Find the natural frequencies and mode shapes of the system shown in Fig. 7.2 for $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m_3 = m$ by the matrix iteration method.

Solution: The mass and stiffness matrices of the system are given in Example 7.2. The flexibility matrix is

$$[a] = [k]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{E.1}$$

and so the dynamical matrix is

$$[k]^{-1}[m] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{E.2}$$

The eigenvalue problem can be stated as

$$[D]\vec{X} = \lambda\vec{X} \tag{E.3}$$

where

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{E.4}$$

and

$$\lambda = \frac{k}{m}\cdot\frac{1}{\omega^2} \tag{E.5}$$

First Natural Frequency: By assuming the first trial eigenvector or mode shape to be

$$\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{E.6}$$

the second trial eigenvector can be obtained:

$$\vec{X}_2 = [D]\vec{X}_1 = \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} \tag{E.7}$$

By making the first element equal to unity, we obtain

$$\vec{X}_2 = 3.0\begin{Bmatrix} 1.0000 \\ 1.6667 \\ 2.0000 \end{Bmatrix} \tag{E.8}$$

and the corresponding eigenvalue is given by

$$\lambda_1 \simeq 3.0 \qquad \text{or} \qquad \omega_1 \simeq 0.5773\sqrt{\frac{k}{m}} \tag{E.9}$$

The subsequent trial eigenvector can be obtained from the relation

$$\vec{X}_{i+1} = [D]\vec{X}_i \tag{E.10}$$

and the corresponding eigenvalues are given by

$$\lambda_1 \simeq X_{1,\,i+1} \tag{E.11}$$

where $X_{1,\,i+1}$ is the first component of the vector $\vec{X}_{i+1}$ before normalization. The various trial eigenvectors and eigenvalues obtained by using Eqs. (E.10) and (E.11) are shown in the table below.

i	$\vec{X}_i$ with $X_{1,i} = 1$	$\vec{X}_{i+1} = [D]\vec{X}_i$	$\lambda_1 \simeq X_{1,i+1}$	ω_1
1	$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix}$	3.0	$0.5773 \sqrt{\dfrac{k}{m}}$
2	$\begin{Bmatrix} 1.00000 \\ 1.66667 \\ 2.00000 \end{Bmatrix}$	$\begin{Bmatrix} 4.66667 \\ 8.33333 \\ 10.33333 \end{Bmatrix}$	4.66667	$0.4629 \sqrt{\dfrac{k}{m}}$
3	$\begin{Bmatrix} 1.0000 \\ 1.7857 \\ 2.2143 \end{Bmatrix}$	$\begin{Bmatrix} 5.00000 \\ 9.00000 \\ 11.2143 \end{Bmatrix}$	5.00000	$0.4472 \sqrt{\dfrac{k}{m}}$
.				
.				
.				
7	$\begin{Bmatrix} 1.00000 \\ 1.80193 \\ 2.24697 \end{Bmatrix}$	$\begin{Bmatrix} 5.04891 \\ 9.09781 \\ 11.34478 \end{Bmatrix}$	5.04891	$0.44504 \sqrt{\dfrac{k}{m}}$
8	$\begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$	$\begin{Bmatrix} 5.04892 \\ 9.09783 \\ 11.34481 \end{Bmatrix}$	5.04892	$0.44504 \sqrt{\dfrac{k}{m}}$

It can be seen that the mode shape and the natural frequency converged (to the fourth decimal place) in eight iterations. Thus the first eigenvalue and the corresponding natural frequency and mode shape are given by

$$\lambda_1 = 5.04892, \qquad \omega_1 = 0.44504 \sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix} \tag{E.12}$$

Second Natural Frequency: To compute the second eigenvalue and the eigenvector, we must first produce a deflated matrix:

$$[D_2] = [D_1] - \lambda_1 \vec{X}^{(1)} \vec{X}^{(1)T} [m] \tag{E.13}$$

This equation, however, calls for a normalized vector $\vec{X}^{(1)}$ satisfying $\vec{X}^{(1)T}[m]\vec{X}^{(1)} = 1$. Let the normalized vector be denoted as

$$\vec{X}^{(1)} = \alpha\begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$$

where α is a constant whose value must be such that

$$\vec{X}^{(1)T}[m]\vec{X}^{(1)} = \alpha^2 m \begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$$

$$= \alpha^2 m(9.29591) = 1 \tag{E.14}$$

from which we obtain $\alpha = 0.32799m^{-1/2}$. Hence the first normalized eigenvector is

$$\vec{X}^{(1)} = m^{-1/2}\begin{Bmatrix} 0.32799 \\ 0.59102 \\ 0.73699 \end{Bmatrix} \tag{E.15}$$

Next we use Eq. (E.13) and form the first deflated matrix:

$$[D_2] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} - 5.04892\begin{Bmatrix} 0.32799 \\ 0.59102 \\ 0.73699 \end{Bmatrix}\begin{Bmatrix} 0.32799 \\ 0.59102 \\ 0.73699 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.45684 & 0.02127 & -0.22048 \\ 0.02127 & 0.23641 & -0.19921 \\ -0.22048 & -0.19921 & 0.25768 \end{bmatrix} \tag{E.16}$$

Since the trial vector can be chosen arbitrarily, we again take

$$\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{E.17}$$

By using the iterative scheme

$$\vec{X}_{i+1} = [D_2]\vec{X}_i \tag{E.18}$$

we obtain $\vec{X}_2$

$$\vec{X}_2 = \begin{Bmatrix} 0.25763 \\ 0.05847 \\ -0.16201 \end{Bmatrix} = 0.25763\begin{Bmatrix} 1.00000 \\ 0.22695 \\ -0.62885 \end{Bmatrix} \tag{E.19}$$

Hence λ_2 can be found from the general relation

$$\lambda_2 \simeq X_{1,i+1} \tag{E.20}$$

as 0.25763. Continuation of this procedure gives the results shown in the table below.

i	$\vec{X}_i$ with $X_{1,i} = 1$	$\vec{X}_{i+1} = [D_2]\vec{X}_i$	$\lambda_2 \simeq X_{1,i+1}$	ω_2
1	$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 0.25763 \\ 0.05847 \\ -0.16201 \end{Bmatrix}$	0.25763	$1.97016\sqrt{\frac{k}{m}}$
2	$\begin{Bmatrix} 1.00000 \\ 0.22695 \\ -0.62885 \end{Bmatrix}$	$\begin{Bmatrix} 0.60032 \\ 0.20020 \\ -0.42773 \end{Bmatrix}$	0.60032	$1.29065\sqrt{\frac{k}{m}}$
.				
.				
.				
10	$\begin{Bmatrix} 1.00000 \\ 0.44443 \\ -0.80149 \end{Bmatrix}$	$\begin{Bmatrix} 0.64300 \\ 0.28600 \\ -0.51554 \end{Bmatrix}$	0.64300	$1.24708\sqrt{\frac{k}{m}}$
11	$\begin{Bmatrix} 1.00000 \\ 0.44479 \\ -0.80177 \end{Bmatrix}$	$\begin{Bmatrix} 0.64307 \\ 0.28614 \\ -0.51569 \end{Bmatrix}$	0.64307	$1.24701\sqrt{\frac{k}{m}}$

Thus the converged second eigenvalue and the eigenvector are

$$\lambda_2 = 0.64307, \qquad \omega_2 = 1.24701\sqrt{\frac{k}{m}}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} 1.00000 \\ 0.44496 \\ -0.80192 \end{Bmatrix} \tag{E.21}$$

Third Natural Frequency: For the third eigenvalue and the eigenvector, we use a similar procedure. The detailed calculations are left as an exercise to the reader. Note that before computing the deflated matrix $[D_3]$, we need to normalize $\vec{X}^{(2)}$ by using Eq. (7.62), which gives

$$\vec{X}^{(2)} = m^{-1/2}\begin{Bmatrix} 0.73700 \\ 0.32794 \\ -0.59102 \end{Bmatrix} \tag{E.22}$$

■

7.6 Jacobi's Method

The matrix iteration method described in the preceding section produces the eigenvalues and eigenvectors of matrix $[D]$ one at a time. Jacobi's method is also an iterative method but produces all the eigenvalues and eigenvectors of $[D]$ simultaneously, where $[D] = [d_{ij}]$ is a real symmetric matrix of order $n \times n$. The method is based on a theorem in linear algebra stating that a real symmetric matrix $[D]$ has only real eigenvalues and that there exists a real orthogonal matrix $[R]$ such that $[R]^T[D][R]$ is diagonal [7.17]. The diagonal elements are the eigenvalues, and the columns of the matrix $[R]$ are the eigenvectors. According to Jacobi's method, the matrix $[R]$ is generated as a product of several rotation matrices [7.18] of the form

$$\underset{n \times n}{[R_1]} = \begin{array}{c} \begin{array}{cccccc} & & i\text{th column} & & j\text{th column} & \end{array} \\ \begin{bmatrix} 1 & 0 & & & & & & \\ 0 & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & \cos\theta & & -\sin\theta & & \\ & & & & \ddots & & & \\ & & & \sin\theta & & \cos\theta & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \\ i\text{th row} \\ \\ j\text{th row} \\ \\ \\ \end{array} \tag{7.65}$$

where all elements other than those appearing in columns and rows i and j are identical with those of the identity matrix $[I]$. If the sine and cosine entries appear in positions (i, i), (i, j), (j, i), and (j, j), then the corresponding elements of $[R_1]^T[D][R_1]$ can be computed as follows:

$$\underline{d}_{ii} = d_{ii}\cos^2\theta + 2d_{ij}\sin\theta\cos\theta + d_{jj}\sin^2\theta \tag{7.66}$$

$$\underline{d}_{ij} = \underline{d}_{ji} = (d_{jj} - d_{ii})\sin\theta\cos\theta + d_{ij}(\cos^2\theta - \sin^2\theta) \tag{7.67}$$

$$\underline{d}_{jj} = d_{ii}\sin^2\theta - 2d_{ij}\sin\theta\cos\theta + d_{jj}\cos^2\theta \tag{7.68}$$

If θ is chosen to be

$$\tan 2\theta = \left(\frac{2d_{ij}}{d_{ii} - d_{jj}}\right) \tag{7.69}$$

then it makes $\underline{d}_{ij} = \underline{d}_{ji} = 0$. Thus each step of Jacobi's method reduces a pair of off-diagonal elements to zero. Unfortunately, in the next step, while the method reduces a new pair of zeros, it introduces nonzero contributions to formerly zero positions. However, successive matrices of the form

$$[R_2]^T[R_1]^T[D][R_1][R_2], \qquad [R_3]^T[R_2]^T[R_1]^T[D][R_1][R_2][R_3], \ \ldots$$

converge to the required diagonal form; the final matrix $[R]$, whose columns give the eigenvectors, then becomes

$$[R] = [R_1][R_2][R_3]\ldots \tag{7.70}$$

EXAMPLE 7.6 Eigenvalue Solution Using Jacobi Method

Find the eigenvalues and eigenvectors of the matrix

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

using Jacobi's method.

Solution: We start with the largest off-diagonal term, $d_{23} = 2$, in the matrix $[D]$ and try to reduce it to zero. From Eq. (7.69),

$$\theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{2d_{23}}{d_{22} - d_{33}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4}{2-3}\right) = -37.981878^\circ$$

$$[R_1] = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.7882054 & 0.6154122 \\ 0.0 & -0.6154122 & 0.7882054 \end{bmatrix}$$

$$[D'] = [R_1]^T[D][R_1] = \begin{bmatrix} 1.0 & 0.1727932 & 1.4036176 \\ 0.1727932 & 0.4384472 & 0.0 \\ 1.4036176 & 0.0 & 4.5615525 \end{bmatrix}$$

Next we try to reduce the largest off-diagonal term of $[D']$—namely, $d'_{13} = 1.4036176$—to zero. Equation (7.69) gives

$$\theta_2 = \frac{1}{2}\tan^{-1}\left(\frac{2d'_{13}}{d'_{11} - d'_{33}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2.8072352}{1.0 - 4.5615525}\right) = -19.122686^\circ$$

$$[R_2] = \begin{bmatrix} 0.9448193 & 0.0 & 0.3275920 \\ 0.0 & 1.0 & 0.0 \\ -0.3275920 & 0.0 & 0.9448193 \end{bmatrix}$$

$$[D''] = [R_2]^T[D'][R_2] = \begin{bmatrix} 0.5133313 & 0.1632584 & 0.0 \\ 0.1632584 & 0.4384472 & 0.0566057 \\ 0.0 & 0.0566057 & 5.0482211 \end{bmatrix}$$

The largest off-diagonal element in $[D'']$ is $d''_{12} = 0.1632584$. θ_3 can be obtained from Eq. (7.69) as

$$\theta_3 = \frac{1}{2}\tan^{-1}\left(\frac{2d''_{12}}{d''_{11} - d''_{22}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{0.3265167}{0.5133313 - 0.4384472}\right) = 38.541515°$$

$$[R_3] = \begin{bmatrix} 0.7821569 & -0.6230815 & 0.0 \\ 0.6230815 & 0.7821569 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$$[D'''] = [R_3]^T[D''][R_3] = \begin{bmatrix} 0.6433861 & 0.0 & 0.0352699 \\ 0.0 & 0.3083924 & 0.0442745 \\ 0.0352699 & 0.0442745 & 5.0482211 \end{bmatrix}$$

Assuming that all the off-diagonal terms in $[D''']$ are close to zero, we can stop the process here. The diagonal elements of $[D''']$ give the eigenvalues (values of $1/\omega^2$) as 0.6433861, 0.3083924, and 5.0482211. The corresponding eigenvectors are given by the columns of the matrix $[R]$, where

$$[R] = [R_1][R_2][R_3] = \begin{bmatrix} 0.7389969 & -0.5886994 & 0.3275920 \\ 0.3334301 & 0.7421160 & 0.5814533 \\ -0.5854125 & -0.3204631 & 0.7447116 \end{bmatrix}$$

The iterative process can be continued for obtaining a more accurate solution. The present eigenvalues can be compared with the exact values: 0.6431041, 0.3079786, and 5.0489173.

■

7.7 Standard Eigenvalue Problem

In the preceding chapter, the eigenvalue problem was stated as

$$[k]\vec{X} = \omega^2[m]\vec{X} \tag{7.71}$$

which can be rewritten in the form of a standard eigenvalue problem [7.19] as

$$[D]\,\vec{X} = \lambda\vec{X} \tag{7.72}$$

where

$$[D] = [k]^{-1}[m] \tag{7.73}$$

and

$$\lambda = \frac{1}{\omega^2} \tag{7.74}$$

Stephen Prokf'yevich Timoshenko (1878–1972), a Russian-born engineer who emigrated to the United States, was one of the most widely known authors of books in the field of elasticity, strength of materials, and vibrations. He held the chair of mechanics at the university of Michigan and later at Stanford University, and he is regarded as the father of engineering mechanics in the United States. The improved theory he presented in 1921 for the vibration of beams has become known as the Timoshenko beam theory. (Courtesy of *Applied Mechanics Reviews.*)

CHAPTER 8

Continuous Systems

Chapter Outline

The vibration analysis of continuous systems, which are also known as *distributed systems*, is considered in this chapter. The equations of motion of continuous systems will be partial differential equations. The equations of motion of several continuous systems, including the transverse vibration of a tightly stretched string or cable, longitudinal vibration of a bar, torsional vibration of a shaft or rod, the lateral vibration of beams, and transverse vibration of a membrane are derived by considering the free-body diagram of an infinitesimally small element of the particular system and applying the Newton's second law of motion. The free-vibration solution of the system is found by assuming harmonic motion and applying the relevant boundary conditions. The solution gives infinite number of natural frequencies and the corresponding mode shapes. The free-vibration displacement of the system is found as a linear superposition of the mode shapes, the constants involved being determined from the known initial conditions of the system. In the case of transverse vibration of a string of infinite length, the traveling-wave solution is presented. In the case of the longitudinal vibration of a bar, the vibration response under an initial force is also found. In the case of the transverse vibration of beams, all the common boundary conditions are summarized and the orthogonality of normal modes is proved. The forced vibration of beams is presented using the mode superposition method. The effect of axial force on the natural frequencies and mode shapes of beams is considered. The *thick beam theory*, also called the *Timoshenko beam theory*, is presented by considering the effects of rotary inertia and shear deformation. The free vibration of rectangular membranes is presented. *Rayleigh's method*, based on Rayleigh's quotient, for finding the approximate fundamental frequencies of continuous systems is outlined. The extension of the method, known as the *Rayleigh-Ritz* method, is outlined for determining approximate values of several frequencies. Finally, MATLAB solutions are presented for the free and forced vibration of typical continuous systems.

Learning Objectives

After you have finished studying this chapter, you should be able to do the following:

- Derive the equation of motion of a continuous system from the free-body diagram of an infinitesimally small element of the system and Newton's second law.
- Find the natural frequencies and mode shapes of the system using harmonic solution.
- Determine the free-vibration solution using a linear superposition of the mode shapes and the initial conditions.
- Find the free-vibration solutions of string, bar, shaft, beam, and membrane problems.
- Express the vibration of an infinite string in the form of traveling waves.
- Determine the forced-vibration solution of continuous systems using mode superposition method.
- Find the effects of axial force, rotary inertia, and shear deformation on the vibration of beams.
- Apply the Rayleigh and Rayleigh-Ritz methods to find the approximate natural frequencies of continuous systems.
- Use MATLAB to find the natural frequencies, mode shapes, and forced response of continuous systems.

8.1 Introduction

We have so far dealt with discrete systems where mass, damping, and elasticity were assumed to be present only at certain discrete points in the system. In many cases, known as *distributed* or *continuous systems,* it is not possible to identify discrete masses, dampers, or springs. We must then consider the continuous distribution of the mass, damping, and elasticity and assume that each of the infinite number of points of the system can vibrate. This is why a continuous system is also called a *system of infinite degrees of freedom.*

If a system is modeled as a discrete one, the governing equations are ordinary differential equations, which are relatively easy to solve. On the other hand, if the system is modeled as a continuous one, the governing equations are partial differential equations, which are more difficult. However, the information obtained from a discrete model of a system may not be as accurate as that obtained from a continuous model. The choice between the two models must be made carefully, with due consideration of factors such as the purpose of the analysis, the influence of the analysis on design, and the computational time available.

In this chapter, we shall consider the vibration of simple continuous systems—strings, bars, shafts, beams, and membranes. A more specialized treatment of the vibration of continuous structural elements is given in references [8.1–8.3]. In general, the frequency equation of a continuous system is a transcendental equation that yields an infinite number of natural frequencies and normal modes. This is in contrast to the behavior of discrete systems, which yield a finite number of such frequencies and modes. We need to apply boundary conditions to find the natural frequencies of a continuous system. The question of boundary conditions does not arise in the case of discrete systems except in an indirect way, because the influence coefficients depend on the manner in which the system is supported.

8.2 Transverse Vibration of a String or Cable

8.2.1 Equation of Motion

Consider a tightly stretched elastic string or cable of length l subjected to a transverse force $f(x, t)$ per unit length, as shown in Fig. 8.1(a). The transverse displacement of the string, $w(x, t)$, is assumed to be small. Equilibrium of the forces in the z direction gives (see Fig. 8.1(b)):

The net force acting on an element is equal to the inertia force acting on the element, or

$$(P + dP)\sin(\theta + d\theta) + f\,dx - P\sin\theta = \rho\,dx\frac{\partial^2 w}{\partial t^2} \tag{8.1}$$

where P is the tension, ρ is the mass per unit length, and θ is the angle the deflected string makes with the x-axis. For an elemental length dx,

$$dP = \frac{\partial P}{\partial x}dx \tag{8.2}$$

$$\sin\theta \simeq \tan\theta = \frac{\partial w}{\partial x} \tag{8.3}$$

and

$$\sin(\theta + d\theta) \simeq \tan(\theta + d\theta) = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx \tag{8.4}$$

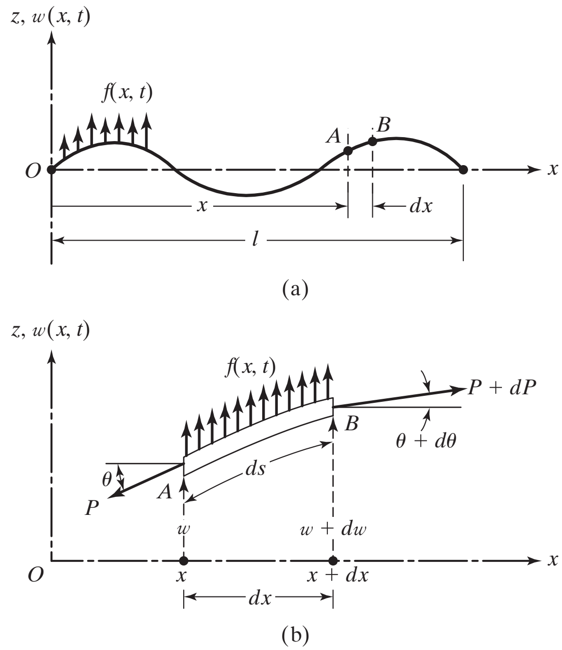


FIGURE 8.1 A vibrating string.

Hence the forced-vibration equation of the nonuniform string, Eq. (8.1), can be simplified to

$$\frac{\partial}{\partial x}\left[P\frac{\partial w(x,t)}{\partial x}\right] + f(x,t) = \rho(x)\frac{\partial^2 w(x,t)}{\partial t^2} \tag{8.5}$$

If the string is uniform and the tension is constant, Eq. (8.5) reduces to

$$P\frac{\partial^2 w(x,t)}{\partial x^2} + f(x,t) = \rho\frac{\partial^2 w(x,t)}{\partial t^2} \tag{8.6}$$

If $f(x,t) = 0$, we obtain the free-vibration equation

$$P\frac{\partial^2 w(x,t)}{\partial x^2} = \rho\frac{\partial^2 w(x,t)}{\partial t^2} \tag{8.7}$$

or

$$c^2\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2} \tag{8.8}$$

where

$$c = \left(\frac{P}{\rho}\right)^{1/2} \tag{8.9}$$

Equation (8.8) is also known as the *wave equation.*

8.2.2 Initial and Boundary Conditions

The equation of motion, Eq. (8.5) or its special forms (8.6) and (8.7), is a partial differential equation of the second order. Since the order of the highest derivative of w with respect to x and t in this equation is two, we need to specify two boundary and two initial conditions in finding the solution $w(x, t)$. If the string has a known deflection $w_0(x)$ and velocity $\dot{w}_0(x)$ at time $t = 0$, the initial conditions are specified as

$$w(x, t = 0) = w_0(x)$$

$$\frac{\partial w}{\partial t}(x, t = 0) = \dot{w}_0(x) \tag{8.10}$$

If the string is fixed at an end, say $x = 0$, the displacement w must always be zero, and so the boundary condition is

$$w(x = 0, t) = 0, \qquad t \geq 0 \tag{8.11}$$

If the string or cable is connected to a pin that can move in a perpendicular direction as shown in Fig. 8.2, the end cannot support a transverse force. Hence the boundary condition becomes

$$P(x)\frac{\partial w(x, t)}{\partial x} = 0 \tag{8.12}$$

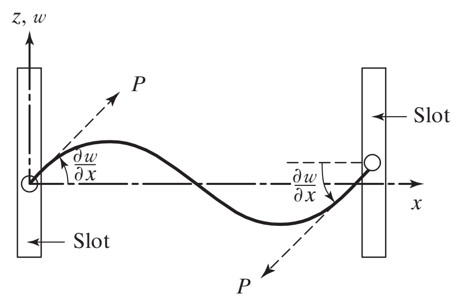


FIGURE 8.2 String connected to pins at the ends.

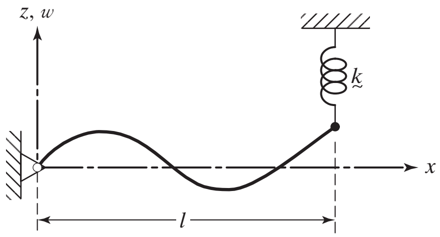


FIGURE 8.3 String with elastic constraint.

If the end $x = 0$ is free and P is a constant, then Eq. (8.12) becomes

$$\frac{\partial w(0, t)}{\partial x} = 0, \qquad t \geq 0 \tag{8.13}$$

If the end $x = l$ is constrained elastically as shown in Fig. 8.3, the boundary condition becomes

$$P(x) \left. \frac{\partial w(x, t)}{\partial x} \right|_{x=l} = -\underset{\sim}{k} w(x, t) \big|_{x=l}, \qquad t \geq 0 \tag{8.14}$$

where $\underset{\sim}{k}$ is the spring constant.

8.2.3 Free Vibration of a Uniform String

The free-vibration equation, Eq. (8.8), can be solved by the method of separation of variables. In this method, the solution is written as the product of a function $W(x)$ (which depends only on x) and a function $T(t)$ (which depends only on t) [8.4]:

$$w(x, t) = W(x)T(t) \tag{8.15}$$

Substitution of Eq. (8.15) into Eq. (8.8) leads to

$$\frac{c^2}{W} \frac{d^2W}{dx^2} = \frac{1}{T} \frac{d^2T}{dt^2} \tag{8.16}$$

Since the left-hand side of this equation depends only on x and the right-hand side depends only on t, their common value must be a constant—say, a—so that

$$\frac{c^2}{W} \frac{d^2W}{dx^2} = \frac{1}{T} \frac{d^2T}{dt^2} = a \tag{8.17}$$

The equations implied in Eq. (8.17) can be written as

$$\frac{d^2W}{dx^2} - \frac{a}{c^2} W = 0 \tag{8.18}$$

$$\frac{d^2T}{dt^2} - aT = 0 \tag{8.19}$$

Since the constant a is generally negative (see Problem 8.9), we can set $a = -\omega^2$ and write Eqs. (8.18) and (8.19) as

$$\frac{d^2W}{dx^2} + \frac{\omega^2}{c^2}W = 0 \tag{8.20}$$

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \tag{8.21}$$

The solutions of these equations are given by

$$W(x) = A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c} \tag{8.22}$$

$$T(t) = C\cos\omega t + D\sin\omega t \tag{8.23}$$

where ω is the frequency of vibration and the constants A, B, C, and D can be evaluated from the boundary and initial conditions.

8.2.4 Free Vibration of a String with Both Ends Fixed

If the string is fixed at both ends, the boundary conditions are $w(0, t) = w(l, t) = 0$ for all time $t \geq 0$. Hence, from Eq. (8.15), we obtain

$$W(0) = 0 \tag{8.24}$$

$$W(l) = 0 \tag{8.25}$$

In order to satisfy Eq. (8.24), A must be zero in Eq. (8.22). Equation (8.25) requires that

$$B\sin\frac{\omega l}{c} = 0 \tag{8.26}$$

Since B cannot be zero for a nontrivial solution, we have

$$\sin\frac{\omega l}{c} = 0 \tag{8.27}$$

Equation (8.27) is called the *frequency* or *characteristic equation* and is satisfied by several values of ω. The values of ω are called the *eigenvalues* (or *natural frequencies* or *characteristic values*) of the problem. The nth natural frequency is given by

$$\frac{\omega_n l}{c} = n\pi, \qquad n = 1, 2, \ldots$$

or

$$\omega_n = \frac{nc\pi}{l}, \qquad n = 1, 2, \ldots \tag{8.28}$$

The solution $w_n(x, t)$ corresponding to ω_n can be expressed as

$$w_n(x, t) = W_n(x)T_n(t) = \sin\frac{n\pi x}{l}\left[C_n \cos\frac{nc\pi t}{l} + D_n \sin\frac{nc\pi t}{l}\right] \tag{8.29}$$

where C_n and D_n are arbitrary constants. The solution $w_n(x, t)$ is called the *nth mode of vibration* or *nth harmonic* or *nth normal mode* of the string. In this mode, each point of the string vibrates with an amplitude proportional to the value of W_n at that point, with the circular frequency $\omega_n = (nc\pi)/l$. The function $W_n(x)$ is called the nth normal mode, or characteristic function. The first three modes of vibration are shown in Fig. 8.4. The mode corresponding to $n = 1$ is called the *fundamental mode,* and ω_1 is called the *fundamental frequency.* The fundamental period is

$$\tau_1 = \frac{2\pi}{\omega_1} = \frac{2l}{c}$$

The points at which $w_n = 0$ for all times are called *nodes.* Thus the fundamental mode has two nodes, at $x = 0$ and $x = l$; the second mode has three nodes, at $x = 0$, $x = l/2$, and $x = l$; etc.

The general solution of Eq. (8.8), which satisfies the boundary conditions of Eqs. (8.24) and (8.25), is given by the superposition of all $w_n(x, t)$:

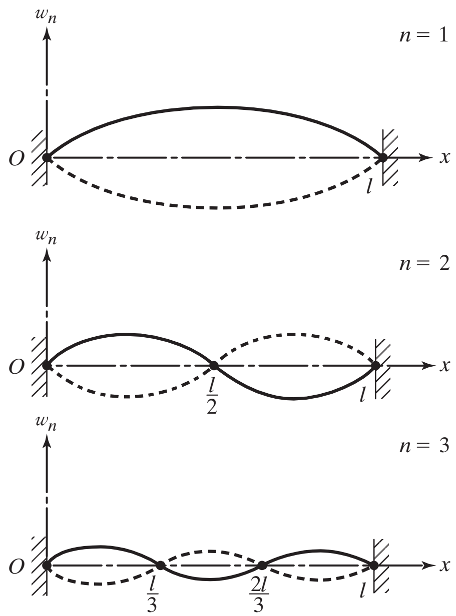


FIGURE 8.4 Mode shapes of a string.

$$w(x, t) = \sum_{n=1}^{\infty} w_n(x, t)$$

$$= \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left[C_n \cos \frac{nc\pi t}{l} + D_n \sin \frac{nc\pi t}{l} \right] \tag{8.30}$$

This equation gives all possible vibrations of the string; the particular vibration that occurs is uniquely determined by the specified initial conditions. The initial conditions give unique values of the constants C_n and D_n. If the initial conditions are specified as in Eq. (8.10), we obtain

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = w_0(x) \tag{8.31}$$

$$\sum_{n=1}^{\infty} \frac{nc\pi}{l} D_n \sin \frac{n\pi x}{l} = \dot{w}_0(x) \tag{8.32}$$

which can be seen to be Fourier sine series expansions of $w_0(x)$ and $\dot{w}_0(x)$ in the interval $0 \leq x \leq l$. The values of C_n and D_n can be determined by multiplying Eqs. (8.31) and (8.32) by $\sin(n\pi x/l)$ and integrating with respect to x from 0 to l:

$$C_n = \frac{2}{l} \int_0^l w_0(x) \sin \frac{n\pi x}{l} dx \tag{8.33}$$

$$D_n = \frac{2}{nc\pi} \int_0^l \dot{w}_0(x) \sin \frac{n\pi x}{l} dx \tag{8.34}$$

Note: The solution given by Eq. (8.30) can be identified as the *mode superposition method* since the response is expressed as a superposition of the normal modes. The procedure is applicable in finding not only the free-vibration solution but also the forced-vibration solution of continuous systems.

EXAMPLE 8.1 Dynamic Response of a Plucked String

If a string of length l, fixed at both ends, is plucked at its midpoint as shown in Fig. 8.5 and then released, determine its subsequent motion.

Solution: The solution is given by Eq. (8.30) with C_n and D_n given by Eqs. (8.33) and (8.34), respectively. Since there is no initial velocity, $\dot{w}_0(x) = 0$, and so $D_n = 0$. Thus the solution of Eq. (8.30) reduces to

$$w(x, t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} \cos \frac{nc\pi t}{l} \tag{E.1}$$

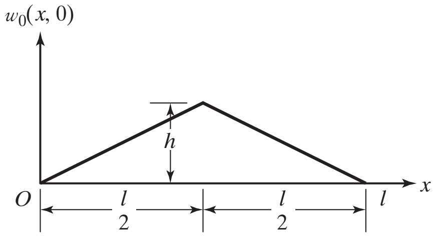


FIGURE 8.5 Initial deflection of the string.

where

$$C_n = \frac{2}{l}\int_0^l w_0(x)\sin\frac{n\pi x}{l}\,dx \tag{E.2}$$

The initial deflection $w_0(x)$ is given by

$$w_0(x) = \begin{cases} \dfrac{2hx}{l} & \text{for } 0 \le x \le \dfrac{l}{2} \\ \dfrac{2h(l-x)}{l} & \text{for } \dfrac{l}{2} \le x \le l \end{cases} \tag{E.3}$$

By substituting Eq. (E.3) into Eq. (E.2), C_n can be evaluated:

$$C_n = \frac{2}{l}\left\{\int_0^{l/2}\frac{2hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2h}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \begin{cases} \dfrac{8h}{\pi^2 n^2}\sin\dfrac{n\pi}{2} & \text{for } n = 1, 3, 5, \ldots \\ 0 & \text{for } n = 2, 4, 6, \ldots \end{cases} \tag{E.4}$$

By using the relation

$$\sin\frac{n\pi}{2} = (-1)^{(n-1)/2}, \qquad n = 1, 3, 5, \ldots \tag{E.5}$$

the desired solution can be expressed as

$$w(x, t) = \frac{8h}{\pi^2}\left\{\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{1}{9}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} + \cdots\right\} \tag{E.6}$$

In this case, no even harmonics are excited.

■

8.2.5 Traveling-Wave Solution

The solution of the wave equation, Eq. (8.8), for a string of infinite length can be expressed as [8.5]

$$w(x, t) = w_1(x - ct) + w_2(x + ct) \tag{8.35}$$

where w_1 and w_2 are arbitrary functions of $(x - ct)$ and $(x + ct)$, respectively. To show that Eq. (8.35) is the correct solution of Eq. (8.8), we first differentiate Eq. (8.35):

$$\frac{\partial^2 w(x, t)}{\partial x^2} = w_1''(x - ct) + w_2''(x + ct) \tag{8.36}$$

$$\frac{\partial^2 w(x, t)}{\partial t^2} = c^2 w_1''(x - ct) + c^2 w_2''(x + ct) \tag{8.37}$$

Substitution of these equations into Eq. (8.8) reveals that the wave equation is satisfied. In Eq. (8.35), $w_1(x - ct)$ and $w_2(x + ct)$ represent waves that propagate in the positive and negative directions of the x-axis, respectively, with a velocity c.

For a given problem, the arbitrary functions w_1 and w_2 are determined from the initial conditions, Eq. (8.10). Substitution of Eq. (8.35) into Eq. (8.10) gives, at $t = 0$,

$$w_1(x) + w_2(x) = w_0(x) \tag{8.38}$$

$$-cw_1'(x) + cw_2'(x) = \dot{w}_0(x) \tag{8.39}$$

where the prime indicates differentiation with respect to the respective argument at $t = 0$ (that is, with respect to x). Integration of Eq. (8.39) yields

$$-w_1(x) + w_2(x) = \frac{1}{c}\int_{x_0}^{x} \dot{w}_0(x')\, dx' \tag{8.40}$$

where x_0 is a constant. Solution of Eqs. (8.38) and (8.40) gives w_1 and w_2:

$$w_1(x) = \frac{1}{2}\left[w_0(x) - \frac{1}{c}\int_{x_0}^{x} \dot{w}_0(x')\, dx'\right] \tag{8.41}$$

$$w_2(x) = \frac{1}{2}\left[w_0(x) + \frac{1}{c}\int_{x_0}^{x} \dot{w}_0(x')\, dx'\right] \tag{8.42}$$

By replacing x by $(x - ct)$ and $(x + ct)$, respectively, in Eqs. (8.41) and (8.42), we obtain the total solution:

$$\begin{aligned} w(x, t) &= w_1(x - ct) + w_2(x + ct) \\ &= \frac{1}{2}[w_0(x - ct) + w_0(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{w}_0(x')\, dx' \end{aligned} \tag{8.43}$$

The following points should be noted:

1. As can be seen from Eq. (8.43), there is no need to apply boundary conditions to the problem.
2. The solution given by Eq. (8.43) can be expressed as

$$w(x, t) = w_D(x, t) + w_V(x, t) \tag{8.44}$$

where $w_D(x, t)$ denotes the waves propagating due to the known initial displacement $w_0(x)$ with zero initial velocity, and $w_V(x, t)$ represents waves traveling due only to the known initial velocity $\dot{w}_0(x)$ with zero initial displacement.

The transverse vibration of a string fixed at both ends excited by the transverse impact of an elastic load at an intermediate point was considered in [8.6]. A review of the literature on the dynamics of cables and chains was given by Triantafyllou [8.7].

8.3 Longitudinal Vibration of a Bar or Rod

8.3.1 Equation of Motion and Solution

Consider an elastic bar of length l with varying cross-sectional area $A(x)$, as shown in Fig. 8.6. The forces acting on the cross sections of a small element of the bar are given by P and $P + dP$ with

$$P = \sigma A = EA\frac{\partial u}{\partial x} \tag{8.45}$$

where σ is the axial stress, E is Young's modulus, u is the axial displacement, and $\partial u/\partial x$ is the axial strain. If $f(x, t)$ denotes the external force per unit length, the summation of the forces in the x direction gives the equation of motion

$$(P + dP) + f\,dx - P = \rho A\,dx\,\frac{\partial^2 u}{\partial t^2} \tag{8.46}$$

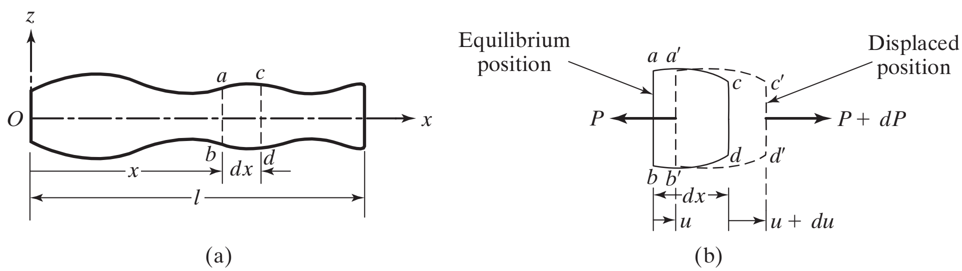


FIGURE 8.6 Longitudinal vibration of a bar.

where ρ is the mass density of the bar. By using the relation $dP = (\partial P/\partial x)\, dx$ and Eq. (8.45), the equation of motion for the forced longitudinal vibration of a nonuniform bar, Eq. (8.46), can be expressed as

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] + f(x,t) = \rho(x)A(x)\frac{\partial^2 u}{\partial t^2}(x,t) \tag{8.47}$$

For a uniform bar, Eq. (8.47) reduces to

$$EA\frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t) = \rho A\frac{\partial^2 u}{\partial t^2}(x,t) \tag{8.48}$$

The free-vibration equation can be obtained from Eq. (8.48), by setting $f = 0$, as

$$c^2\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) \tag{8.49}$$

where

$$c = \sqrt{\frac{E}{\rho}} \tag{8.50}$$

Note that Eqs. (8.47) to (8.50) can be seen to be similar to Eqs. (8.5), (8.6), (8.8), and (8.9), respectively. The solution of Eq. (8.49), which can be obtained as in the case of Eq. (8.8), can thus be written as

$$u(x,t) = U(x)T(t) \equiv \left(\underset{\sim}{A}\cos\frac{\omega x}{c} + \underset{\sim}{B}\sin\frac{\omega x}{c}\right)(C\cos\omega t + D\sin\omega t) \tag{8.51}$$ [1]

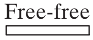
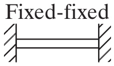
End Conditions of Bar	Boundary Conditions	Frequency Equation	Mode Shape (Normal Function)	Natural Frequencies
Fixed-free 	$u(0,t) = 0$ $\frac{\partial u}{\partial x}(l,t) = 0$	$\cos\frac{\omega l}{c} = 0$	$U_n(x) = C_n \sin\frac{(2n+1)\pi x}{2l}$	$\omega_n = \frac{(2n+1)\pi c}{2l};$ $n = 0, 1, 2, \ldots$
Free-free 	$\frac{\partial u}{\partial x}(0,t) = 0$ $\frac{\partial u}{\partial x}(l,t) = 0$	$\sin\frac{\omega l}{c} = 0$	$U_n(x) = C_n \cos\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l};$ $n = 0, 1, 2, \ldots$
Fixed-fixed 	$u(0,t) = 0$ $u(l,t) = 0$	$\sin\frac{\omega l}{c} = 0$	$U_n(x) = C_n \cos\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l};$ $n = 1, 2, 3, \ldots$

FIGURE 8.7 Common boundary conditions for a bar in longitudinal vibration.

[1]We use $\underset{\sim}{A}$ and $\underset{\sim}{B}$ in this section; A is used to denote the cross-sectional area of the bar.

where the function $U(x)$ represents the normal mode and depends only on x and the function $T(t)$ depends only on t. If the bar has known initial axial displacement $u_0(x)$ and initial velocity $\dot{u}_0(x)$, the initial conditions can be stated as

$$u(x, t = 0) = u_0(x)$$
$$\frac{\partial u}{\partial t}(x, t = 0) = \dot{u}_0(x) \tag{8.52}$$

The common boundary conditions and the corresponding frequency equations for the longitudinal vibration of uniform bars are shown in Fig. 8.7.

EXAMPLE 8.2 Boundary Conditions for a Bar

A uniform bar of cross-sectional area A, length l, and Young's modulus E is connected at both ends by springs, dampers, and masses, as shown in Fig. 8.8(a). State the boundary conditions.

Solution: The free-body diagrams of the masses m_1 and m_2 are shown in Fig. 8.8(b). From this, we find that at the left end $(x = 0)$, the force developed in the bar due to positive u and $\partial u/\partial x$ must be equal to the sum of spring, damper, and inertia forces:

$$AE\frac{\partial u}{\partial x}(0, t) = k_1 u(0, t) + c_1\frac{\partial u}{\partial t}(0, t) + m_1\frac{\partial^2 u}{\partial t^2}(0, t) \tag{E.1}$$

Similarly at the right end $(x = l)$, the force developed in the bar due to positive u and $\partial u/\partial x$ must be equal to the negative sum of spring, damper, and inertia forces:

$$AE\frac{\partial u}{\partial x}(l, t) = -k_2 u(l, t) - c_2\frac{\partial u}{\partial t}(l, t) - m_2\frac{\partial^2 u}{\partial t^2}(l, t) \tag{E.2}$$

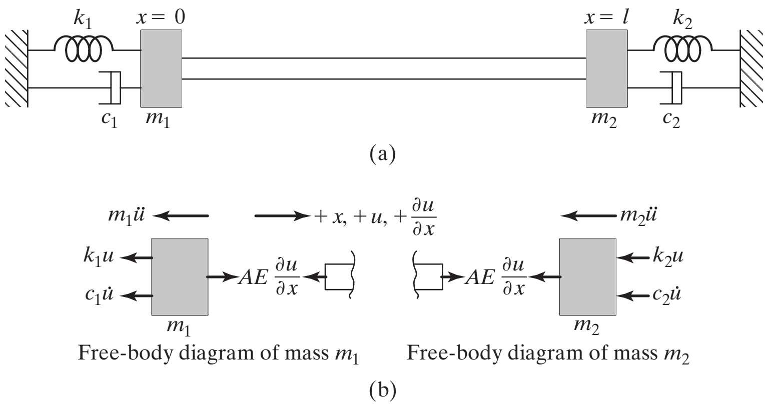


FIGURE 8.8 Bar connected to springs-masses-dampers at ends.

■

8.3.2 Orthogonality of Normal Functions

The normal functions for the longitudinal vibration of bars satisfy the orthogonality relation

$$\int_0^l U_i(x) U_j(x)\, dx = 0 \tag{8.53}$$

where $U_i(x)$ and $U_j(x)$ denote the normal functions corresponding to the ith and jth natural frequencies ω_i and ω_j, respectively. When $u(x, t) = U_i(x) T(t)$ and $u(x, t) = U_j(x) T(t)$ are assumed as solutions, Eq. (8.49) gives

$$c^2 \frac{d^2 U_i(x)}{dx^2} + \omega_i^2 U_i(x) = 0 \qquad \text{or} \qquad c^2 U_i''(x) + \omega_i^2 U_i(x) = 0 \tag{8.54}$$

and

$$c^2 \frac{d^2 U_j(x)}{dx^2} + \omega_j^2 U_j(x) = 0 \qquad \text{or} \qquad c^2 U_j''(x) + \omega_j^2 U_j(x) = 0 \tag{8.55}$$

where $U_i'' = \dfrac{d^2 U_i}{dx^2}$ and $U_j'' = \dfrac{d^2 U_j}{dx^2}$. Multiplication of Eq. (8.54) by U_j and Eq. (8.55) by U_i gives

$$c^2 U_i'' U_j + \omega_i^2 U_i U_j = 0 \tag{8.56}$$

$$c^2 U_j'' U_i + \omega_j^2 U_j U_i = 0 \tag{8.57}$$

Subtraction of Eq. (8.57) from Eq. (8.56) and integration from 0 to l results in

$$\int_0^l U_i U_j \, dx = -\frac{c^2}{\omega_i^2 - \omega_j^2} \int_0^l (U_i'' U_j - U_j'' U_i)\, dx$$

$$= -\frac{c^2}{\omega_i^2 - \omega_j^2} [U_i' U_j - U_j' U_i] \Big|_0^l \tag{8.58}$$

The right-hand side of Eq. (8.58) can be proved to be zero for any combination of boundary conditions. For example, if the bar is fixed at $x = 0$ and free at $x = l$,

$$u(0, t) = 0, \qquad t \geq 0 \qquad \text{or} \qquad U(0) = 0 \tag{8.59}$$

$$\frac{\partial u}{\partial x}(l, t) = 0, \qquad t \geq 0 \qquad \text{or} \qquad U'(l) = 0 \tag{8.60}$$

Thus $(U_i' U_j - U_j' U_i)\big|_{x=l} = 0$ due to U' being zero (Eq. (8.60)) and $(U_i' U_j - U_j' U_i)\big|_{x=0} = 0$ due to U being zero (Eq. (8.59)). Equation (8.58) thus reduces to Eq. (8.53), which is also known as the *orthogonality principle for the normal functions.*

EXAMPLE 8.3 Free Vibrations of a Fixed-Free Bar

Find the natural frequencies and the free-vibration solution of a bar fixed at one end and free at the other.

Solution: Let the bar be fixed at $x = 0$ and free at $x = l$, so that the boundary conditions can be expressed as

$$u(0, t) = 0, \qquad t \geq 0 \tag{E.1}$$

$$\frac{\partial u}{\partial x}(l, t) = 0, \qquad t \geq 0 \tag{E.2}$$

The use of Eq. (E.1) in Eq. (8.51) gives $\underset{\sim}{A} = 0$, while the use of Eq. (E.2) gives the frequency equation

$$\underset{\sim}{B}\frac{\omega}{c}\cos\frac{\omega l}{c} = 0 \qquad \text{or} \qquad \cos\frac{\omega l}{c} = 0 \tag{E.3}$$

The eigenvalues or natural frequencies are given by

$$\frac{\omega_n l}{c} = (2n+1)\frac{\pi}{2}, \qquad n = 0, 1, 2, \ldots$$

or

$$\omega_n = \frac{(2n+1)\pi c}{2l}, \qquad n = 0, 1, 2, \ldots \tag{E.4}$$

Thus the total (free-vibration) solution of Eq. (8.49) can be written, using the mode superposition method, as

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi x}{2l}\left[C_n \cos\frac{(2n+1)\pi ct}{2l} + D_n \sin\frac{(2n+1)\pi ct}{2l}\right] \end{aligned} \tag{E.5}$$

where the values of the constants C_n and D_n can be determined from the initial conditions, as in Eqs. (8.33) and (8.34):

$$C_n = \frac{2}{l}\int_0^l u_0(x)\sin\frac{(2n+1)\pi x}{2l}\,dx \tag{E.6}$$

$$D_n = \frac{4}{(2n+1)\pi c}\int_0^l \dot{u}_0(x)\sin\frac{(2n+1)\pi x}{2l}\,dx \tag{E.7}$$

■

EXAMPLE 8.4 Natural Frequencies of a Bar Carrying a Mass

Find the natural frequencies of a bar with one end fixed and a mass attached at the other end, as in Fig. 8.9.

Solution: The equation governing the axial vibration of the bar is given by Eq. (8.49) and the solution by Eq. (8.51). The boundary condition at the fixed end ($x = 0$)

$$u(0, t) = 0 \tag{E.1}$$

leads to $\underset{\sim}{A} = 0$ in Eq. (8.51). At the end $x = l$, the tensile force in the bar must be equal to the inertia force of the vibrating mass M, and so

$$AE\frac{\partial u}{\partial x}(l, t) = -M\frac{\partial^2 u}{\partial t^2}(l, t) \tag{E.2}$$

With the help of Eq. (8.51), this equation can be expressed as

$$AE\frac{\omega}{c}\cos\frac{\omega l}{c}(C\cos\omega t + D\sin\omega t) = M\omega^2\sin\frac{\omega l}{c}(C\cos\omega t + D\sin\omega t)$$

That is,

$$\frac{AE\omega}{c}\cos\frac{\omega l}{c} = M\omega^2\sin\frac{\omega l}{c}$$

or

$$\alpha\tan\alpha = \beta \tag{E.3}$$

where

$$\alpha = \frac{\omega l}{c} \tag{E.4}$$

and

$$\beta = \frac{AEl}{c^2 M} = \frac{A\rho l}{M} = \frac{m}{M} \tag{E.5}$$

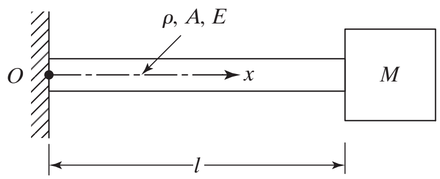


FIGURE 8.9 Bar carrying an end mass.

TABLE 8.1

	Values of the Mass Ratio β				
	0.01	**0.1**	**1.0**	**10.0**	**100.0**
Value of α_1 $\left(\omega_1 = \dfrac{\alpha_1 c}{l}\right)$	0.1000	0.3113	0.8602	1.4291	1.5549
Value of α_2 $\left(\omega_2 = \dfrac{\alpha_2 c}{l}\right)$	3.1448	3.1736	3.4267	4.3063	4.6658

where m is the mass of the bar. Equation (E.3) is the frequency equation (in the form of a transcendental equation) whose solution gives the natural frequencies of the system. The first two natural frequencies are given in Table 8.1 for different values of the parameter β.

Note: If the mass of the bar is negligible compared to the mass attached, $m \simeq 0$,

$$c = \left(\frac{E}{\rho}\right)^{1/2} = \left(\frac{EAl}{m}\right)^{1/2} \to \infty \quad \text{and} \quad \alpha = \frac{\omega l}{c} \to 0$$

In this case

$$\tan\frac{\omega l}{c} \simeq \frac{\omega l}{c}$$

and the frequency equation (E.3) can be taken as

$$\left(\frac{\omega l}{c}\right)^2 = \beta$$

This gives the approximate value of the fundamental frequency

$$\omega_1 = \frac{c}{l}\beta^{1/2} = \frac{c}{l}\left(\frac{\rho A l}{M}\right)^{1/2} = \left(\frac{EA}{lM}\right)^{1/2} = \left(\frac{g}{\delta_s}\right)^{1/2}$$

where

$$\delta_s = \frac{Mgl}{EA}$$

represents the static elongation of the bar under the action of the load Mg.

■

EXAMPLE 8.5 Vibrations of a Bar Subjected to Initial Force

A bar of uniform cross-sectional area A, density ρ, modulus of elasticity E, and length l is fixed at one end and free at the other end. It is subjected to an axial force F_0 at its free end, as shown in Fig. 8.10(a). Study the resulting vibrations if the force F_0 is suddenly removed.

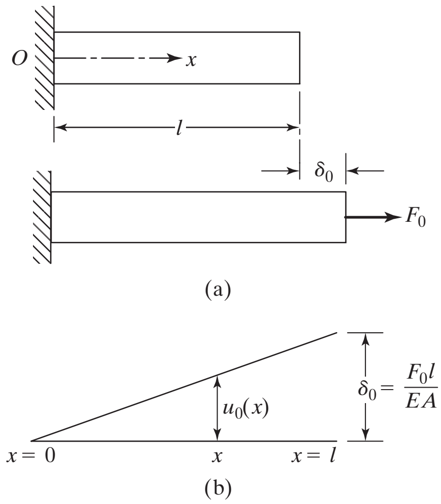


FIGURE 8.10 Bar subjected to an axial force at end.

Solution: The tensile strain induced in the bar due to F_0 is

$$\varepsilon = \frac{F_0}{EA}$$

Thus the displacement of the bar just before the force F_0 is removed (initial displacement) is given by (see Fig. (8.10b))

$$u_0 = u(x, 0) = \varepsilon x = \frac{F_0 x}{EA}, \qquad 0 \le x \le l \tag{E.1}$$

Since the initial velocity is zero, we have

$$\dot{u}_0 = \frac{\partial u}{\partial t}(x, 0) = 0, \qquad 0 \le x \le l \tag{E.2}$$

The general solution of a bar fixed at one end and free at the other end is given by Eq. (E.5) of Example 8.3:

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= \sum_{n=0}^{\infty} \sin \frac{(2n+1)\pi x}{2l} \left[C_n \cos \frac{(2n+1)\pi ct}{2l} + D_n \sin \frac{(2n+1)\pi ct}{2l} \right] \end{aligned} \tag{E.3}$$

where C_n and D_n are given by Eqs. (E.6) and (E.7) of Example 8.3. Since $\dot{u}_0 = 0$, we obtain $D_n = 0$. By using the initial displacement of Eq. (E.1) in Eq. (E.6) of Example 8.3, we obtain

$$C_n = \frac{2}{l}\int_0^l \frac{F_0 x}{EA} \cdot \sin\frac{(2n+1)\pi x}{2l} dx = \frac{8F_0 l}{EA\pi^2}\frac{(-1)^n}{(2n+1)^2} \tag{E.4}$$

Thus the solution becomes

$$u(x, t) = \frac{8F_0 l}{EA\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l}\cos\frac{(2n+1)\pi ct}{2l} \tag{E.5}$$

Equations (E.3) and (E.5) indicate that the motion of a typical point at $x = x_0$ on the bar is composed of the amplitudes

$$C_n \sin\frac{(2n+1)\pi x_0}{2l}$$

corresponding to the circular frequencies

$$\frac{(2n+1)\pi c}{2l}$$

■

8.4 Torsional Vibration of a Shaft or Rod

Figure 8.11 represents a nonuniform shaft subjected to an external torque $f(x, t)$ per unit length. If $\theta(x, t)$ denotes the angle of twist of the cross section, the relation between the torsional deflection and the twisting moment $M_t(x, t)$ is given by [8.8]

$$M_t(x, t) = GJ(x)\frac{\partial\theta}{\partial x}(x, t) \tag{8.61}$$

where G is the shear modulus and $GJ(x)$ is the torsional stiffness, with $J(x)$ denoting the polar moment of inertia of the cross section in the case of a circular section. If the mass polar moment of inertia of the shaft per unit length is I_0, the inertia torque acting on an element of length dx becomes

$$I_0\, dx \frac{\partial^2\theta}{\partial t^2}$$

If an external torque $f(x, t)$ acts on the shaft per unit length, the application of Newton's second law yields the equation of motion:

$$(M_t + dM_t) + f\,dx - M_t = I_0\, dx\frac{\partial^2\theta}{\partial t^2} \tag{8.62}$$

By expressing dM_t as

$$\frac{\partial M_t}{\partial x}dx$$

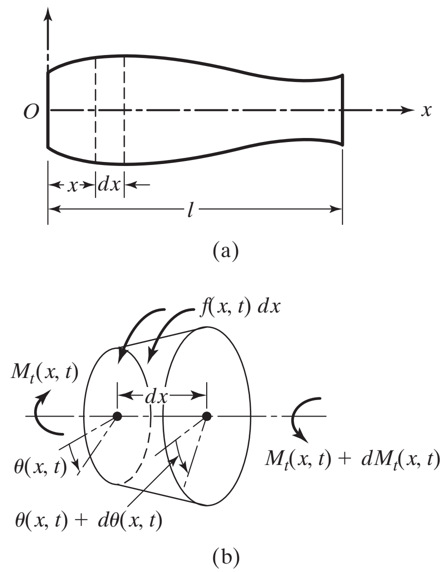


FIGURE 8.11 Torsional vibration of a shaft.

and using Eq. (8.61), the forced torsional vibration equation for a nonuniform shaft can be obtained:

$$\frac{\partial}{\partial x}\left[GJ(x)\frac{\partial \theta}{\partial x}(x,t)\right] + f(x,t) = I_0(x)\frac{\partial^2 \theta}{\partial t^2}(x,t) \tag{8.63}$$

For a uniform shaft, Eq. (8.63) takes the form

$$GJ\frac{\partial^2 \theta}{\partial x^2}(x,t) + f(x,t) = I_0\frac{\partial^2 \theta}{\partial t^2}(x,t) \tag{8.64}$$

which, in the case of free vibration, reduces to

$$c^2\frac{\partial^2 \theta}{\partial x^2}(x,t) = \frac{\partial^2 \theta}{\partial t^2}(x,t) \tag{8.65}$$

where

$$c = \sqrt{\frac{GJ}{I_0}} \tag{8.66}$$

Notice that Eqs. (8.63) to (8.66) are similar to the equations derived in the cases of transverse vibration of a string and longitudinal vibration of a bar. If the shaft has a uniform cross section, $I_0 = \rho J$. Hence Eq. (8.66) becomes

$$c = \sqrt{\frac{G}{\rho}} \tag{8.67}$$

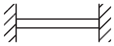
End Conditions of Shaft	Boundary Conditions	Frequency Equation	Mode Shape (Normal Function)	Natural Frequencies
Fixed-free 	$\theta(0, t) = 0$ $\frac{\partial \theta}{\partial x}(l, t) = 0$	$\cos\frac{\omega l}{c} = 0$	$\theta(x) = C_n \sin\frac{(2n+1)\,\pi x}{2l}$	$\omega_n = \frac{(2n+1)\,\pi c}{2l}$; $n = 0, 1, 2, \ldots$
Free-free 	$\frac{\partial \theta}{\partial x}(0, t) = 0$ $\frac{\partial \theta}{\partial x}(l, t) = 0$	$\sin\frac{\omega l}{c} = 0$	$\theta(x) = C_n \cos\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l}$; $n = 0, 1, 2, \ldots$
Fixed-fixed 	$\theta(0, t) = 0$ $\theta(l, t) = 0$	$\sin\frac{\omega l}{c} = 0$	$\theta(x) = C_n \cos\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l}$; $n = 1, 2, 3, \ldots$

FIGURE 8.12 Boundary conditions for uniform shafts (rods) subjected to torsional vibration.

If the shaft is given an angular displacement $\theta_0(x)$ and an angular velocity $\dot{\theta}_0(x)$ at $t = 0$, the initial conditions can be stated as

$$\theta(x, t = 0) = \theta_0(x)$$
$$\frac{\partial \theta}{\partial t}(x, t = 0) = \dot{\theta}_0(x) \tag{8.68}$$

The general solution of Eq. (8.65) can be expressed as

$$\theta(x, t) = \left(A \cos\frac{\omega x}{c} + B \sin\frac{\omega x}{c}\right)(C \cos \omega t + D \sin \omega t) \tag{8.69}$$

The common boundary conditions for the torsional vibration of uniform shafts are indicated in Fig. 8.12 along with the corresponding frequency equations and the normal functions.

EXAMPLE 8.6 Natural Frequencies of a Milling Cutter

Find the natural frequencies of the plane milling cutter shown in Fig. 8.13 when the free end of the shank is fixed. Assume the torsional rigidity of the shank as GJ and the mass moment of inertia of the cutter as I_0.

Solution: The general solution is given by Eq. (8.69). From this equation, by using the fixed boundary condition $\theta(0, t) = 0$, we obtain $A = 0$. The boundary condition at $x = l$ can be stated as

$$GJ \frac{\partial \theta}{\partial x}(l, t) = -I_0 \frac{\partial^2 \theta}{\partial t^2}(l, t) \tag{E.1}$$

That is,

$$BGJ \frac{\omega}{c} \cos\frac{\omega l}{c} = BI_0 \omega^2 \sin\frac{\omega l}{c}$$

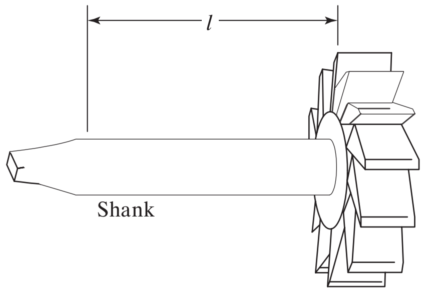


FIGURE 8.13 Plane milling cutter.

or

$$\frac{\omega l}{c} \tan \frac{\omega l}{c} = \frac{J \rho l}{I_0} = \frac{\underset{\sim}{J}_{\text{rod}}}{I_0} \tag{E.2}$$

where $\underset{\sim}{J}_{\text{rod}} = J\rho l$. Equation (E.2) can be expressed as

$$\alpha \tan \alpha = \beta \qquad \text{where } \alpha = \frac{\omega l}{c} \qquad \text{and} \qquad \beta = \frac{\underset{\sim}{J}_{\text{rod}}}{I_0} \tag{E.3}$$

The solution of Eq. (E.3), and thus the natural frequencies of the system, can be obtained as in the case of Example 8.4.

■

8.5 Lateral Vibration of Beams

8.5.1 Equation of Motion

Consider the free-body diagram of an element of a beam shown in Fig. 8.14, where $M(x, t)$ is the bending moment, $V(x, t)$ is the shear force, and $f(x, t)$ is the external force per unit length of the beam. Since the inertia force acting on the element of the beam is

$$\rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t)$$

the force equation of motion in the z direction gives

$$-(V + dV) + f(x, t)\, dx + V = \rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t) \tag{8.70}$$

where ρ is the mass density and $A(x)$ is the cross-sectional area of the beam. The moment equation of motion about the y-axis passing through point O in Fig. 8.14 leads to

$$(M + dM) - (V + dV)\, dx + f(x, t)\, dx \frac{dx}{2} - M = 0 \tag{8.71}$$

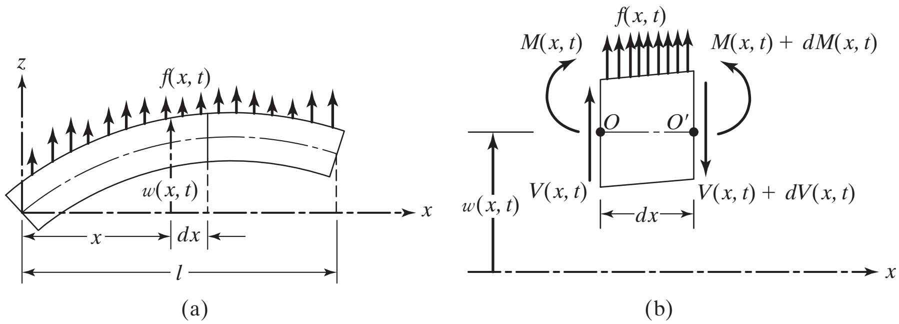


FIGURE 8.14 A beam in bending.

By writing

$$dV = \frac{\partial V}{\partial x}\,dx \qquad \text{and} \qquad dM = \frac{\partial M}{\partial x}\,dx$$

and disregarding terms involving second powers in dx, Eqs. (8.70) and (8.71) can be written as

$$-\frac{\partial V}{\partial x}(x,t) + f(x,t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) \tag{8.72}$$

$$\frac{\partial M}{\partial x}(x,t) - V(x,t) = 0 \tag{8.73}$$

By using the relation $V = \partial M/\partial x$ from Eq. (8.73), Eq. (8.72) becomes

$$-\frac{\partial^2 M}{\partial x^2}(x,t) + f(x,t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) \tag{8.74}$$

From the elementary theory of bending of beams (also known as the *Euler-Bernoulli* or *thin beam theory*), the relationship between bending moment and deflection can be expressed as [8.8]

$$M(x,t) = EI(x)\frac{\partial^2 w}{\partial x^2}(x,t) \tag{8.75}$$

where E is Young's modulus and $I(x)$ is the moment of inertia of the beam cross section about the y-axis. Inserting Eq. (8.75) into Eq. (8.74), we obtain the equation of motion for the forced lateral vibration of a nonuniform beam:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}(x,t)\right] + \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t) \tag{8.76}$$

For a uniform beam, Eq. (8.76) reduces to

$$EI\frac{\partial^4 w}{\partial x^4}(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t) \tag{8.77}$$

For free vibration, $f(x,t) = 0$, and so the equation of motion becomes

$$c^2\frac{\partial^4 w}{\partial x^4}(x,t) + \frac{\partial^2 w}{\partial t^2}(x,t) = 0 \tag{8.78}$$

where

$$c = \sqrt{\frac{EI}{\rho A}} \tag{8.79}$$

8.5.2 Initial Conditions

Since the equation of motion involves a second-order derivative with respect to time and a fourth-order derivative with respect to x, two initial conditions and four boundary conditions are needed for finding a unique solution for $w(x,t)$. Usually, the values of lateral displacement and velocity are specified as $w_0(x)$ and $\dot{w}_0(x)$ at $t = 0$, so that the initial conditions become

$$\begin{aligned} w(x, t=0) &= w_0(x) \\ \frac{\partial w}{\partial t}(x, t=0) &= \dot{w}_0(x) \end{aligned} \tag{8.80}$$

8.5.3 Free Vibration

The free-vibration solution can be found using the method of separation of variables as

$$w(x,t) = W(x)T(t) \tag{8.81}$$

Substituting Eq. (8.81) into Eq. (8.78) and rearranging leads to

$$\frac{c^2}{W(x)}\frac{d^4W(x)}{dx^4} = -\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = a = \omega^2 \tag{8.82}$$

where $a = \omega^2$ is a positive constant (see Problem 8.45). Equation (8.82) can be written as two equations:

$$\frac{d^4W(x)}{dx^4} - \beta^4 W(x) = 0 \tag{8.83}$$

$$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0 \tag{8.84}$$

where

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A\omega^2}{EI} \tag{8.85}$$

The solution of Eq. (8.84) can be expressed as

$$T(t) = A\cos\omega t + B\sin\omega t \tag{8.86}$$

where A and B are constants that can be found from the initial conditions. For the solution of Eq. (8.83), we assume

$$W(x) = Ce^{sx} \tag{8.87}$$

where C and s are constants, and derive the auxiliary equation as

$$s^4 - \beta^4 = 0 \tag{8.88}$$

The roots of this equation are

$$s_{1,2} = \pm\beta, \qquad s_{3,4} = \pm i\beta \tag{8.89}$$

Hence the solution of Eq. (8.83) becomes

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x} \tag{8.90}$$

where C_1, C_2, C_3, and C_4 are constants. Equation (8.90) can also be expressed as

$$W(x) = C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x \tag{8.91}$$

or

$$\begin{aligned} W(x) &= C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x) \\ &\quad + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x) \end{aligned} \tag{8.92}$$

where C_1, C_2, C_3, and C_4, in each case, are different constants. The constants C_1, C_2, C_3, and C_4 can be found from the boundary conditions. The natural frequencies of the beam are computed from Eq. (8.85) as

$$\omega = \beta^2\sqrt{\frac{EI}{\rho A}} = (\beta l)^2\sqrt{\frac{EI}{\rho A l^4}} \tag{8.93}$$

The function $W(x)$ is known as the *normal mode* or *characteristic function* of the beam and ω is called the *natural frequency of vibration.* For any beam, there will be an infinite number of normal modes with one natural frequency associated with each normal mode. The unknown constants C_1 to C_4 in Eq. (8.91) or (8.92) and the value of β in Eq. (8.93) can be determined from the boundary conditions of the beam as indicated below.

8.5.4 Boundary Conditions

The common boundary conditions are as follows:

1. *Free end:*

$$\text{Bending moment} = EI\frac{\partial^2 w}{\partial x^2} = 0$$

$$\text{Shear force} = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{8.94}$$

2. *Simply supported (pinned) end:*

$$\text{Deflection} = w = 0, \qquad \text{Bending moment} = EI\frac{\partial^2 w}{\partial x^2} = 0 \tag{8.95}$$

3. *Fixed (clamped) end:*

$$\text{Deflection} = 0, \qquad \text{Slope} = \frac{\partial w}{\partial x} = 0 \tag{8.96}$$

The frequency equations, the mode shapes (normal functions), and the natural frequencies for beams with common boundary conditions are given in Fig. 8.15 [8.13, 8.17]. We shall now consider some other possible boundary conditions for a beam.

4. *End connected to a linear spring, damper, and mass* (Fig. 8.16(a)): When the end of a beam undergoes a transverse displacement w and slope $\partial w/\partial x$. with velocity $\partial w/\partial t$ and acceleration $\partial^2 w/\partial t^2$, the resisting forces due to the spring, damper, and mass are proportional to w, $\partial w/\partial t$, and $\partial^2 w/\partial t^2$, respectively. This resisting force is balanced by the shear force at the end. Thus

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = a\left[kw + c\frac{\partial w}{\partial t} + m\frac{\partial^2 w}{\partial t^2}\right] \tag{8.97}$$

where $a = -1$ for the left end and $+1$ for the right end of the beam. In addition, the bending moment must be zero; hence

$$EI\frac{\partial^2 w}{\partial x^2} = 0 \tag{8.98}$$

5. *End connected to a torsional spring, torsional damper, and rotational inertia* (Fig. 8.16(b)): In this case, the boundary conditions are

$$EI\frac{\partial^2 w}{\partial x^2} = a\left[k_t\frac{\partial w}{\partial x} + c_t\frac{\partial^2 w}{\partial x \partial t} + I_0\frac{\partial^3 w}{\partial x \partial t^2}\right] \tag{8.99}$$

where $a = +1$ for the left end and -1 for the right end of the beam, and

$$\frac{\partial}{\partial x}\left[EI\frac{\partial^2 w}{\partial x^2}\right] = 0 \tag{8.100}$$

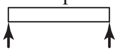
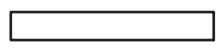
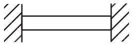
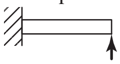
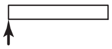
End Conditions of Beam	Frequency Equation	Mode Shape (Normal Function)	Value of $\beta_n l$
Pinned-pinned 	$\sin \beta_n l = 0$	$W_n(x) = C_n[\sin \beta_n x]$	$\beta_1 l = \pi$ $\beta_2 l = 2\pi$ $\beta_3 l = 3\pi$ $\beta_4 l = 4\pi$
Free-free 	$\cos \beta_n l \cdot \cosh \beta_n l = 1$	$W_n(x) = C_n[\sin \beta_n x + \sinh \beta_n x + \alpha_n (\cos \beta_n x + \cosh \beta_n x)]$ where $\alpha_n = \left(\dfrac{\sin \beta_n l - \sinh \beta_n l}{\cosh \beta_n l - \cos \beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$ ($\beta l = 0$ for rigid-body mode)
Fixed-fixed 	$\cos \beta_n l \cdot \cosh \beta_n l = 1$	$W_n(x) = C_n[\sinh \beta_n x_n x - \sin \beta_n x + \alpha_n (\cosh \beta_n x - \cos \beta_n x)]$ where $\alpha_n = \left(\dfrac{\sinh \beta_n l - \sin \beta_n l}{\cos \beta_n l - \cosh \beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$
Fixed-free 	$\cos \beta_n l \cdot \cosh \beta_n l = -1$	$W_n(x) = C_n[\sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x)]$ where $\alpha_n = \left(\dfrac{\sin \beta_n l + \sinh \beta_n l}{\cos \beta_n l + \cosh \beta_n l}\right)$	$\beta_1 l = 1.875104$ $\beta_2 l = 4.694091$ $\beta_3 l = 7.854757$ $\beta_4 l = 10.995541$
Fixed-pinned 	$\tan \beta_n l - \tanh \beta_n l = 0$	$W_n(x) = C_n[\sin \beta_n x - \sinh \beta_n x + \alpha_n (\cosh \beta_n x - \cos \beta_n x)]$ where $\alpha_n = \left(\dfrac{\sin \beta_n l - \sinh \beta_n l}{\cos \beta_n l - \cosh \beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$
Pinned-free 	$\tan \beta_n l - \tanh \beta_n l = 0$	$W_n(x) = C_n[\sin \beta_n x + \alpha_n \sinh \beta_n x]$ where $\alpha_n = \left(\dfrac{\sin \beta_n l}{\sinh \beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$ ($\beta l = 0$ for rigid-body mode)

FIGURE 8.15 Common boundary conditions for the transverse vibration of a beam.

8.5.5 Orthogonality of Normal Functions

The normal functions $W(x)$ satisfy Eq. (8.83):

$$c^2 \frac{d^4 W}{dx^4}(x) - \omega^2 W(x) = 0 \tag{8.101}$$

Let $W_i(x)$ and $W_j(x)$ be the normal functions corresponding to the natural frequencies ω_i and $\omega_j (i \neq j)$, so that

$$c^2 \frac{d^4 W_i}{dx^4} - \omega_i^2 W_i = 0 \tag{8.102}$$

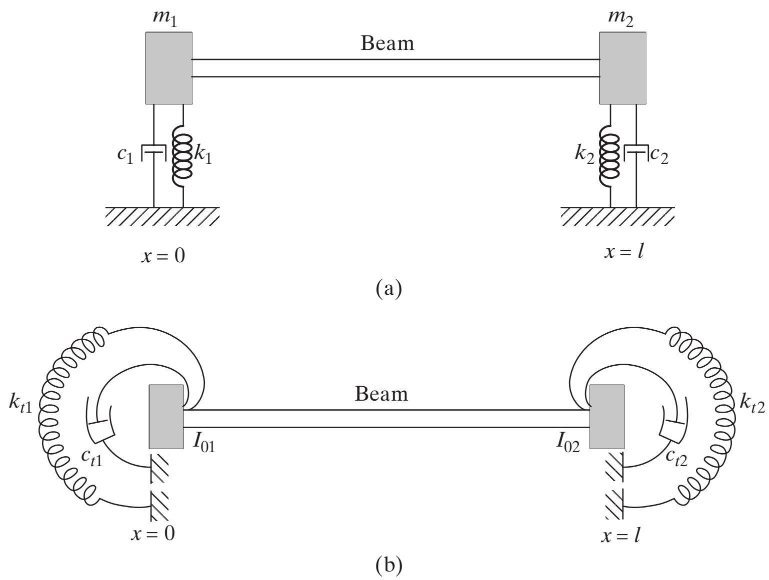


FIGURE 8.16 Beams connected with springs-dampers-masses at ends.

and

$$c^2 \frac{d^4 W_j}{dx^4} - \omega_j^2 W_j = 0 \tag{8.103}$$

Multiplying Eq. (8.102) by W_j and Eq. (8.103) by W_i, subtracting the resulting equations one from the other, and integrating from 0 to l gives

$$\int_0^l \left[c^2 \frac{d^4 W_i}{dx^4} W_j - \omega_i^2 W_i W_j \right] dx - \int_0^l \left[c^2 \frac{d^4 W_j}{dx^4} W_i - \omega_j^2 W_j W_i \right] dx = 0$$

or

$$\int_0^l W_i W_j \, dx = -\frac{c^2}{\omega_i^2 - \omega_j^2} \int_0^l (W_i'''' W_j - W_i W_j'''') \, dx \tag{8.104}$$

where a prime indicates differentiation with respect to x. The right-hand side of Eq. (8.104) can be evaluated using integration by parts to obtain

$$\int_0^l W_i W_j \, dx = -\frac{c^2}{\omega_i^2 - \omega_j^2} [W_i W_j''' - W_j W_i''' + W_j' W_i'' - W_i' W_j''] \Big|_0^l \tag{8.105}$$

The right-hand side of Eq. (8.105) can be shown to be zero for any combination of free, fixed, or simply supported end conditions. At a free end, the bending moment and shear force are equal to zero so that

$$W'' = 0, \qquad W''' = 0 \tag{8.106}$$

For a fixed end, the deflection and slope are zero:

$$W = 0, \qquad W' = 0 \tag{8.107}$$

At a simply supported end, the bending moment and deflection are zero:

$$W'' = 0, \qquad W = 0 \tag{8.108}$$

Since each term on the right-hand side of Eq. (8.105) is zero at $x = 0$ or $x = l$ for any combination of the boundary conditions in Eqs. (8.106) to (8.108), Eq. (8.105) reduces to

$$\int_0^l W_i W_j \, dx = 0 \tag{8.109}$$

which proves the orthogonality of normal functions for the transverse vibration of beams.

EXAMPLE 8.7

Natural Frequencies of a Fixed-Pinned Beam

Determine the natural frequencies of vibration of a uniform beam fixed at $x = 0$ and simply supported at $x = l$.

Solution: The boundary conditions can be stated as

$$W(0) = 0 \tag{E.1}$$

$$\frac{dW}{dx}(0) = 0 \tag{E.2}$$

$$W(l) = 0 \tag{E.3}$$

$$EI\frac{d^2W}{dx^2}(l) = 0 \quad \text{or} \quad \frac{d^2W}{dx^2}(l) = 0 \tag{E.4}$$

Condition (E.1) leads to

$$C_1 + C_3 = 0 \tag{E.5}$$

in Eq. (8.91), while Eqs. (E.2) and (8.91) give

$$\left.\frac{dW}{dx}\right|_{x=0} = \beta[-C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x]_{x=0} = 0$$

or

$$\beta[C_2 + C_4] = 0 \tag{E.6}$$

Thus the solution, Eq. (8.91), becomes

$$W(x) = C_1(\cos \beta x - \cosh \beta x) + C_2(\sin \beta x - \sinh \beta x) \tag{E.7}$$

Applying conditions (E.3) and (E.4) to Eq. (E.7) yields

$$C_1(\cos \beta l - \cosh \beta l) + C_2(\sin \beta l - \sinh \beta l) = 0 \tag{E.8}$$

$$-C_1(\cos \beta l + \cosh \beta l) - C_2(\sin \beta l + \sinh \beta l) = 0 \tag{E.9}$$

For a nontrivial solution of C_1 and C_2, the determinant of their coefficients must be zero—that is,

$$\begin{vmatrix} (\cos \beta l - \cosh \beta l) & (\sin \beta l - \sinh \beta l) \\ -(\cos \beta l + \cosh \beta l) & -(\sin \beta l + \sinh \beta l) \end{vmatrix} = 0 \tag{E.10}$$

Expanding the determinant gives the frequency equation

$$\cos \beta l \sinh \beta l - \sin \beta l \cosh \beta l = 0$$

or

$$\tan \beta l = \tanh \beta l \tag{E.11}$$

The roots of this equation, $\beta_n l$, give the natural frequencies of vibration

$$\omega_n = (\beta_n l)^2 \left(\frac{EI}{\rho A l^4}\right)^{1/2}, \qquad n = 1, 2, \ldots \tag{E.12}$$

where the values of $\beta_n l$, $n = 1, 2, \ldots$ satisfying Eq. (E.11) are given in Fig. 8.15. If the value of C_2 corresponding to β_n is denoted as C_{2n}, it can be expressed in terms of C_{1n} from Eq. (E.8) as

$$C_{2n} = -C_{1n}\left(\frac{\cos \beta_n l - \cosh \beta_n l}{\sin \beta_n l - \sinh \beta_n l}\right) \tag{E.13}$$

Hence Eq. (E.7) can be written as

$$W_n(x) = C_{1n}\left[(\cos \beta_n x - \cosh \beta_n x) - \left(\frac{\cos \beta_n l - \cosh \beta_n l}{\sin \beta_n l - \sinh \beta_n l}\right)(\sin \beta_n x - \sinh \beta_n x)\right] \tag{E.14}$$

The normal modes of vibration can be obtained by the use of Eq. (8.81)

$$w_n(x, t) = W_n(x)(A_n \cos \omega_n t + B_n \sin \omega_n t) \tag{E.15}$$

with $W_n(x)$ given by Eq. (E.14). The general or total solution of the fixed-simply supported beam can be expressed by the sum of the normal modes:

$$w(x, t) = \sum_{n=1}^{\infty} w_n(x, t) \tag{E.16}$$

■

8.5.6 Forced Vibration

The forced-vibration solution of a beam can be determined using the mode superposition principle. For this, the deflection of the beam is assumed as

$$w(x, t) = \sum_{n=1}^{\infty} W_n(x) q_n(t) \tag{8.110}$$

where $W_n(x)$ is the nth normal mode or characteristic function satisfying the differential equation (Eq. 8.101)

$$EI \frac{d^4 W_n(x)}{dx^4} - \omega_n^2 \rho A W_n(x) = 0; \qquad n = 1, 2, \ldots \tag{8.111}$$

and $q_n(t)$ is the generalized coordinate in the nth mode. By substituting Eq. (8.110) into the forced-vibration equation, Eq. (8.77), we obtain

$$EI \sum_{n=1}^{\infty} \frac{d^4 W_n(x)}{dx^4} q_n(t) + \rho A \sum_{n=1}^{\infty} W_n(x) \frac{d^2 q_n(t)}{dt^2} = f(x, t) \tag{8.112}$$

In view of Eq. (8.111), Eq. (8.112) can be written as

$$\sum_{n=1}^{\infty} \omega_n^2 W_n(x) q_n(t) + \sum_{n=1}^{\infty} W_n(x) \frac{d^2 q_n(t)}{dt^2} = \frac{1}{\rho A} f(x, t) \tag{8.113}$$

By multiplying Eq. (8.113) throughout by $W_m(x)$, integrating from 0 to l, and using the orthogonality condition, Eq. (8.109), we obtain

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n^2 q_n(t) = \frac{1}{\rho A b} Q_n(t) \tag{8.114}$$

where $Q_n(t)$ is called the generalized force corresponding to $q_n(t)$

$$Q_n(t) = \int_0^l f(x, t) W_n(x)\, dx \tag{8.115}$$

and the constant b is given by

$$b = \int_0^l W_n^2(x)\, dx \tag{8.116}$$

Equation (8.114) can be identified to be, essentially, the same as the equation of motion of an undamped single-degree-of-freedom system. Using the Duhamel integral, the solution of Eq. (8.114) can be expressed as

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t$$

$$+ \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau)\, d\tau \tag{8.117}$$

where the first two terms on the right-hand side of Eq. (8.117) represent the transient or free vibration (resulting from the initial conditions) and the third term denotes the steady-state vibration (resulting from the forcing function). Once Eq. (8.117) is solved for $n = 1, 2, \ldots$, the total solution can be determined from Eq. (8.110).

EXAMPLE 8.8 Forced Vibration of a Simply Supported Beam

Find the steady-state response of a pinned-pinned beam subject to a harmonic force $f(x, t) = f_0 \sin \omega t$ applied at $x = a$, as shown in Fig. 8.17.

Solution: *Approach:* Mode superposition method.
The normal mode functions of a pinned-pinned beam are given by (see Fig. 8.15; also Problem 8.33)

$$W_n(x) = \sin \beta_n x = \sin \frac{n\pi x}{l} \tag{E.1}$$

where

$$\beta_n l = n\pi \tag{E.2}$$

The generalized force $Q_n(t)$, given by Eq. (8.115), becomes

$$Q_n(t) = \int_0^l f(x, t) \sin \beta_n x \, dx = f_0 \sin \frac{n\pi a}{l} \sin \omega t \tag{E.3}$$

The steady-state response of the beam is given by Eq. (8.117)

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) \, d\tau \tag{E.4}$$

where

$$b = \int_0^l W_n^2(x) \, dx = \int_0^l \sin^2 \beta_n x \, dx = \frac{l}{2} \tag{E.5}$$

The solution of Eq. (E.4) can be expressed as

$$q_n(t) = \frac{2 f_0}{\rho A l} \frac{\sin \frac{n\pi a}{l}}{\omega_n^2 - \omega^2} \sin \omega t \tag{E.6}$$

Thus the response of the beam is given by Eq. (8.110):

$$w(x, t) = \frac{2 f_0}{\rho A l} \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 - \omega^2} \sin \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \sin \omega t \tag{E.7}$$

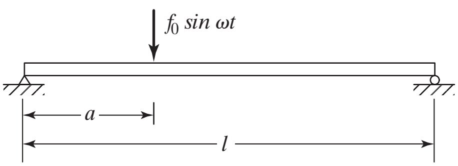


FIGURE 8.17 Pinned-pinned beam under harmonic force.

■

8.5.7 Effect of Axial Force

The problem of vibrations of a beam under the action of axial force finds application in the study of vibrations of cables and guy wires. For example, although the vibrations of a cable can be found by treating it as an equivalent string, many cables have failed due to fatigue caused by alternating flexure. The alternating flexure is produced by the regular shedding of vortices from the cable in a light wind. We must therefore consider the effects of axial force and bending stiffness on lateral vibrations in the study of fatigue failure of cables.

To find the effect of an axial force $P(x, t)$ on the bending vibrations of a beam, consider the equation of motion of an element of the beam, as shown in Fig. 8.18. For the vertical motion, we have

$$-(V + dV) + f\,dx + V + (P + dP)\sin(\theta + d\theta) - P\sin\theta = \rho A\,dx\,\frac{\partial^2 w}{\partial t^2} \tag{8.118}$$

and for the rotational motion about 0,

$$(M + dM) - (V + dV)\,dx + f\,dx\,\frac{dx}{2} - M = 0 \tag{8.119}$$

For small deflections,

$$\sin(\theta + d\theta) \simeq \theta + d\theta = \theta + \frac{\partial \theta}{\partial x}dx = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx$$

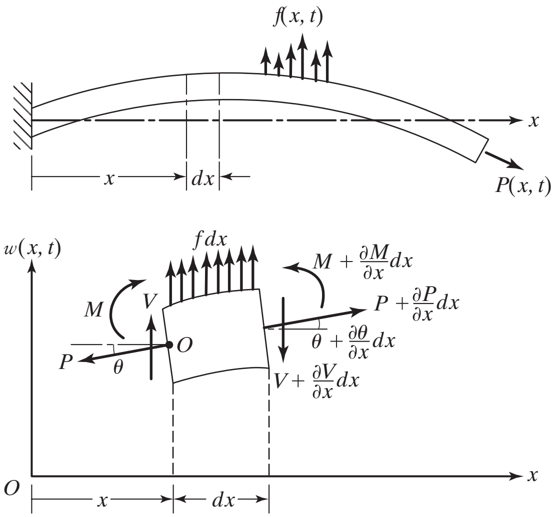


FIGURE 8.18 An element of a beam under axial load.

With this, Eqs. (8.118), (8.119), and (8.75) can be combined to obtain a single differential equation of motion:

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right] + \rho A\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = f \tag{8.120}$$

For the free vibration of a uniform beam, Eq. (8.120) reduces to

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = 0 \tag{8.121}$$

The solution of Eq. (8.121) can be obtained using the method of separation of variables as

$$w(x, t) = W(x)(A\cos\omega t + B\sin\omega t) \tag{8.122}$$

Substitution of Eq. (8.122) into Eq. (8.121) gives

$$EI\frac{d^4 W}{dx^4} - P\frac{d^2 W}{dx^2} - \rho A\omega^2 W = 0 \tag{8.123}$$

By assuming the solution $W(x)$ to be

$$W(x) = Ce^{sx} \tag{8.124}$$

in Eq. (8.123), the auxiliary equation can be obtained:

$$s^4 - \frac{P}{EI}s^2 - \frac{\rho A\omega^2}{EI} = 0 \tag{8.125}$$

The roots of Eq. (8.125) are

$$s_1^2, s_2^2 = \frac{P}{2EI} \pm \left(\frac{P^2}{4E^2I^2} + \frac{\rho A\omega^2}{EI}\right)^{1/2} \tag{8.126}$$

and so the solution can be expressed as (with absolute value of s_2)

$$W(x) = C_1\cosh s_1 x + C_2\sinh s_1 x + C_3\cos s_2 x + C_4\sin s_2 x \tag{8.127}$$

where the constants C_1 to C_4 are to be determined from the boundary conditions.

EXAMPLE 8.9 Beam Subjected to an Axial Compressive Force

Find the natural frequencies of a simply supported beam subjected to an axial compressive force.

Solution: The boundary conditions are

$$W(0) = 0 \tag{E.1}$$

$$\frac{d^2W}{dx^2}(0) = 0 \tag{E.2}$$

$$W(l) = 0 \tag{E.3}$$

$$\frac{d^2W}{dx^2}(l) = 0 \tag{E.4}$$

Equations (E.1) and (E.2) require that $C_1 = C_3 = 0$ in Eq. (8.127), and so

$$W(x) = C_2 \sinh s_1 x + C_4 \sin s_2 x \tag{E.5}$$

The application of Eqs. (E.3) and (E.4) to Eq. (E.5) leads to

$$\sinh s_1 l \cdot \sin s_2 l = 0 \tag{E.6}$$

Since $\sinh s_1 l > 0$ for all values of $s_1 l \neq 0$, the only roots to this equation are

$$s_2 l = n\pi, \qquad n = 0, 1, 2, \ldots \tag{E.7}$$

Thus Eqs. (E.7) and (8.126) give the natural frequencies of vibration:

$$\omega_n = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}}\left(n^4 + \frac{n^2 P l^2}{\pi^2 EI}\right)^{1/2} \tag{E.8}$$

Since the axial force P is compressive, P is negative. Further, from strength of materials, the smallest Euler buckling load for a simply supported beam is given by [8.9]

$$P_{\text{cri}} = \frac{\pi^2 EI}{l^2} \tag{E.9}$$

Thus Eq. (E.8) can be written as

$$\omega_n = \frac{\pi^2}{l^2}\left(\frac{EI}{\rho A}\right)^{1/2}\left(n^4 - n^2\frac{P}{P_{\text{cri}}}\right)^{1/2} \tag{E.10}$$

The following observations can be made from the present example:

1. If $P = 0$, the natural frequency will be same as that of a simply supported beam given in Fig. 8.15.
2. If $EI = 0$, the natural frequency (see Eq. (E.8)) reduces to that of a taut string.
3. If $P > 0$, the natural frequency increases as the tensile force stiffens the beam.
4. As $P \rightarrow P_{\text{cri}}$, the natural frequency approaches zero for $n = 1$.

■

8.5.8 Effects of Rotary Inertia and Shear Deformation

If the cross-sectional dimensions are not small compared to the length of the beam, we need to consider the effects of rotary inertia and shear deformation. The procedure, presented by Timoshenko [8.10], is known as the *thick beam theory* or *Timoshenko beam theory*. Consider the element of the beam shown in Fig. 8.19. If the effect of shear deformation is disregarded, the tangent to the deflected center line $O'T$ coincides with the normal to the face $Q'R'$ (since cross sections normal to the center line remain normal even

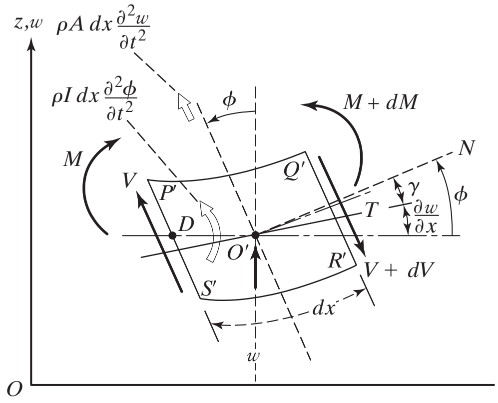


FIGURE 8.19 An element of Timoshenko beam.

after deformation). Due to shear deformation, the tangent to the deformed center line $O'T$ will not be perpendicular to the face $Q'R'$. The angle γ between the tangent to the deformed center line $(O'T)$ and the normal to the face $(O'N)$ denotes the shear deformation of the element. Since positive shear on the right face $Q'R'$ acts downward, we have, from Fig. 8.19,

$$\gamma = \phi - \frac{\partial w}{\partial x} \tag{8.128}$$

where ϕ denotes the slope of the deflection curve due to bending deformation alone. Note that because of shear alone, the element undergoes distortion but no rotation.

The bending moment M and the shear force V are related to ϕ and w by the formulas[2]

$$M = EI \frac{\partial \phi}{\partial x} \tag{8.129}$$

and

$$V = kAG\gamma = kAG\left(\phi - \frac{\partial w}{\partial x}\right) \tag{8.130}$$

where G denotes the modulus of rigidity of the material of the beam and k is a constant, also known as *Timoshenko's shear coefficient*, which depends on the shape of

[2]Equation (8.129) is similar to Eq. (8.75). Equation (8.130) can be obtained as follows:

$$\text{Shear force} = \text{Shear stress} \times \text{Area} = \text{Shear strain} \times \text{Shear modulus} \times \text{Area}$$

or

$$V = \gamma GA$$

This equation is modified as $V = kAG\gamma$ by introducing a factor k on the right-hand side to take care of the shape of the cross section.

the cross section. For a rectangular section the value of k is 5/6; for a circular section it is 9/10 [8.11].

The equations of motion for the element shown in Fig. 8.19 can be derived as follows:

1. For translation in the z direction:

$$\begin{aligned} -[V(x, t) + dV(x, t)] + f(x, t)\, dx + V(x, t) \\ = \rho A(x)\, dx \frac{\partial^2 w}{\partial t^2}(x, t) \\ \equiv \text{Translational inertia of the element} \end{aligned} \tag{8.131}$$

2. For rotation about a line passing through point D and parallel to the y-axis:

$$\begin{aligned} [M(x, t) + dM(x, t)] + [V(x, t) + dV(x, t)]\, dx \\ + f(x, t)\, dx \frac{dx}{2} - M(x, t) \\ = \rho I(x)\, dx \frac{\partial^2 \phi}{\partial t^2} \equiv \text{Rotary inertia of the element} \end{aligned} \tag{8.132}$$

Using the relations

$$dV = \frac{\partial V}{\partial x} dx \qquad \text{and} \qquad dM = \frac{\partial M}{\partial x} dx$$

along with Eqs. (8.129) and (8.130) and disregarding terms involving second powers in dx, Eqs. (8.131) and (8.132) can be expressed as

$$-kAG\left(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + f(x, t) = \rho A \frac{\partial^2 w}{\partial t^2} \tag{8.133}$$

$$EI \frac{\partial^2 \phi}{\partial x^2} - kAG\left(\phi - \frac{\partial w}{\partial x}\right) = \rho I \frac{\partial^2 \phi}{\partial t^2} \tag{8.134}$$

By solving Eq. (8.133) for $\partial \phi / \partial x$ and substituting the result in Eq. (8.134), we obtain the desired equation of motion for the forced vibration of a uniform beam:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho I\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2\, \partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4}$$

$$+\frac{EI}{kAG}\frac{\partial^2 f}{\partial x^2} - \frac{\rho I}{kAG}\frac{\partial^2 f}{\partial t^2} - f = 0 \tag{8.135}$$

For free vibration, $f = 0$, and Eq. (8.135) reduces to

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2\,\partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4} = 0 \tag{8.136}$$

The following boundary conditions are to be applied in the solution of Eq. (8.135) or (8.136):

1. Fixed end:

$$\phi = w = 0$$

2. Simply supported end:

$$EI\frac{\partial \phi}{\partial x} = w = 0$$

3. Free end:

$$kAG\left(\frac{\partial w}{\partial x} - \phi\right) = EI\frac{\partial \phi}{\partial x} = 0$$

EXAMPLE 8.10 Natural Frequencies of a Simply Supported Beam

Determine the effects of rotary inertia and shear deformation on the natural frequencies of a simply supported uniform beam.

Solution: By defining

$$\alpha^2 = \frac{EI}{\rho A} \quad \text{and} \quad r^2 = \frac{I}{A} \tag{E.1}$$

Eq. (8.136) can be written as

$$\alpha^2\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - r^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2\,\partial t^2} + \frac{\rho r^2}{kG}\frac{\partial^4 w}{\partial t^4} = 0 \tag{E.2}$$

We can express the solution of Eq. (E.2) as

$$w(x, t) = C\sin\frac{n\pi x}{l}\cos\omega_n t \tag{E.3}$$

which satisfies the necessary boundary conditions at $x = 0$ and $x = l$. Here, C is a constant and ω_n is the nth natural frequency. By substituting Eq. (E.3) into Eq. (E.2), we obtain the frequency equation:

$$\omega_n^4\left(\frac{\rho r^2}{kG}\right) - \omega_n^2\left(1 + \frac{n^2\pi^2 r^2}{l^2} + \frac{n^2\pi^2 r^2}{l^2}\frac{E}{kG}\right) + \left(\frac{\alpha^2 n^4 \pi^4}{l^4}\right) = 0 \tag{E.4}$$

It can be seen that Eq. (E.4) is a quadratic equation in ω_n^2, and for any given n there are two values of ω_n that satisfy Eq. (E.4). The smaller value corresponds to the bending deformation mode, while the larger one corresponds to the shear deformation mode.

The values of the ratio of ω_n given by Eq. (E.4) to the natural frequency given by the classical theory (in Fig. 8.15) are plotted for three values of E/kG in Fig. 8.20 [8.22].[3]

Note the following aspects of rotary inertia and shear deformation:

1. If the effect of rotary inertia alone is considered, the resulting equation of motion does not contain any term involving the shear coefficient k. Hence we obtain (from Eq. (8.136)):

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial x^2\,\partial t^2} = 0 \tag{E.5}$$

In this case the frequency equation (E.4) reduces to

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4\left(1 + \dfrac{n^2\pi^2 r^2}{l^2}\right)} \tag{E.6}$$

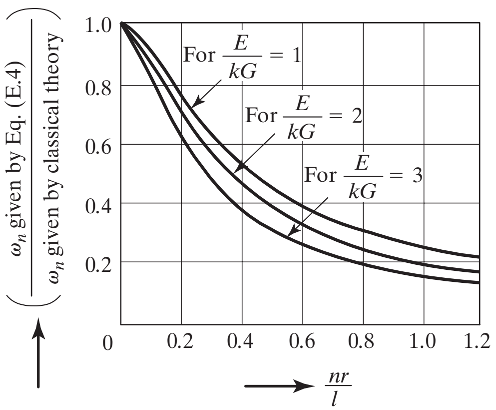


FIGURE 8.20 Variation of frequency.

---

[3]The theory used for the derivation of the equation of motion (8.76), which disregards the effects of rotary inertia and shear deformation, is called the *classical* or *Euler-Bernoulli* or *thin beam theory.*

2. If the effect of shear deformation alone is considered, the resulting equation of motion does not contain the terms originating from $\rho I(\partial^2\phi/\partial t^2)$ in Eq. (8.134). Thus we obtain the equation of motion

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \frac{EI\rho}{kG}\frac{\partial^4 w}{\partial x^2\,\partial t^2} = 0 \tag{E.7}$$

and the corresponding frequency equation

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4\left(1 + \dfrac{n^2\pi^2 r^2}{l^2}\dfrac{E}{kG}\right)} \tag{E.8}$$

3. If both the effects of rotary inertia and shear deformation are disregarded, Eq. (8.136) reduces to the classical equation of motion, Eq. (8.78),

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \tag{E.9}$$

and Eq. (E.4) to

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4} \tag{E.10}$$

■

8.5.9 Other Effects

The transverse vibration of tapered beams is presented in references [8.12, 8.14]. The natural frequencies of continuous beams are discussed by Wang [8.15]. The dynamic response of beams resting on elastic foundation is considered in reference [8.16]. The effect of support flexibility on the natural frequencies of beams is presented in [8.18, 8.19]. A treatment of the problem of natural vibrations of a system of elastically connected Timoshenko beams is given in reference [8.20]. A comparison of the exact and approximate solutions of vibrating beams is made by Hutchinson [8.30]. The steady-state vibration of damped beams is considered in reference [8.21].

8.6 Vibration of Membranes

A membrane is a plate that is subjected to tension and has negligible bending resistance. Thus a membrane bears the same relationship to a plate as a string bears to a beam. A drumhead is an example of a membrane.

8.6.1 Equation of Motion

To derive the equation of motion of a membrane, consider the membrane to be bounded by a plane curve S in the xy-plane, as shown in Fig. 8.21. Let $f(x, y, t)$ denote the pressure loading acting in the z direction and P the intensity of tension at a point that is equal to the product of the tensile stress and the thickness of the membrane. The magnitude of P is usually

This shows that the addition of a 10.0561-oz weight in the left plane at 145.5548° and a 5.8774-oz weight in the right plane at 248.2559° from the reference position will balance the turbine rotor. It is implied that the balance weights are added at the same radial distance as the trial weights. If a balance weight is to be located at a different radial position, the required balance weight is to be modified in inverse proportion to the radial distance from the axis of rotation.

■

9.5 Whirling of Rotating Shafts

In the previous section, the rotor system—the shaft as well as the rotating body—was assumed to be rigid. However, in many practical applications, such as turbines, compressors, electric motors, and pumps, a heavy rotor is mounted on a lightweight, flexible shaft that is supported in bearings. There will be unbalance in all rotors due to manufacturing errors. These unbalances as well as other effects, such as the stiffness and damping of the shaft, gyroscopic effects, and fluid friction in bearings, will cause a shaft to bend in a complicated manner at certain rotational speeds, known as the whirling, whipping, or critical speeds. Whirling is defined as the rotation of the plane made by the line of centers of the bearings and the bent shaft. We consider the aspects of modeling the rotor system, critical speeds, response of the system, and stability in this section [9.13–9.14].

9.5.1 Equations of Motion

Consider a shaft supported by two bearings and carrying a rotor or disc of mass m at the middle, as shown in Fig. 9.11. We shall assume that the rotor is subjected to a steady-state excitation due to mass unbalance. The forces acting on the rotor are the inertia force due to the acceleration of the mass center, the spring force due to the elasticity of the shaft, and the external and internal damping forces.[3]

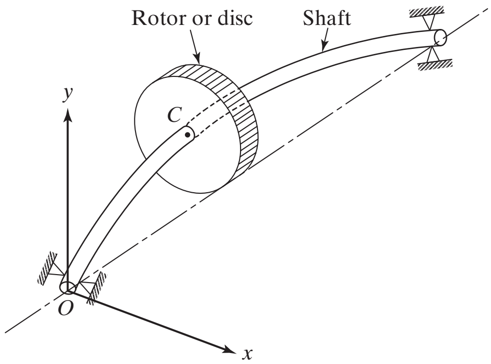


FIGURE 9.11 Shaft carrying a rotor.

[3]Any rotating system responds in two different ways to damping or friction forces, depending upon whether the forces rotate with the shaft or not. When the positions at which the forces act remain fixed in space, as in the case of damping forces (which cause energy losses) in the bearing support structure, the damping is called *stationary* or *external damping*. On the other hand, if the positions at which they act rotate with the shaft in space, as in the case of internal friction of the shaft material, the damping is called *rotary* or *internal damping*.

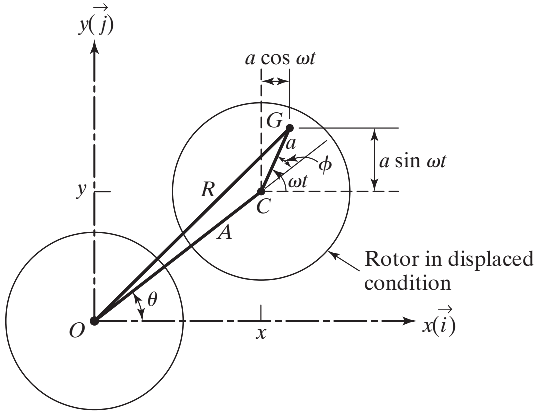


FIGURE 9.12 Rotor with eccentricity.

Let O denote the equilibrium position of the shaft when balanced perfectly, as shown in Fig. 9.12. The shaft (line CG) is assumed to rotate with a constant angular velocity ω. During rotation, the rotor deflects radially by a distance $A = OC$ (in steady state). The rotor (disc) is assumed to have an eccentricity a so that its mass center (center of gravity) G is at a distance a from the geometric center, C. We use a fixed coordinate system (x and y fixed to the earth) with O as the origin for describing the motion of the system. The angular velocity of the line OC, $\dot{\theta} = d\theta/dt$, is known as the whirling speed and, in general, is not equal to ω. The equations of motion of the rotor (mass m) can be written as

$$\begin{aligned}\text{Inertia force } (\vec{F_i}) &= \text{Elastic force } (\vec{F_e}) \\ &\quad + \text{Internal damping force } (\vec{F_{di}}) \\ &\quad + \text{External damping force } (\vec{F_{de}})\end{aligned} \tag{9.25}$$

The various forces in Eq. (9.25) can be expressed as follows:

$$\text{Inertia force: } \vec{F_i} = m\ddot{\vec{R}} \tag{9.26}$$

where $\vec{R}$ denotes the radius vector of the mass center G given by

$$\vec{R} = (x + a\cos\omega t)\vec{i} + (y + a\sin\omega t)\vec{j} \tag{9.27}$$

with x and y representing the coordinates of the geometric center C and $\vec{i}$ and $\vec{j}$ denoting the unit vectors along the x and y coordinates, respectively. Equations (9.26) and (9.27) lead to

$$\vec{F_i} = m[(\ddot{x} - a\omega^2\cos\omega t)\vec{i} + (\ddot{y} - a\omega^2\sin\omega t)\vec{j}] \tag{9.28}$$

$$\text{Elastic force: } \vec{F}_e = -k(x\vec{i} + y\vec{j}) \tag{9.29}$$

where k is the stiffness of the shaft.

$$\text{Internal damping force: } \vec{F}_{di} = -c_i[(\dot{x} + \omega y)\vec{i} + (\dot{y} + \omega x)\vec{j}] \tag{9.30}$$

where c_i is the internal or rotary damping coefficient:

$$\text{External damping force: } \vec{F}_{de} = -c(\dot{x}\vec{i} + \dot{y}\vec{j}) \tag{9.31}$$

where c is the external damping coefficient. By substituting Eqs. (9.28) to (9.31) into Eq. (9.25), we obtain the equations of motion in scalar form:

$$m\ddot{x} + (c_i + c)\dot{x} + kx - c_i\omega y = m\omega^2 a \cos \omega t \tag{9.32}$$

$$m\ddot{y} + (c_i + c)\dot{y} + ky - c_i\omega x = m\omega^2 a \sin \omega t \tag{9.33}$$

These equations of motion, which describe the lateral vibration of the rotor, are coupled and are dependent on the speed of the steady-state rotation of the shaft, ω. By defining a complex quantity w as

$$w = x + iy \tag{9.34}$$

where $i = (-1)^{1/2}$, and by adding Eq. (9.32) to Eq. (9.33) multiplied by i, we obtain a single equation of motion:

$$m\ddot{w} + (c_i + c)\dot{w} + kw - i\omega c_i w = m\omega^2 a e^{i\omega t} \tag{9.35}$$

9.5.2 Critical Speeds

A critical speed is said to exist when the frequency of the rotation of a shaft equals one of the natural frequencies of the shaft. The undamped natural frequency of the rotor system can be obtained by solving Eqs. (9.32), (9.33), or (9.35), retaining only the homogeneous part with $c_i = c = 0$. This gives the natural frequency of the system (or critical speed of the undamped system):

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \tag{9.36}$$

When the rotational speed is equal to this critical speed, the rotor undergoes large deflections, and the force transmitted to the bearings can cause bearing failures. A rapid transition of the rotating shaft through a critical speed is expected to limit the whirl amplitudes, while a slow transition through the critical speed aids the development of large amplitudes. Reference [9.15] investigates the behavior of the rotor during acceleration and deceleration through critical speeds. A FORTRAN computer program for calculating the critical speeds of rotating shafts is given in reference [9.16].